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Extended Legendre Wavelet Operational Matrix of Integration

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Abstract: This paper first presents the extended Legendre wavelet (ELW) defined on interval (-*r*, *r*) (*r* is a rational number). Second, the integral operator matrix is calculated by using the ELW. Finally, the ELW and operational matrix obtained are applied to solving a ordinal differential equation (ODE). The good results of this numerical experiment demonstrate that this method is valid and applicable.

Keywords: Legendre Wavelet, Extended Legendre Wavelet, Integral Operator Matrix

1. Introduction

It is very important to select a suitable basis function in numerical methods. Generally, orthogonal functions have received considerable attentions in solving analysis, optimal control and various problems of dynamic systems [1-2]. Especially, applications of Legendre wavelet integral and derivative operators for numerical approximations of the ODEs can be found in the references [1-11]. The essence of this method is that the ODEs are converted to a system of algebraic equations by using the operational matrices of integration or derivative [1]. For example, Razzaghi Yousefi [1] derived Legendre wavelet operational matrix of integration which is defined on the interval $[0,\ 1)$ and this approach is implemented to solve the ODE. Pandey et al. [3] proposed Legendre wavelet operational matrix of derivative technique to solving the ODE.

In this paper, we focus on the calculation of the integral operator by the ELW. Firstly, using the translation property of Legendre wavelet, a conception of the ELW defined on the interval $(-r,\ r)$ is presented. Secondly, the Legendre wavelet operational matrix of integration is calculated on sub-interval other than on the whole interval $[0,\ 1)$ as in the literature [1]. The main advantage of this computation is that the operational matrix of integration is lower dimensional matrix compared with other methods. Thus, it simplifies for solution system of algebraic equations with less storage space and execution time.

The organization of the paper is as follows. In Section 2, descriptions of Legendre wavelet bases and its rich properties are demonstrated, then the ELW is proposed. Section 3 computes the ELW operational matrix of integration on the interval $[l/a^{-n},\ (l+1)/a^{-n})$. Section 4 applies the the ELW operational matrix of integration to solving a linear equation. The good results of the numerical experiment show that our method is very effective. Conclusions of the proposed method in this paper are given at the end in Section 5.

2. Properties of Legendre Wavelet

In this section, Legendre wavelet [2] is introduced and its properties are analyzed and then the translation property is demonstrated by using the property of its structure.

2.1. Legendre Wavelet

For decomposition level $n=0,\ 1,\ 2,\cdots$ and translation $l=0,\ 1,\ 2,\cdots,2^n-1$, we define the sub-interval, i.e., element $I_{nl}=[2^{-n}l,\ 2^{-n}(l+1))$. For $p=1,2,\cdots$, define $V_{p,n}$ as a subspace of piece-wise polynomial functions satisfying

$V_{p,n}=\{f: f\big|_{I_{nl}}$ is a polynomial of degree strictly less than p; and f vanishes elsewhere$\}$.

We now start to review Legendre polynomials and Legendre wavelet bases. Let $L_k(x)$ denote Legendre polynomial of degree k, which is defined as follows

$$L_0(x)=1\,,\ \ L_1(x)=x\,,$$

$$L_{k+2}(x)=\frac{2k+3}{k+2}xL_{k+1}(x)-\frac{k+1}{k+2}L_k(x)\,. \tag{1}$$

Then, at the level of resolution $n=0$, let $\varphi_k(x)$ denote Legendre scale functions defined as

$$\varphi_k(x)=\begin{cases}\sqrt{2k+1}L_k(2x-1), & x\in[0,\ 1],\\ 0, & x\notin[0,\ 1].\end{cases} \tag{2}$$

The whole set $\{\varphi_k\}_{k=0}^{p-1}$ forms an orthonormal basis for $V_{p,0}$. Generally, the subspace $V_{p,n}$ is spanned by $2^n p$ functions which are obtained from $\varphi_0,\cdots,\varphi_{k-1}$ by dilation and translation, i.e.,

$$V_{p,n}:=V_{p,nl}=\text{span}\,\{\varphi_{k,nl}(x)=2^{n/2}\varphi_k(2^n x-l),\quad 0\le k\le p-1,\quad 0\le l\le 2^n-1\}\,. \tag{3}$$

In order to intuitively understand Legendre scale functions, we let $p=3$, the scale $n=2$, then Figure 1 plots the basis functions $\varphi_{k,2l}(x)$ which defined on the whole interval $[0,\ 1)$.

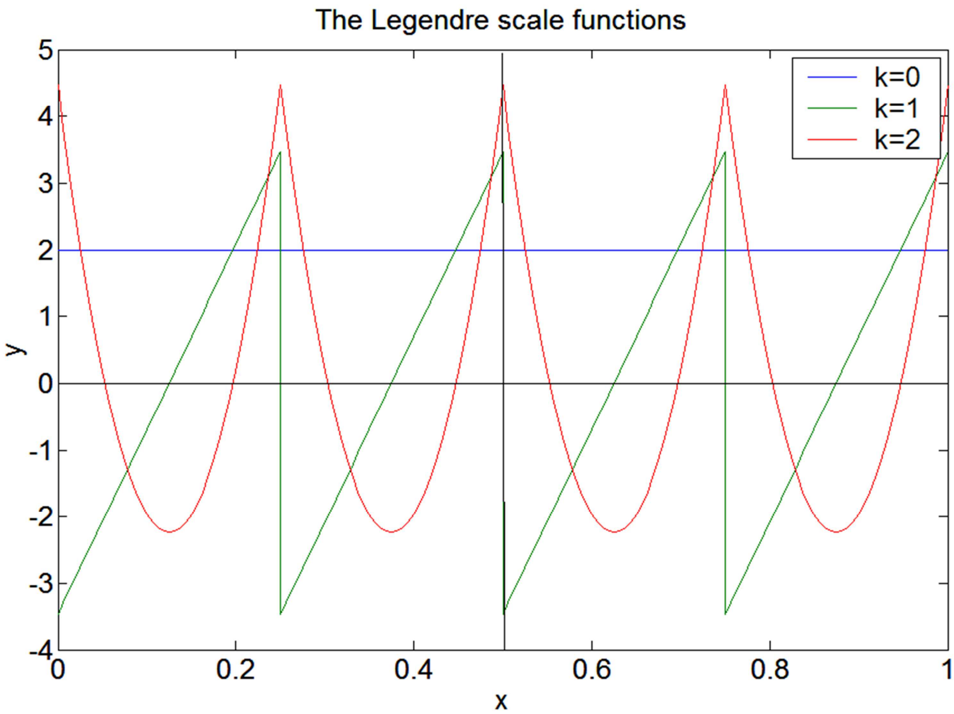

Figure 1. *The six Legendre wavelet bases.*

2.2. Translation Property of Legendre Wavelet

Actually, Legendre wavelet approximates a function by piece-wise Legendre polynomials, which are defined on the sub-interval I_{nl}. Consequently, Legendre wavelets that are defined on the interval $[0,\ 1)$ can be obtained by using a translation operator transformation on Legendre wavelet defined on the sub-interval.

Here, a concept of translation operator is defined as

$$T(l)f:\ f(x)\to g(x)\,. \tag{4}$$

The operators $T_{right}(l)$, $T_{left}(l)$ denote the function $g(x)$ obtained from the function $f(x)$ by translation l times by a certain unit (2^{-n} or a^{-n} in this paper) on the right and left, respectively.

Lemma. Legendre wavelet defined on the whole interval $[0,\ 1)$ can be obtained by using the translation operator $T_{right}(l)$ transformation on Legendre wavelet defined on the sub-interval $[0,\ 2^{-n})$, i.e.,

$$\varphi_{k,nl}=T_{right}(l)[\varphi_{k,n0}(x)]\,, \tag{5}$$

which $\varphi_{k,n0}(x)=2^{n/2}\varphi_k(2^n x),\quad 0\le k\le p-1\}$ and $l=1,\cdots,2^n-1$.

Furthermore, the interval of the definition can be extended by using the translation property of Legendre wavelet. Thus, a concept of the DLW is presented.

2.3. Extended Legendre Wavelet

In this section, a concept of the ELW is demonstrated. Now, if the interval $[0,\ 1)$ is divided by a positive integer a

$(a \geq 2)$, then Legendre wavelet defined on the interval $[0,\ a^{-n})$ can be obtained by

$$\varphi_{k,n0a}(x) = \begin{cases} a^{n/2}\sqrt{2k+1}L_k(2a^n x-1), & 0 \leq x < a^{-n}, \\ 0, & otherwise. \end{cases} \qquad (6)$$

Similar to Lemma, Legendre wavelet defined on the interval $[0,\ 1)$ is obtained as the form of

$$\varphi_{k,nl}(x) = T_{right}(l)[\varphi_{k,n0a}(x)], \qquad (7)$$

where $l = 0,\ 1, \cdots, a^n - 1$.

Theorem. Legendre wavelet defined on the interval $(-r,\ r)$ ($r = l' a^{-n}$ is a rational number) can be obtained as the form of

$$\varphi_{k,nl}(x) = T(l')[\varphi_{k,n0a}(x)], \qquad (8)$$

where $l = -l', \cdots, 0,\ 1, \cdots, l'-1$ and l' is a positive integer such that $r = l' a^{-n}$.

Based on the result in (8), we obtain the ELW defined on the interval $(-r,\ r)$. For example, we let $p = 3$, $n = 1$, $a = 2$ and $l' = 2$, then we plot the ELW on the interval $(-1,\ 1)$ as Figure 2

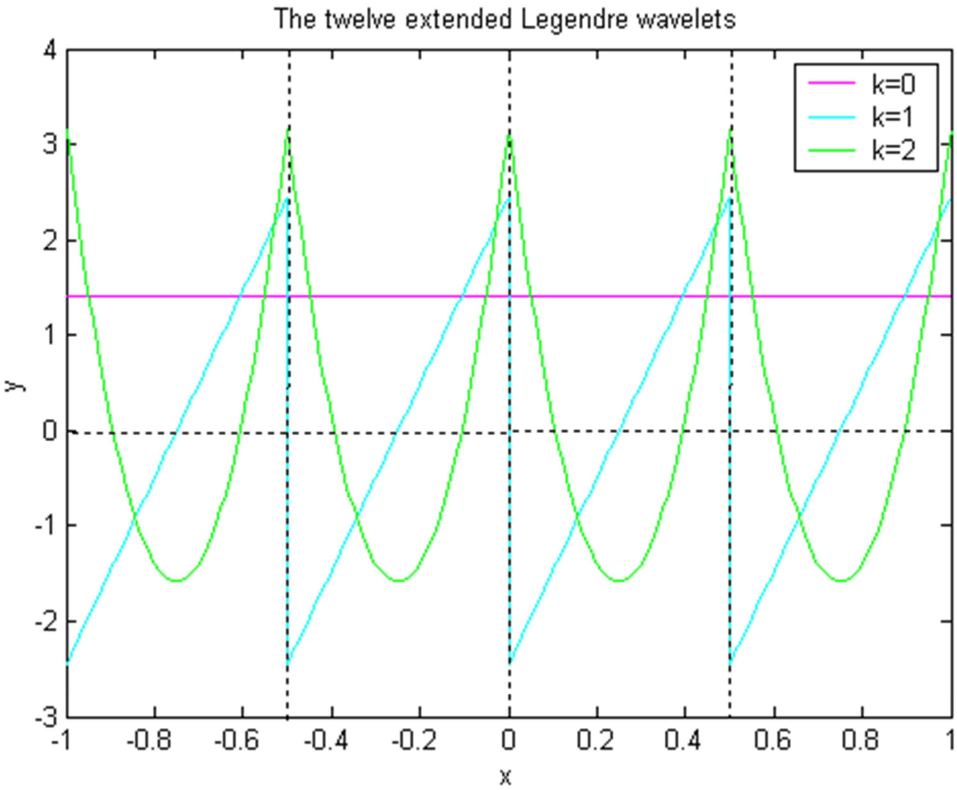

Figure 2. *The twelve extended Legendre wavelet bases.*

In addition, the ELW base is an orthogonal set. It is easy to prove this conclusion by using simple variable substitute.

3. Operational Matrix of Integration by the ELW

In this section, we first calculate the integral operator matrix by Legendre wavelet defined on the interval I_{nl}. Second, we use similar method to compute the ELW operational matrix of integration on the interval $[l/a^{-n},\ (l+1)/a^{-n})$, where $-l' \leq l \leq l'-1$.

Generally, we let $\Phi_{k,nl} = [\varphi_{0,nl},\ \varphi_{1,nl}, \cdots, \varphi_{p-1,nl}]^T$ and for any $x \in [l/2^{-n},\ (l+1)/2^{-n})$ we obtain

$$\int_{l/2^{-n}}^{x} \Phi_{k,nl}(x)dx := P_{p\times p}\Phi_{k,nl}(x), \qquad (9)$$

where $P_{p\times p}$ is the integral operator matrix and its elements are computed by the formula as

$$(P)_{k+1,k'+1} = \int_{l/2^{-n}}^{(l+1)/2^{-n}} [\int_{l/2^{-n}}^{x} \varphi_{k,nl}(x)dx]\varphi_{k',nl}(x)dx. \qquad (10)$$

Then, we have

$$P_{p\times p}=2^{-n-1}\begin{pmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & 0 & \cdots & 0 \\ -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{5}\times\sqrt{3}}{5\times 3} & 0 & \cdots & 0 \\ 0 & -\frac{\sqrt{5}\times\sqrt{3}}{5\times 3} & 0 & \frac{\sqrt{7}\times\sqrt{5}}{7\times 5} & \cdots & 0 \\ 0 & 0 & -\frac{\sqrt{7}\times\sqrt{5}}{7\times 5} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \frac{\sqrt{2p+1}\times\sqrt{2p-1}}{(2p+1)\times(2p-1)} \\ 0 & 0 & 0 & \cdots & -\frac{\sqrt{2p+1}\times\sqrt{2p-1}}{(2p+1)\times(2p-1)} & 0 \end{pmatrix}. \tag{11}$$

For all l, it is easy to prove that the elements of the integral operator matrix $P_{p\times p}$ are the same on each sub-interval I_{nl} by simple variable substitute.

Finally, we can compute the integral operator matrix by the ELW on the the interval $[l/a^{-n},\ (l+1)/a^{-n})$ by using the above similar technique. We let $\Phi_{k,nla}=[\varphi_{0,nla},\ \varphi_{1,nla},\cdots,\varphi_{p-1,nla}]^T$ and for any $x\in[l/a^{-n},\ (l+1)/a^{-n})$ we obtain

$$\int_{l/a^{-n}}^{x}\Phi_{k,nla}(x)dx := EP_{p\times p}\Phi_{k,nla}(x), \tag{12}$$

where $EP_{p\times p}$ is the ELW integral operator matrix and its elements are computed by the formula as

$$(EP)_{k+1,k'+1}=\int_{l/a^{-n}}^{(l+1)/a^{-n}}[\int_{l/a^{-n}}^{x}\varphi_{k,nla}(x)dx]\varphi_{k',nla}(x)dx. \tag{13}$$

4. Application on Solving the ODE

In this section, we consider the linear differential equations of the form

$$y'+y=x,\qquad y(-1)=0,\qquad x\in[-1,\ -0.5) \tag{14}$$

with exact solution

$$y(x)=2e^{-x-1}+x-1. \tag{15}$$

We let $p=3$, $n=1$ and $l'=-2$ and assume that

$$y(x):=C_{l'}^{T}\Phi_{p,nl'}(x), \tag{16}$$

where $C_{l'}^{T}=[c_{0,1,-2},c_{1,1,-2}c_{2,1,-2}]$ and $\Phi_{p,nl'}(x)=[\varphi_{0,1,-2}(x),\varphi_{1,1,-2}(x),\varphi_{2,1,-2}(x)]^T$. Now, for every $x\in[-1,\ -0.5)$, we integrate on this interval

$$\int_{-1}^{x}y'dx+\int_{-1}^{x}ydx=\int_{-1}^{x}xdx. \tag{17}$$

We first compute the approximation $x=[\frac{-3}{4\sqrt{2}},\ \frac{1}{4\sqrt{6}},\ 0]\Phi_{p,nl'}(x)=q^T\Phi_{p,nl'}(x)$, which q is a vector. Second, using the result in (12), we can convert the ODE (16) to a system of algebraic equations

$$(E+P^T)C_{l'}=P^Tq \tag{18}$$

for $C_{l'}$. we solve this equation (17) and obtain the ELW approximation solution of E.q. (14), i.e., the ELW coefficients

$$C_{l'}^{T}=[\ \text{-0.1245}\ \ \text{-0.0579}\ \ \text{0.0103}]. \tag{19}$$

Table 1 and Figure 3 illustrate the numerical results for the example.

Table 1. *Exact and estimated values of* $y(x)$.

y \ x	-0.95	-0.9	-0.85	-0.65	-0.55
Exact	-0.0475	-0.0903	-0.1286	-0.2406	-0.2747
Estimated	-0.0476	-0.0897	-0.1278	-0.2413	-0.2745

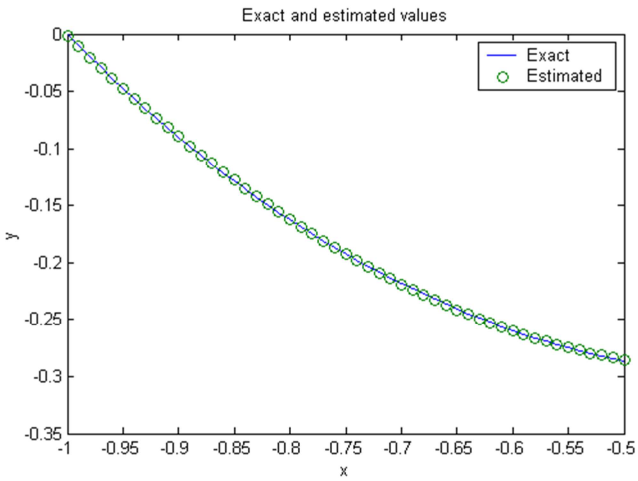

Figure 3. *Result for the example with* $p=3,\ n=1,\ l'=-2$.

This simple example demonstrates the validity and applicability of the ELW operational matrix of integration to solving the ODE.

5. Conclusion

In this article, a new numerical method is developed using the ELW operational matrix of integration for the ODE in (14). The numerical result shows that the method is efficient and accurate. The DLWG method is applicable to both linear and nonlinear problems of the ODEs.

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