Optimization of Spatial Rotations of the Spacecraft Controlled by Inertial Actuators

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Abstract
The problem of correct choosing the time of optimal rotation from an arbitrary initial position into prescribed final angular position is investigated. The case, when slew maneuver is carried out with minimal magnitude of spacecraft angular momentum, is considered. Calculated dependences for constructing optimal program of reorientation (in sense of minimum angular momentum) are written as the law of variation of the angular velocity vector. Optimal control is in the class of regular motions. Dynamics of spacecraft rotation during the slew maneuver is similar to the known method of control, designed earlier, which includes as much as possible fast acceleration of a spacecraft, rotation with constant modulus of angular momentum and as much as possible fast cancellation of angular momentum. The formalized equations are presented, and computational expressions for determination of optimal duration of reorientation maneuver are obtained for known mass-inertial characteristics of a spacecraft, if attitude control is made by inertial actuators (system of control moment gyroscopes, gyrodynes). The knowledge of fit range of preferable duration of a slew maneuver helps to plan correctly the flight program of spacecraft controlled by the powered gyroscopes. The condition for determining the moment of the beginning of deceleration which uses current parameters of motion (information on angular position of a spacecraft and measurements of angular velocity) was given, what considerably increases accuracy of spacecraft move into a required position. Example of optimal spacecraft rotation and results of mathematical simulation adds the made theoretical descriptions, and illustrates reorientation process in visual form.

1. Introduction
Below we solve the problem of a choosing the duration of maneuver of spacecraft transfer into position of the required orientation (reorientation maneuver of a spacecraft). As spatial reorientation we mean transfer of axes $OXYZ$ bounded to the spacecraft body from one known angular position into another known (typically prescribed) angular position in a finite amount of time $T$. In this case, parameters of a slew maneuver (for example, components of slew quaternion) are known in advance, even before the beginning of maneuver; any initial angular mismatches can be possible (from a few degrees up to 180°). The angular orientation of the right-hand coordinate system $OXYZ$ (as well as its initial and final positions, $OX_1Y_1Z_1$ and $OX_fY_fZ_f$) is determined relative to a chosen coordinate system (reference basis $I$). In most cases the reference basis is inertial coordinate system $OX_iY_iZ_i$ (ICS).

Many papers [1-13] are dedicated to questions of controlled rotations of solid body
around its centre of mass. In particular, the kinematic problem of a slew maneuver is studied in detail [3] where solution was resulted for variant when angular velocity vector is limited by modulus. In [4], the questions of optimal rotation of a spacecraft by criteria of fastest response and minimum of energy expense for a case when the region of admissible values of the controlling moment is limited by sphere are considered, and itself spacecraft rotates around the vector of finite slew. Interesting results are presented in [5] where control is constructed by method of the combined synthesis by criterion of the generalized work. Later works [6, 7] are known also (including decisions on the basis of algorithms of indistinct logic [7]) which also describe the constructing the controls stabilizing rotation of a spacecraft around the instantaneous Euler axis (we will notice, that the majority of decisions [3-8] corresponds to spacecraft rotation around motionless axis) though the rotation in the plane of the least slew angle is not always optimum in practice (but only in individual special cases), however exactly it was not executed.

In this paper, it is supposed that inertial actuators, in particular, powered gyroscopes [9] (or gyrodynes), are used for control of spacecraft attitude, and size of angular momentum of spacecraft becomes essential-critical, therefore we must take into account this size at control of spacecraft rotation. Earlier, the control method for spacecraft reorientation [1] has been designed, which takes into account restriction of angular momentum of a spacecraft. This method is accepted as model motion (as reference). From theoretical studies [1, 2, 12] is known that level of spacecraft’s angular momentum and time of a slew maneuver $T$ are interdependent; if angular momentum is small (i.e. less) then duration of maneuver is more large. However at presence of perturbations acting upon the spacecraft, the choice of time $T$ is not so obvious (further we will show that available stock of the angular momentum of gyro-system has significant role at timing $T$). This investigation is dedicated to finding of optimal duration of spacecraft reorientation (within the mode [1]) at which the reserved stock of angular momentum of gyroscopes system would be maximum.

### 2. Equations of Motion and Statement of the Control Problem

It is assumed that the spacecraft attitude control is realized using actuators that generate torques relative to all three principal central axis of inertia of the spacecraft. Angular motion of the spacecraft as a solid body we describe by the equations written down in quaternion variables [3].

$$2\dot{\Lambda} = \Lambda \times \omega_k$$  \hspace{1cm} (1)

where $\omega$ is the vector of absolute angular velocity of a spacecraft; $\Lambda$ is the quaternion reflected spacecraft orientation relative to inertial basis $I$. The equation (1) has the boundary conditions $\Lambda(0) = \Lambda_0$ and $\Lambda(T) = \Lambda_f$. The quaternions $\Lambda_0$ and $\Lambda_f$ that specify directions of spacecraft axes at initial and final moments of time have any arbitrary a priori given values satisfying the condition $|\Lambda_0| = |\Lambda_f| = 1$. Spacecraft motion is considered optimum if the value

$$\max_{0 < t < T} \sqrt{J_1 \omega_1^2 + J_2 \omega_2^2 + J_3 \omega_3^2}$$  \hspace{1cm} (2)

is minimal ($T$ is the time of termination of the slew maneuver), where $\omega_i$ are the projections of the absolute angular velocity on axes of the body’s coordinate system, $J_i$ are the spacecraft central principal moments of inertia. Optimal control of a spatial rotation consists in transfer of a spacecraft from the position $\Lambda_0$ into the position $\Lambda_f$ according to the equation (1) with the minimum value of the index (2).

For the spacecrafts, controlled by powered gyroscopes (gyrodynes or flywheels), minimization of angular momentum of a spacecraft body is very important. Now powered gyroscopes (gyrodynes) have received good application as executive mechanisms of spacecraft attitude system. Their use in regime of a slew maneuver demands that the summary angular momentum of the gyro-system would not exceed the admissible value. At control of spacecraft attitude with use of inertial actuators (control moment gyroscopes), the vector of angular momentum of the gyro-system should be within preset limited region $S$, and the extending of this vector beyond the region $S$ leads to loss of controllability of a spacecraft; the size of available angular momentum determines the controlling possibilities of gyro-system. Control of spacecraft rotation is achieved by redistribution of the angular momentum between the system of gyrodynes and spacecraft body [9]; total angular momentum of a spacecraft as solid body with rotating masses is equal or close to zero. When designing, analyzing, elaborating, and modeling the algorithms of attitude control of a spacecraft with powered gyroscopes it is assumed that region $S$ of available angular momentum of the system of powered gyroscopes is limited by a sphere with radius $R_g$. Such statement is used by many developers, designers and researchers [14–17]; it is valid for a large number (if not for the majority) of spacecraft. Obviously, for the guaranteed presence of angular momentum $G$ of powered gyroscopes's system within region $S$ it is necessary that $|G|$ was as small as possible. Hence, it is necessary, that during operated rotation of a spacecraft from position $\Lambda_0$ in position $\Lambda_f$ the modulus of angular momentum of spacecraft body was minimum possible (since $L + G = 0$, where $L$ is the angular momentum of the spacecraft body, and $G$ is the angular momentum of the system of powered gyroscopes). From here we understand a sense of the formulated problem of control with minimization of the index (2), because the stock of the angular momentum of the gyro-system, determined by difference $R_g - |L|$ will be maximal in this case only.

### 3. Formalization of optimal Control of Spacecraft’s Slew Maneuver

Let us shortly write down basic relationships and equations.
describing optimal motion of a spacecraft. The accepted functional (2) does not contain (in an explicit form) components $M_i$ of the moment of forces. Therefore the projections of angular velocity $\omega_i (i = 1, 3)$ are considered as the controlling variables (controls). If there were no restrictions of the moment of forces, then optimal rotation of a spacecraft (in sense of criterion (2)) satisfies the equations [2]

$$\omega_i = b_i p_i / J^2_i,$$

$$\dot{p}_i = \omega_i p_{2i} - \omega_{2i} p_i, \quad \dot{p}_{2i} = \omega_i p_{1i} - \omega_{1i} p_i, \quad \dot{p}_3 = \omega_2 p_1 - \omega_1 p_2, \quad (4)$$

where $p_i, p_{2i}, p_3$ are projections of certain vector $p$ onto axes related with spacecraft; $b > 0$ is scalar value (from (4) we show that the vector $p$ is immovable relative to inertial basis $I$; $|p| = 1$. The problem of determining the optimal control consists in solving the system of equations of angular motion (1) and the equations (4) under the condition that optimal vector-function $\omega (t)$ is chosen basing on the requirement (3) and this solution satisfies the boundary conditions $\Lambda(0) = \Lambda_0, \quad \Lambda(T) = \Lambda_t$. The angular momentum $L$ and the magnitude $b$ are connected by the formula:

$$L^2 = b^2 (p_i^2 / J_1^2 + p_{2i}^2 / J_2^2 + p_3^2 / J_3^2).$$

In case of unlimited moments $M_i$, spacecraft rotates with $|L| = \text{const}$ at entire interval of motion $0 < t < T$; optimal rotation of a spacecraft is carried out with constant modulus of angular momentum $L_m$. Concrete value $L_m$ is unequivocally determined by time of maneuver termination $T$. The optimal vectors $\omega$ and $p$ are connected by relation [2]

$$\omega_i = \frac{L_m p_i}{J^2_i \sqrt{p_{1i}^2 / J_1^2 + p_{2i}^2 / J_2^2 + p_3^2 / J_3^2}} \quad (i = 1, 3), \quad (5)$$

at satisfying the conditions $\Lambda(0) = \Lambda_0, \quad \Lambda(T) = \Lambda_t$ for solution to equations (1) $\Lambda(t)$, where $L_m > 0$ is the magnitude (modulus) of angular momentum with which the slew maneuver of a spacecraft is made. The equations for angular velocities $\omega_i$ can be formalized in following form:

$$\dot{\omega}_1 = \omega_2 \omega_3 (J_2^3 - J_3^3) / J_1^2, \quad \dot{\omega}_2 = \omega_1 \omega_3 (J_1^3 - J_2^3) / J_3^2, \quad \dot{\omega}_3 = \omega_1 \omega_2 (J_1^3 - J_3^3) / J_2^2.$$  

During optimal maneuver (without restrictions for $M_i$), the desired solution $\omega(t)$ has the following properties

$$J_1^2 \omega_1^2 + J_2^2 \omega_2^2 + J_3^2 \omega_3^2 = R = \text{const}, \quad J_1^3 \omega_1^2 + J_2^3 \omega_2^2 + J_3^3 \omega_3^2 = D = \text{const}.$$  

Optimal control of spatial rotation consists of a short-term imparting the initial conditions of motion (the calculated angular velocity) to the spacecraft at the beginning of a slew maneuver, maintenance of spacecraft rotation with demanded (programmed) angular velocity $\omega(t)$ at which the modulus of angular momentum of a spacecraft has constant value $|L| = \text{const}$, and a short-term suppressing of available angular velocity to zero at the moment of time $t = T$, when $\Lambda(t) = \Lambda_t$ (at achievement by spacecraft of final position $\Lambda_t$). Key problem is the finding the law of variation of the vector $p(t)$ and value $L_m$ that at result of solving the system of the equations (1), (4), (5) with initial condition $\Lambda(0) = \Lambda_0$, the boundary condition $\Lambda(T) = \Lambda_t$ was satisfied at right endpoint (the determination of the vector $p(0)$ is a separate and rather complicated problem).

The problems in which the boundary values $\omega(0) = \omega(T) = 0$ (such conditions of spacecraft rotation are most typical) are of practical importance. Of course, at the times $t = 0$ and $t = T$ the angular velocity for a nominal program of spacecraft rotation, determined by equations (5), are not equal to zero. Consequently, transferring segments are unavoidable: acceleration of rotation as a transfer from the state of rest (when $\omega = 0$) to the regime of rotation with an angular momentum of maximal magnitude $H_0$, and braking, i.e., reduction of the spacecraft’s angular momentum down to zero; and for that to finish a maneuver during given time $T$ should be $H_0 > L_m$. Between acceleration of rotation and braking, the equations (4) and

$$\omega_i = \frac{H_0 p_i}{J^2_i \sqrt{p_{1i}^2 / J_1^2 + p_{2i}^2 / J_2^2 + p_3^2 / J_3^2}} \quad (i = 1, 3, H_0 = \text{const}) \quad (7)$$

are satisfied.

If the conditions of slew maneuver $\Lambda_m, \Lambda_t$, and the time $T$ are such that times of acceleration and braking are very small (in comparison with the total time of a maneuver $T$) and we may to neglect them, then one can consider as impulsive processes both imparting necessary angular momentum $H_0$ to the spacecraft and reducing available angular momentum down to zero, and almost during the entire maneuver (between acceleration and braking) $|L| = \text{const} = H_0$ with satisfaction of the equations (7), (4). When finding optimal solutions $p(t), \omega(t)$ the value of vector $p$ at the moment of instant $t = 0$ is determinative.

If the controlling moment $M$ is limited, then a boost of spacecraft angular momentum to the required level $|L| = H_0$ at the beginning of the slew maneuver and damping of available angular momentum to zero at the end of reorientation maneuver occupy some finite (distinct from zero) time. Of interest is the general case, when conditions of slew maneuver $\Lambda_m$ and $\Lambda_t$ are such that one cannot neglect the transition segments (acceleration and braking). Let the vector $M$ satisfy the condition

$$M_1^2 + M_2^2 + M_3^2 \leq m_0^2,$$

where $M_i$ are the projections of the moment of forces $M$ on the central principal axes of the spacecraft; $m_0$ – the maximum admissible magnitude of the controlling moment.

The laws of the fastest imparting and reduction of the angular velocity under the constraint (8) are known [10]. At the segment of acceleration, optimal control has the form

$$M = m_0 J \omega / \sqrt{J \omega},$$

where $J = \text{diag}(J_1, J_2, J_3)$ is the inertia tensor of spacecraft.
optimal motion, angular momentum of a spacecraft does not change the direction in inertial coordinate system, because the controlling moment $\mathbf{M}$ and angular momentum $\mathbf{L}$ have same direction. The magnitude of angular momentum varies according to the law $|\mathbf{L}| = m_0 \mathbf{L}$. At the segment of braking, optimal control we write in following form

$$\mathbf{M} = -m_0 J \omega / |J \omega|.$$  

At optimal motion, angular momentum of a spacecraft does not change the direction in inertial coordinate system, and the controlling moment $\mathbf{M}$ makes with the angular momentum an angle of 180 degree. The angular momentum varies according to the law $|\mathbf{L}| = L_{br} - m_0 (I - t_0)$, where $L_{br} = |J \omega(t_0)|$; $t_0$ is time of beginning of rotation damping. Since the instant $t_0$ when the equality $|J \omega(t_0)| = H_0$ becomes valid, optimal will be the motion (4), (7) at which the equality $|J \omega(t)| = H_0$ take place. It is obvious that because of existence of the boundary condition $\omega(T) = 0$ such a time instant $t_0$ ($t_0 < T$) should exist from which the angular momentum is begun to be reduced at a maximum controlling moment $\mathbf{M} = -m_0 \mathbf{L} / |\mathbf{L}|$ (the instant $t_0$ is chosen so that by the moment of a absolute stop $\omega = 0$ the spacecraft would occupy the desired angular position $\omega_t$. At the intervals of rotation acceleration and braking, the controlling moment $\mathbf{M}$ has ultimate maximum, and condition (8) is a strict equality, and within interval between acceleration and braking, the equations (4), (7) and equality $|\mathbf{L}| = \text{const} = H_0$ are valid. As result, the trajectory of spacecraft rotation $\Lambda(t)$ is partitioned in three components: $\Lambda(0) = \Lambda(t_0)$, $\Lambda(t_0) = \Lambda(t_0)$, and $\Lambda(t_0) + \Lambda(I)$. Let us represent the quaternion of a slew in the form:

$$\Lambda_s = \tilde{\Lambda}_{in} \circ \Lambda_T = \Lambda_{\Delta \omega} \circ \Lambda_{\Delta \omega} \circ \Delta \Lambda_{br},$$

where $\Delta \Lambda_{\omega} = \tilde{\Lambda}_{in} \circ \Lambda(t_0)$ is the quaternion of spacecraft rotation during the acceleration segment; $\Delta \Lambda_{br} = \tilde{\Lambda}(t_0) \circ \Lambda_T$ is the quaternion of spacecraft rotation during braking; $\Delta \Lambda_{\Delta \omega} = \tilde{\Lambda}(t_0) \circ \Lambda(t_0)$ is the quaternion of rotation during spacecraft rotation with maximum angular momentum $H_0$.

The optimal motion of a spacecraft consists of segments on which the controlling moment maximum in magnitude acts (segments of acceleration and braking), and of a segment of rotation with constant (in modulus) angular momentum, equal to the designed value $H_0$ supporting the reaching of final position $\omega_t$ in the given time $T$. On the segment of maximal controlling moment the angular momentum vector $\mathbf{L}$ has a permanent direction in the inertial space, but it is variable in magnitude (increase up to preset value on the acceleration segment, and decrease to zero on the braking segment), while the controlling moment $\mathbf{M}$ is immovable with respect to the reference basis $\mathbf{I}$ (the vectors $\mathbf{M}$ and $\mathbf{L}$ are parallel). During the spacecraft rotation with maximum angular momentum the parameters of motion are determined by equations (4), (7). In this case, the angular momentum vector $\mathbf{L}$ has a constant magnitude $H_0$, but its direction varies from a position preset at spacecraft acceleration to a position required during spacecraft braking. The spacecraft motion during its slew maneuver proceeds according to the following program of angular momentum variation: an increase of the magnitude of vector $\mathbf{L}$ from zero to $H_0$ with a maximum velocity (with maximum moment $\mathbf{M}$) at the invariable direction relative to the reference basis $\mathbf{I}$, next follows rotation of vector $\mathbf{L}$ with a constant modulus of $H_0$ according to the optimal law determined by equations (4), (7); and, finally, a decrease of the modulus of vector $\mathbf{L}$ down to zero with maximum possible velocity (with maximum moment $\mathbf{M}$) at the invariable direction relative to the reference basis $\mathbf{I}$. This program fully determines the spacecraft motion in the process of transfer from the state $\Lambda = \Lambda_{in}, \omega = 0$ to the state $\Lambda = \Lambda_t, \omega = 0$, since the equations (1) and the equality $\omega(t) = J^{-1} \mathbf{L}(t)$ take place.

Since initial and final angular velocities are exact equal and the magnitude of controlling moment is constant $|\mathbf{M}| = \text{const} = m_0$, duration of stages of acceleration and braking will be identical. The optimal solution $\omega(t)$ during segment of nominal motion (between acceleration and braking) possesses the properties (6), the vectors $\mathbf{M}$ and $\mathbf{L}$ are orthogonal, the magnitude of angular momentum is maximum and constant $|\mathbf{L}| = \text{const} = H_0$. Dynamics of motion is fully similar to solution of problem of spacecraft’s optimal rotation which was obtained in [1]. If the magnitude $H_0$ of maximum possible angular momentum of a spacecraft is known, then after solving the problem of maximally fast response we shall find the time of a maneuver $T$, if the time $T$ of termination of a maneuver is given, then value of parameter $H_0$ in control law is subject to definition.

The optimal control of angular position of a spacecraft can be realized by the method which is similar to the known mode [1]. Since at spacecraft braking the torque $\mathbf{M}$ is directed strictly against angular momentum $\mathbf{L}$, the instant when braking begins can be predicted with a good accuracy. Duration of rotation damping is $\tau = |\mathbf{L}| / m_0$. The moment of the beginning of braking segment is determined by the condition:

$$4 \arcsin \left( \frac{K \sqrt{q_1^2 + q_2^2}}{\sqrt{(J_1 \omega_1)^2 + (J_3 \omega_3)^2} m_0 \sqrt{(J_3 \omega_3)^2 + (J_5 \omega_5)^2}} \right) = \frac{K^2 \sqrt{\omega_2^2 + \omega_3^2}}{m_0 \sqrt{(J_3 \omega_3)^2 + (J_5 \omega_5)^2}},$$

where $q_j$ are the components of quaternion of mismatch $\tilde{\Lambda}(t) \circ \Lambda_T$ ($j = 0, 1, 2, 3$); $K = |J \omega|$ is the magnitude of spacecraft’s angular momentum. At braking segment, cancellation of angular momentum is make according to linear law: $|\mathbf{L}(t)| = H_0 - m_0 (I - t_0)$, where $t_0$ is the instant of the beginning of braking.

Determination of the time instant $t_0$ according to actual (measured values) kinematic parameters of motion (angular
mismatch and angular velocity) improves the accuracy of guiding the spacecraft to the required state \( \Lambda = \Lambda_f, \omega = 0 \).

4. Example of Mathematical Simulation of Optimal Rotation of a Spacecraft

The aspects of optimization of the terminal control that matches, in the prescribed time \( T \), the right-handed coordinate system \( OXTZ \) fixed to the spacecraft body with the basis whose attitude in the inertial space is given are above investigated. Now, in this section, we describe the numerical finding of the optimal control of the spacecraft maneuver. By way of example, consider the slewing maneuver of the spacecraft by 150° from the initial angular position \( \Lambda_m \) in which the spacecraft axes coincide with the axes of the reference basis \( I \) to the prescribed angular position \( \Lambda_f \). It was assumed in this case that the initial and final angular velocities are zero: \( \omega(0) = \omega(T) = 0 \). The elements of the quaternion \( \Lambda_t \) describing the desired attitude of the spacecraft after its slewing are assumed to be \( \lambda_0 = 0.258819, \lambda_1 = 0.723196, \lambda_2 = 0.5, \) and \( \lambda_3 = 0.4 \). We determine the optimal angular velocity control program \( \omega(t) \) for the spacecraft maneuver from the attitude \( \Lambda(0) = \Lambda_m \) to the attitude \( \Lambda(T) = \Lambda_f \). We assume that the central principal moments of inertia of the spacecraft are \( J_1 = 108652 \) kg \( \cdot \) m\(^2\), \( J_2 = 208049 \) kg \( \cdot \) m\(^2\), and \( J_3 = 220303 \) kg \( \cdot \) m\(^2\); maximum magnitude of the controlling moment is \( m_0 = 200 \) N \( \cdot \) m; and maneuver duration is assumed equal \( T = 150 \) s. Notice that the slew quaternion \( \Lambda_s \) corresponds to the case when the final rotation vector [4] (Euler’s axis) makes approximately the same angle with the axis \( OX \) as with the plane perpendicular to \( OX \). This is a fairly difficult case of reorientation of a solid body.

As a result of solving the boundary problem corresponding to the transfer of the spacecraft from the attitude \( \Lambda(0) = \Lambda_m \) to the attitude \( \Lambda(T) = \Lambda_f \), we obtained the rated value for vector \( \mathbf{p}(0) = \{0.20548, -0.15337, 0.96657\} \) (for possibility of solving the system of the equations (4), (7)). The desired accuracy is attained at the fifth approximation step of the vector \( \mathbf{p}(0) \) to the solution.

The results of mathematical simulation for the slew maneuver under the optimal control are presented in Figures 1-4. Figure 1 shows the dynamics of the optimal angular velocities \( \omega_1(t), \omega_2(t), \) and \( \omega_3(t) \) in the body-fixed coordinate system in the course of the slew maneuver. One clearly see partition of the entire maneuver into three characteristic stages: acceleration of a spacecraft (gain of angular velocity), spacecraft motion with constant module of angular momentum (segment of nominal rotation), and spacecraft braking (reduction of angular velocity to zero). The law of forming the control also confirms this. The behavior of the rate of changing the angular momentum value is of a relay character, the duration of the segments of acceleration \( \tau_{ac} \) (when \( K > 0 \)) and braking \( \tau_{br} \) (when \( K < 0 \)) is identical and equals \( \tau = 14.2 \) s. The maximum magnitude of spacecraft angular momentum during optimal rotation (the programmed level) is \( H_0 = 2835 \) N \( \cdot \) m \( \cdot \) s. The variation of the components \( \lambda_0(t), \lambda_1(t), \lambda_2(t), \) and \( \lambda_3(t) \) of the quaternion \( \Lambda(t) \), which determines the current attitude of the spacecraft in the course of the slew maneuver, is illustrated in Figure 2. Figure 3 shows the plots of the components \( I_1(t), I_2(t), \) and \( I_3(t) \) of the ort of angular momentum \( \mathbf{L} \) as functions of time. A distinctive feature is that the projection \( I_1 \) varies insignificantly (its variation is much less than the variation of \( I_2 \) and \( I_3 \)). This indicates that \( OX \) is the longitudinal axis of the spacecraft. It is also worth noting that the variables \( \omega_i(t) \) and \( I_i(t) \) corresponding to the longitudinal axis of the spacecraft have a constant sign (for all combinations of the boundary values \( \Lambda_m \) and \( \Lambda_f \)).

Finally, Figure 4 shows the variation of magnitude of controlling moment in the course of the slew maneuver. All three stages of the maneuver are well pronounced in it. During acceleration and braking processes the controlling moment is maximum possible. In the time interval between acceleration and braking, the controlling moment magnitude is substantially less than during gain and reduction of the spacecraft angular velocity; controlling moment \( \mathbf{M} \) is determined according to dynamic Euler’s equations in which the time functions of angular velocities \( \omega_i(t) \) are a solution to system of equations (4) with expressions (7). At the stages of acceleration and braking, the controlling moment module is constant; during spacecraft rotation with maximum angular momentum the controlling moment magnitude varies slightly (for a dynamically symmetric spacecraft it is constant).

![Figure 1. Variation of angular velocities in the course of optimal slew maneuver.](image1)

![Figure 2. Optimal variation of components of the attitude quaternion \( \Lambda \).](image2)
probability of involving any other (except the gyrodyne) means of attitude control (for example, jet engines). For the case with zero boundary conditions \( \mathbf{q}(0) = \mathbf{q}(T) = 0 \) only a single type of motion is realized: the first phase - acceleration of spacecraft rotation with maximal controlling moment \( |\mathbf{M}| = m_0 \) up to \( |\mathbf{L}| = H_0 \), next the site of spacecraft motion with constant in magnitude of angular momentum \( |\mathbf{L}| = H_0 \) (with satisfaction of equalities (6)), and then the symmetric segment of spacecraft braking with maximal controlling moment \( |\mathbf{M}| = m_0 \) up to a full stop of the spacecraft \( (\mathbf{M} \parallel \mathbf{L}) \). Changing the magnitude of angular momentum \( \mathbf{G} \) of the system of powered gyroscopes during the slew maneuver is such that the condition \( d\mathbf{G}/dt = \text{const} \) is true on the segments of acceleration and braking (since the moment \( \mathbf{M} \) of the controlling forces is much more perturbations moment \( \mathbf{M}_{\text{per}} \)). And, in most cases, it is possible to assume

\[
|d\mathbf{G}/dt|_{t < \tau_{\text{ac}}} = |d\mathbf{G}/dt|_{t > \tau_{\text{br}}},
\]

where \( \tau_{\text{ac}} \) is the instant of termination of acceleration; \( \tau_{\text{br}} \) is the instant of the beginning of braking. For the hypothetical case when \( \mathbf{M}_{\text{per}} = 0 \), acceleration of spacecraft rotation can be carried out up to situation \( |\mathbf{L}| = R_0 \), because in this ideal case \( |\mathbf{L}| = \|\mathbf{G}\| \), and within the interval between acceleration and braking we have \( |d\mathbf{G}/dt| = 0 \) (here \( R_0 \) is radius of the sphere entered in region \( S \) of available values of the angular momentum \( \mathbf{G} \) of powered gyroscopes system).

In real conditions of flight \( \mathbf{M}_{\text{per}} \neq 0 \), and therefore \( \mathbf{L} + \mathbf{G} \neq 0 \), and hence in general case \( |d\mathbf{G}/dt| \neq 0 \) on the segment of nominal rotation (when \( |\mathbf{L}(t)| = \text{const} \)). At the presence of perturbations moments \( \mathbf{M}_{\text{per}} \neq 0 \) there is a problem - what value should be time of a maneuver \( T \) (or level \( H_0 \) that up to termination of maneuver the possible increase in size \( |\mathbf{G}| \) was less \( R_0 - H_0 \) (dependence \( H_0 \) from \( T \) is the monotonously decreasing function). Though perturbations \( \mathbf{M}_{\text{per}} \) can "help" the rotation of a spacecraft (in this case \( d\mathbf{G}/dt < 0 \)), but to guarantee such evolution of events throughout all interval of time \( \left[\tau_{\text{ac}}, \tau_{\text{br}}\right] \) is impossible. Therefore at constructing the designed program of control, we must take into account the worst scenario - to assume the perturbations \( \mathbf{M}_{\text{per}} \) greatest possible in magnitude and directed against angular momentum \( \mathbf{L} \) of spacecraft body. Then \( d\mathbf{G}/dt = \mathbf{M}_{\text{per}} \) and

\[
\max \ |d\mathbf{G}/dt| = M_{\text{per}} \text{cal},
\]

where \( M_{\text{per}} \text{cal} \) is the calculated (the greatest possible) magnitude of perturbation moment \( \mathbf{M}_{\text{per}} \) (i.e. \( \mathbf{M}_{\text{per}} \leq M_{\text{per}} \text{cal} \)). The increment of magnitude of gyro-system angular momentum \( \mathbf{G} \) during spacecraft rotation with \( |\mathbf{L}| = \text{const} \) will following

\[
\Delta \mathbf{G} = \int_{\tau_{\text{ac}}}^{\tau_{\text{br}}} \mathbf{M}_{\text{per}} \ dt \leq (\tau_{\text{br}} - \tau_{\text{ac}}) M_{\text{per}} \text{cal} = (T - \tau_{\text{ac}} - \tau_{\text{br}}) M_{\text{per}} \text{cal},
\]

where \( \tau_{\text{ac}} \) and \( \tau_{\text{br}} \) are the durations of acceleration and braking; \( G = \|\mathbf{G}\| \). Thus, the relation

\[
G \left( \tau_{\text{ac}} \right) + (T - \tau_{\text{ac}} - \tau_{\text{br}}) M_{\text{per}} \text{cal} \leq R_0
\]

should be satisfied, where \( R_0 \) is the known value before preset.

**5. Problem of Choosing the Optimal Time of the Slew Maneuver**

For a spacecrafts with inertial actuators (powered gyroscopes), is important not only to minimize the angular momentum during the slew maneuver, but also to preset properly (maximum correctly) time \( T \) of maneuver ending. The problem consists in determination of such duration of a slew \( T \) (and the value of the parameter \( H_0 \)), that during the spacecraft motion around its center of mass, evolution of vector \( \mathbf{H} \) of the total angular momentum of the system of gyrodyne will not bring this vector beyond the limits of region \( S \) of admissible values (no saturating of the system of gyrodyne will occur), and no necessity will emerge in unloading, i.e., in removal of accumulated angular momentum of the gyrodyne system by applying the moment of forces of another nature (magnetic [4], produced by switching the attitude control jet engines, etc.). Such motions of spacecraft are considered as allowable (in the sense of controlling the spacecraft orientation without unloading the system of gyrodyne).

At solving the formulated problem (choice of optimal time of slew maneuver) two moments are initial - (a) slew maneuver of a spacecraft occurs according to the method [1], and (b) the stock of the angular momentum of the gyrodyne system should be maximum (at the found parameters of slew maneuver \( H_0 \) and \( T \)); last requirement allows to reduce

![Figure 3. Variation of the variables \( l_1, l_2, l_3 \) under optimal control.](image1)

![Figure 4. Character of modulus of the controlling moment \( \mathbf{M} \) during optimal rotation.](image2)
At the moment of the termination of acceleration $G(\tau_0) = [L(\tau_0)]$ (since $|M_{\text{per}}| < \eta_0$, "Saturation" of system of powered gyroscopes can take place in the limiting case, if the equality $[L(\tau_0)] = R_0 - (T - \tau_\infty - \tau_0) M_{\text{cal}}$ is valid. It is necessary to note, that if the time of slew maneuver $T$ less, then probability of achievement by the moment $\tau_0$ of the braking beginning of the threshold level $|G(\tau)| = R_0$ will less, it would seem. However with reduction of time $T$, the required angular momentum $[L(\tau_0)]$ increases, and the stock $\Delta R = R_0 - [L(\tau_\infty)]$ decreases, that raises probability of presence of gyro-system's "saturation" in the course of slew maneuver (probability of satisfaction of equality $[G] = R_0$ increases). Here the problem of spacecraft rotation with minimum angular momentum $[L]$ during given time $T$ is very important (for that to increase the stock of angular momentum $R_0 - [L]$ for its use for indemnification of prospective perturbation moments $M_{\text{per}}$).

At construction of the optimum program of management, we can vary only two parameters – time of a maneuver $T$ or magnitude of spacecraft's angular momentum $[L(\tau_\infty)]$ (other characteristics are preset by conditions of slew maneuver and cannot be changed). We will write down the equation setting connection between time of rotation $T$, an estimation of the maximum magnitude of perturbation moments $M_{\text{cal}}$ and the computed value of spacecraft's angular momentum $[L(\tau_\infty)]$.

For the rotation of solid body along the trajectory which satisfies the equations (3)–(4), the equality

$$\int_0^T [L] dt = \text{const} = S_L$$

is true, where the value $S_L$ is determined exclusively by quaternion of a slew $\Lambda_s = \Lambda_\infty \circ \Lambda_T$ and inertial characteristics of a spacecraft $J_1, J_2, J_3$ [11]. Having designated $L_0 = [L(\tau_\infty)]$, and if we believe that on the segments of acceleration and braking, the modulus of angular momentum varies under linear law $[dL]/[dt] = m$, where $m$ is the maximum speed of varying the modulus of angular momentum, we will receive following relationships:

$$2 \int_0^T [L] dt = L_0 (2T - \tau_\infty - \tau_0) \text{ or } L_0 (T - L_0/m) = S_L \quad (\text{since } \tau_\infty = \tau_\infty = L_0/m).$$

Two statements of a problem are possible: (a) at the presence of constraint $T \leq T_{\text{cal}}$ we use criterion $L_0 \to \min$ or $M_{\text{cal}} T \to \max$; and (b) if right endpoint is free, then we receive the system of two equations in which the time of a maneuver $T$ is specified from the condition $M_{\text{cal}} \to \max$. These equations have the following form

$$L_0 + M_{\text{cal}} T = R_0, \quad L_0 (T - L_0/m) = S_L \text{ or } T = (R_0 - L_0)/M_{\text{cal}}, \quad L_0^2/m - L_0 T + S_L = 0;$$

whence

$$L_0^2 (1/M_{\text{cal}} + 1/m) - L_0 R_0/M_{\text{cal}} + S_L = 0.$$

The solution of last equation is

$$L_0 = 0.5 \left( R_0 m + \sqrt{R_0^2 m^2 - 4S_L m M_{\text{cal}} (m + M_{\text{cal}})/\left(m + M_{\text{cal}}\right)} \right).$$

If $m \to \infty$, then values $\tau_\infty \to 0$, $\tau_0 \to 0$ and $S_L = L_0 T$; therefore $R_0 - L_0 = M_{\text{cal}} T = M_{\text{cal}} S_L / L_0$, and hence

$$L_0 = (R_0 \pm \sqrt{R_0^2 - 4S_L M_{\text{cal}}}) / 2$$

(it is obvious that $L_0 < R_0$).

It is obvious that $m > M_{\text{cal}}$ and there are two solutions of quadratic equation (in both $L_0 > 0$; smaller value $L_0$ corresponds to higher border $T_{\text{max}}$, more big value $L_0$ corresponds to the lower border $T_{\text{min}}$ a slew maneuver for minimum time). Time $T$ we appoint in range $T_{\text{min}} < T < T_{\text{max}}$. Outside of interval borders $[T_{\text{min}}, T_{\text{max}}]$ the rotation without "unloading" of gyro-system is not guaranteed, as "saturation" can occur or because of big $L_0$ which was imparted after acceleration of rotation (it is close to $R_0$), owing to small time $T$, or because of the big duration of a stage of nominal rotation (between acceleration and braking), and, as a consequence, the perturbations accumulated up during the slew maneuver "will eat" all stock $R_0 - L_0$ of angular momentum of the gyro-system. Taking into account that $m >> |M_{\text{per}}|$, we will give the approximate estimations (precisely enough reflecting true required levels $L_0$ and $T$).

Since

$$\sqrt{R_0^2 m^2 - 4S_L M_{\text{cal}} (m + M_{\text{cal}})} = R_0 m - 2S_L m M_{\text{cal}} / R_0,$$

we can assume as optimum $L_0 = R_0 / 2 - S_L M_{\text{cal}} / R_0$ and $T = 2S_L / R_0 + R_0 / 2m$.

If the magnitude $M_{\text{cal}}$ is not known or can vary its value over a wide range, then we must appoint time of a maneuver $T$ using the condition of maximum of parameter $M_{\text{cal}}$ for which the inequality

$$R_0^2 m > 4 S_L M_{\text{cal}} (m + M_{\text{cal}})$$

is satisfied. The function $S_L M_{\text{cal}} (m + M_{\text{cal}})$ increases with increase of $M_{\text{cal}}$. Hence, it is possible to assume that the roots of quadratic equation in which $L_0$ is single unknown value, are converged nearer to each other with increase of the parameter $M_{\text{cal}}$ and the range $[T_{\text{min}}, T_{\text{max}}]$ concerning preferable values of the slew maneuver time $T$ will be narrowed. Let us find such critical value of parameter $M_{\text{cal}}$ (we will designate it $M_{\text{cal}}$) at which the rotation from position $\Lambda_\infty$ into position $\Lambda_T$ is still possible without infringement of requirement $|G| \leq R_0$. For this purpose, it is necessary to solve the equation

$$R_0^2 m - 4 S_L M_{\text{cal}} (m + M_{\text{cal}}) = 0$$

with respect to $M_{\text{cal}}$ (all other values in this equation are known, they were preset by conditions of the problem of optimal maneuver). As result, we have
Control of spacecraft attitude using inertial actuators (in particular, by system of powered gyroscopes) is carried out often enough. Efficiency of use of inertial devices (control moment gyroscopes) in regime of spatial rotation is defined by not only the accepted algorithm of spacecraft attitude control but by time of the ending the reorientation maneuver. For the excepting the saturation of gyro-system its total angular momentum should not exceed admissible value (the slew maneuver is carried out due to redistribution of the angular momentum of spacecraft body). For situation that rotation will occur without the unloading of gyro-system, the modulus of controlling moment remains to constant. This circumstance allows us to use the described control method of slew maneuver in a case when the controlling moment is limited by sphere. Basing on properties of optimal rotation [1], the described control method of spacecraft maneuver is exclusively important to apply in cases when the controlling moment is limited by sphere.

Optimization of spacecraft reorientation when motion is controlled by inertial actuators (for example, gyrodynes) includes two main tasks. First task is traditional optimization problem which consists in finding the mode of rotation with minimal magnitude of angular momentum. Second problem is new, it consists in calculating the most correct duration of slew maneuver under which necessity of "unloading" of gyro-system is excluded. In this paper, both problems were successfully solved. Numerical example and results of mathematical simulation of optimal spacecraft rotation, as illustration in visual form, are given. They add the made theoretical descriptions and confirm the efficiency and practical feasibility of the control method developed.

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