Equivalent Characterizations and Structure Theorem of Right C-wrpp Semigroups

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Abstract
The aim of this paper is to study the right dual of left C-wrpp semigroup, that is, the strongly wrpp semigroup whose set of idempotents forms a right regular band and the relation \(L^∗ \lor R\) is a congruence. We call this kind of strongly wrpp semigroups right C-wrpp semigroups. This paper generalizes results of right C-rpp semigroups. Some properties and characterizations of right C-wrpp semigroups are investigated.

1. Introduction
A semigroup \(S\) is an rpp semigroup if all of its principal right ideals \(aS\) regarded as right \(S\)-systems, are projective (see [3]-[5]). These classes of semigroups and its special subclasses have been studied by Fountain. He also defined a generalized Green’s relation \(L'\) on a semigroup \(S\) by \(aL'b\) if and only if the elements \(a, b\) of \(S\) are related by the Green’s relation \(L\) in some oversemigroup of \(S\) in [5]. In particular, if \((e, f) \in L'\) for some idempotent elements \(e, f \in S\), then \((e, f) \in L\). Next, he showed that a monoid \(S\) is rpp if and only if each \(L'\)-class of \(S\) contains an idempotent. Thus, a semigroup \(S\) is an rpp semigroup if each \(L'\)-class of \(S\) contains at least one idempotent. An rpp semigroup having all of its idempotents lying in its center is called a C-rpp semigroup. It is well known that a semigroup \(S\) is a C-rpp semigroup if and only if \(S\) is a strongly semilattice of left cancellative monoids. Thus, C-rpp semigroups are natural generalizations of Clifford semigroups.

For rpp semigroups, the concept of strongly rpp semigroups was first introduced by Guo-Shum-Zhu, that is, an rpp semigroup in which each \(a s_{+}^*\) contains a unique idempotent \(a^+ \in L_+ \cap E(S)\) such that \(a^+ a = a\), where \(E(S)\) is set of idempotents of \(S\). Strongly rpp semigroups and their special cases have been extensively studied by many authors (see [6]-[10], [14]-[17]). In particular, we call a strongly rpp semigroup \(S\) a left C-rpp semigroup if \(L'\) is a congruence and \(eS \subseteq eS\) for all \(e \in E(S)\). For such semigroups, Guo has proved that an rpp semigroup is a left C-rpp semigroup if and only if \(S\) is a semilattice of direct products of left cancellative monoids and left zero bands. Thus, a left C-rpp semigroup is clearly a generalized C-rpp semigroup and left C-semigroup.

On the other hand, we call a strongly rpp semigroup \(S\) a right C-rpp semigroup if \(L' \lor R\) is a congruence and \(Se \subseteq eS\) for all \(e \in E(S)\). It was observed that the concept
of right C-rpp semigroups is not a dual of left C-rpp semigroups by Guo [9]. The structure of right C-rpp semigroups has been recently described by Shum and Ren in [17]. It is noteworthy that many properties of left C-rpp semigroups are not dual to those of right C-rpp semigroups (see [9], [17]).

Tang [18] has introduced a new set of Green’s two stars relations on a semigroup \( S \) by modifying Green’s one star relations on semigroups. Let \( R \) be a Green’s relation on a semigroup \( S \). A relation \( L^* \) on \( S \) by: for some \( a,b\in S \), \((a,b)\in L^*\) if and only if \((ax,ay)\in R \Leftrightarrow (bx,by)\in R \) for all \( x,y\in S \). In view of this new Green’s two stars relation, he defined the concept of wrpp semigroups, that is, a wrpp semigroup in which each \( L^* \)-class contains at least one idempotent. A wrpp semigroup is a C-wrpp semigroup if all the idempotents of \( S \) are central. We have known that a C-wrpp semigroup can be described as a semilattice of the direct product of left \(-\)-semigroups and left zero bands, \( aS L a \) for all \( a\in S \). According to Du-Shum [3] we define:

\[ (aS L a) = \text{the smallest left } \ast \text{-ideal of } S \]

where \( (aS L a) \) is the smallest ideal of \( S \) containing \( a \) such that \( (aS L a) \) is the union of some \( L^*\)-classes or some \( R\)-classes. For the sake of simplicity, we denote the \( L^*\)-class (resp., \( R^*\)-class, \( H^*\)-class, \( D^*\)-class and \( J^*\)-class) of \( S \) containing \( a \) by \( I^*_a \) (resp., \( R^*_a \), \( H^*_a \), \( D^*_a \) and \( J^*_a \)). The Green’s “egg box” diagram for Green’s relation still holds for these Green’s’ relations. We have

\[ \text{Lemma 1} [2] \text{ The equalities } L^* \circ R^* = R^* \circ L^* \text{ and } D^* \circ J^* = J^* \circ D^* \text{ hold on a semigroup.} \]

On the other hand, the Green’s’ relations on \( S \) are similar to the Green’s relations on \( S \), for instance, we have \( D^* \subseteq J^* \). Moreover, we have the following lemma:

\[ \text{Lemma 2} [2] \text{ Let } S \text{ be a semigroup, } a,b\in S \text{. Then } b \in J^*(a) \text{ if and only if there exists } a_0,a_1,\ldots,a_n \in S \text{ with } a = a_0, b = a_n \text{ and } x_1,x_2,\ldots,x_n, y_1,y_2,\ldots,y_n \in S \text{ such that } a_i L a_{i+1} y_i \text{ for all } i = 1,2,\ldots,n. \]

A \( L^\ast\)-class may contain more than one regular \( \ast \)-class. This is because if \( eL^\ast a \) for some idempotent \( e \in E(S) \) and \( a \in S \), then the relation \( a = ae \) need not always hold. For example, if \( S = M(G; I, A; P) \) is a Rees matrix semigroup over a group \( G \) with \( |A| \geq 2 \), then it is not difficult to see that \( S \) is an \( L^\ast\)-simple semigroup containing \( |A| \text{ regular } L\text{-classes. But if } S \text{ is a quasi strongly wrpp semigroup, then we have} \]

\[ \text{Lemma 3 Let } S \text{ be a quasi strongly wrpp semigroup and } a,b \in \text{Reg}S. \text{ Then } aL^\ast b \text{ if and only if } aL^\ast b. \]

\[ \text{Proof Necessity. We only need verify that } aL^\ast b. \text{ If } aL^\ast b, \text{ then } aL^\ast b. \text{ By the quasi strong wrpp property of } S, \text{ we have } aL^\ast b \text{ and } aL^\ast b. \text{ Thus } aL^\ast b. \text{ Sufficiency. It is clear and we omit the proof.} \]

\[ \text{Lemma 4 Let } S \text{ be a quasi strongly wrpp semigroup. Then each } D^\ast \text{-class contains at most one regular } D \text{-class.} \]

\[ \text{Proof According to Lemma 1, we have } D^\ast = R^* \circ L^\ast = R \circ L^\ast. \text{ Let } a,b \in \text{RegS}. \text{ If } aD^\ast b, \text{ then there exists } c \text{ such that } aRcL^\ast b. \text{ By regularity of } a, \text{ we obtain that } c \text{ is a regular element of } S. \text{ According to Lemma 3, we know that clb. Hence adb, so } D^\ast \subseteq \text{RegS} = D^\ast \subseteq \text{RegS}. \]

Definition 1 A wrpp semigroup \( S \) is called a right C-wrpp semigroup, if \( S \) satisfies the following conditions:

1. \( L^\ast = L^\ast \), \( R^\ast = R \), \( D^\ast = L^\ast \circ R^\ast \), \( H^\ast = L^\ast \land R^\ast \), \( J^\ast \Leftrightarrow J^\ast(a) = J^\ast(b) \)

2. Basic Definitions and Results

We first introduce some definitions and results that are useful in the sequel.

In order to describe wrpp semigroups, Du-Shum introduced the following (+)-Green’s relations. For elements \( a, b \) in a semigroup \( S \), we define:

\[ L^\ast = L^\ast \]
\[ D^\ast = L^\ast \circ R^\ast \]
\[ H^\ast = L^\ast \land R^\ast \]
\[ J^\ast \Leftrightarrow J^\ast(a) = J^\ast(b) \]
(1) $S$ is a quasi strong wrpp semigroup;
(2) $D^{(+)}$ is a congruence on $S$;
(3) $(\forall e \in E(S)) Se \subseteq eS$.

We call a band a right regular band if it satisfies the identity $ef = fe$. We now cite the following lemma:

**Lemma 5** [15] The following statements are equivalent on a band $B$:

1. $B$ is a right regular band;
2. $R$ is a congruence;
3. $B$ is a semilattice of right zero bands.

An immediate result of this lemma is:

**Corollary** If $B$ is a right regular band, then each $L$-class of $B$ contains precisely one idempotent.

### 3. Characterizations of Right C-wrpp Semigroups

In this section, we shall describe some characterizations of right C-wrpp semigroups and hence generalize the main results of right C-rpp semigroups obtained by Guo in [9]. The results obtained in [9] will be amplified and strengthened.

**Lemma 6** Let $S$ be a right C-wrpp semigroup. Then the following hold:

1. $E(S)$ is a right regular band;
2. $\text{Reg} S$ is a right C-semigroup.

**Proof** (1) Let $e, f \in E(S)$. Since $Se \subseteq eS$, then there exists $x$ such that $fe = ex$, so $efe = efx = ex = fe$. Hence $(ef)^2 = ef$, it implies that $E(S)$ is a band and a right regular band.

(2) According to (1), $E(S)$ is a band, so $\text{Reg} S$ is a right C-semigroup of $S$. And for all $e \in E(S)$, $(\text{Reg} S)e \subseteq eS \cap \text{Reg} S = e(\text{Reg} S)$, consequently, $\text{Reg} S$ is a right C-semigroup (see [20]).

**Theorem 1** The following statements are equivalent:

1. $S$ is a right C-wrpp semigroup;
2. $S$ is a strong wrpp semigroup such that $D^{(+)}$ is a semilattice congruence, and $D^{\text{Reg}S} = R^{\text{Reg}S}$;
3. $S$ is a semilattice of $D^{(+)}$-simple strong wrpp semigroups, and $D^{\text{Reg}S} = R^{\text{Reg}S}$;
4. $S$ is a semilattice of $S_a = M_a \times \Lambda_a$ for $\alpha \in Y$, where $M_a$ is a left $-R$ cancellative monoid, $\Lambda_a$ is a right zero band.

**Proof** (1) $\Rightarrow$ (2). Let $S$ be a right wrpp semigroup. Then $D^{(+)}$ is a congruence of $S$. Let $a, b \in S, e, f \in E(S)$, and $aL^e, bL^f$. Then clearly $abD^{(+)}ef \Rightarrow aD^{(+)}ef$. But $D^{(+)}$ is a congruence, we have $a^2D^{(+)}e$, so $a^2D^{(+)}a$. Notice that $E(S)$ is a right regular band, it leads to $abD^{(+)}ef \Rightarrow efDf \Rightarrow D^{(+)}ba$. Consequently, $D^{(+)}$ is a semilattice congruence.

According to Lemma 6, $\text{Reg} S$ is a right C-semigroup. Therefore, $\text{Reg} S / R$ is a semilattice, and $D^{\text{Reg}S} = R^{\text{Reg}S}$ by Lemma 6. We easily prove that $D^{\text{Reg}S} = R^{\text{Reg}S}$. By quasi wrpp property of $S$, and Lemma 4, we know that each $D^{(+)}$-class exactly contains one regular $D$-class, and each $D^{(+)}$-class exactly contains one regular $R$-class, it means that each $L^{(+)}$-class in each $D^{(+)}$-class contains a unique idempotent which is a left identity of this $L^{(+)}$-class. Again, $Se \subseteq eS$ for all $e \in E(S)$, so this unique idempotent is also a right identity of above $L^{(+)}$-class. Hence $S$ is a strongly wrpp semigroup.

(2) $\Rightarrow$ (3). Let $S = \bigcup_{\alpha \in \Lambda} S_a$ be a semilattice decomposition corresponding to the semilattice congruence $D^{(+)}$. Obviously, for an arbitrary subsemigroup $T$ of $S$, we have $L^{(+)}_a \supseteq L^{(+)}_T$. Hence the elements of $S_a$ having $L^{(+)}_a$ relation in $S$ also have $L^{(+)}$ relation in $S_a$. By $D^{\text{Reg}S} = R^{\text{Reg}S}$, and Lemma 4, it implies that each $S_a$ only contains one regular $R$-class. Therefore, the elements of $S_a$ having $R$ relation in $S$ and also have $R$ relation in $S_a$, so each $D^{(+)}$-class $S_a$ is $D^{(+)}$-simple, and is a strongly wrpp semigroup.

(3) $\Rightarrow$ (4). Let $S$ be a semilattice decomposition $S = \bigcup_{\alpha \in \Lambda} S_a$, where $S_a$ is a $D^{(+)}$-simple strong wrpp semigroup. Let $\Lambda_a = E(S_a)$. According to $D^{\text{Reg}S} = R^{\text{Reg}S}$, we know that each $L^{(+)}_a$-class $L^{(+)}_a$ of $S_a$ contains a unique idempotent $e_a$, and $\Lambda_a$ is a right zero band. Next we shall verify that $S_a e = L^{(+)}_a$. Let $a \in S_a$. Then $aD^{(+)}e_a$, so $a^2De_a$. Since $D^{\text{Reg}S} = R^{\text{Reg}S}$, it leads to $a^2Re_a$. Hence $e_a = a^2eL^{(+)}ae_a$, so $S_a e \subseteq L^{(+)}_a$. Conversely, if $aL^{(+)}e_a$ $(e_a \in E(\Lambda_a))$, then $a = ae_a$, and $aD^{(+)}e_a$ is easily observed that $a \in S_a$, it means $L^{(+)}_a \subseteq S_a e$. Thus $L^{(+)}_a = S_a e$. By strong wrpp property of $S_a$, we have $L^{(+)}_a = S_a e = eS_a e$, which is a monoid with identity element $e$ for all $e \in E(\Lambda_a)$. We claim that $S_a e$ is left $-R$ cancellative. In fact, for all $eae, ebe, ece \in eS_a e = S_a e$, if $(e(a)(ae)(e(b)(e))(e)) \in R$, notice that $e = eL^{(+)}e$, then $(eae, ebe) \in R$. Now define a mapping:

$$\Phi : S_a e \times \Lambda_a \to S_a, \Phi(a, e) = ae$$

for any fixed $e_a \in \Lambda_a$. Then we deduce that $\Phi(a, e)(b, f) = \Phi(ab, f) = ab = ae bf = \Phi(a, e) \Phi(b, f)$. Thus $\Phi$ is a semigroup homomorphism.

We now show that $\Phi$ is a semigroup isomorphism. By virtue of the strongly wrpp property of $S_a$, for all $x \in S_a$, there exists $e \in \Lambda_a$ such that $x = xe = xe_e$. By the definition of $\Phi$, this means that $\Phi(xe_e, e) = x$, and hence $\Phi$ is an epimorphism. To prove $\Phi$ is a monomorphism, we assume that $\Phi(a, e) = \Phi(b, f)$. Then we have $ae = bf$. Since $\Lambda_a$ is a right zero band, we have $ae_0 = a = e_0 = bfe_0 = be_0$. This implies that $a = b$ for all $a, b \in S_a e_0$. Invoking the strongly wrpp property of $S_a$, we...
obtain that \( e = f \). This shows that \( \Phi \) is a monomorphism as well. Thus \( S_\alpha \cong S_\alpha \times \Lambda_\alpha \). The proof is completed.

1em (4) \( \Rightarrow \) (1). Let \( S \) be a semilattice of \( S_\alpha = M_\alpha \times \Lambda_\alpha \) for \( \alpha \in Y \), where \( M_\alpha \) is a left- \( R \) cancellative monoid, \( \Lambda_\alpha \) is a right zero band. Then \( E(M_\alpha \times \Lambda_\alpha) = E_\alpha = \{(1_\alpha, i_\alpha) | i_\alpha \in \Lambda_\alpha \} \), where \( 1_\alpha \) is unique identity of left- \( R \) monoid \( M_\alpha \). We now show that \( S_\alpha = M_\alpha \times \Lambda_\alpha \) is a \( D^{(\alpha)} \)-class. Let \( a, b \in S_\alpha \), then \( aD^{(\alpha)}b \). Then there exists \( c \in S_\alpha \) such that \( aL^{(\alpha)}cR^{(\alpha)}b \). Since \( cR^{(\alpha)}b \) if and only if \( \beta = \gamma \), so \( c \in S_\beta \). Hence we have \( cRa(\beta, j_\beta) \) for any \( (\beta, j_\beta) \in E(\beta \times \alpha) \), it implies that \( aR_{\beta}a(\alpha, j_\alpha) \). This means that \( \alpha \leq \beta \). Similarly, we can verify that \( \beta \leq \alpha \). Hence we conclude that \( \alpha = \beta \). Because \( S_\alpha \) is just a \( D^{(\alpha)} \)-class of \( S \), \( D^{(\alpha)} \) must be a semilattice congruence on \( S \).

1em Next, we need verify that \( Se \subseteq eS \). Let \( (1_\alpha, i_\alpha) \in E(\alpha, j_\alpha) \subseteq E(\alpha, j_\alpha) \), \( \alpha, \beta \in Y \) with \( \alpha \geq \beta \). Then \( (1_\alpha, i_\alpha)(1_\beta, j_\beta) \in E_{\alpha \beta} \). In fact, \( (1_\alpha, i_\alpha)(1_\beta, j_\beta) \in E_{\alpha \beta} \), then \( (1_\beta, j_\beta)(1_\alpha, i_\alpha) \in E_{\alpha \beta} \), so

\[
(1_\alpha, i_\alpha)(1_\beta, j_\beta) = (1_\alpha, i_\alpha)(1_\beta, j_\beta)(1_\alpha, i_\alpha) = (1_\alpha, i_\alpha)(1_\beta, j_\beta) = (1_\alpha, i_\alpha)(1_\beta, j_\beta) = (1_\alpha, i_\alpha)(1_\beta, j_\beta).
\]

that is, \( (1_\alpha, i_\alpha)(1_\beta, j_\beta) \in E_{\alpha \beta} \). Clearly, \( \beta \geq \alpha \beta \), by using above analogous methods, we obtain that \( (1_\alpha, i_\alpha)(1_\beta, j_\beta) \in E_{\alpha \beta} \) for any \( k_{\alpha \beta} \in \Lambda_{\alpha \beta} \). Hence, we have

\[
(1_\beta, j_\beta)(1_\alpha, i_\alpha) = (1_\beta, j_\beta)(1_\alpha, i_\alpha)(1_\beta, j_\beta) = (1_\alpha, i_\alpha)(1_\beta, j_\beta).
\]

This means that \( E(S) \) is a right regular band. Now let \( (a, i_a) \in S_\alpha \), then

\[
(a, i_a)(1_\beta, j_\beta) = (a, i_a)(1_\beta, j_\beta)(a, i_a)(1_\beta, j_\beta) = (a, i_a)(1_\beta, j_\beta).
\]

This verifies that \( Se \subseteq eS \).

Summing up the above results, then \( S \) is a right \( C \)-wpp semigroup.

Corollary 2 Let \( S \) be a right \( C \)-wpp semigroup. Then \( D^{(\alpha)} = J^{(\alpha)} \).

Proof Because \( D^{(\alpha)} \subseteq J^{(\alpha)} \), we only need to prove that \( J^{(\alpha)} \subseteq D^{(\alpha)} \). Suppose that \( aD^{(\alpha)}b \). Then \( b \in J^{(\alpha)}(a) \). By Lemma 2, there exists \( a_0, a_1, \ldots, a_n \in S \) with \( a = a_0, b = a_n \) and \( x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_n \in S \) such that \( aL^{(\alpha)}x_1a_{i-1}y_i, \ldots, x_ny_1a_n \).

for all \( i = 1, 2, \ldots, n \). Since \( L^{(\alpha)} \subseteq D^{(\alpha)} \) and \( D^{(\alpha)} \) is a congruence, we have \( bD^{(\alpha)}x_1a_{i-1}y_i, \ldots, x_ny_1a_n \). By Theorem 1, we know that \( D^{(\alpha)} \) is a semilattice congruence. We denote semilattice \( S / D^{(\alpha)} \) by \( Y \). Index \( D^{(\alpha)} \)-class in virtue of the elements \( \{\alpha, \beta, \gamma, \ldots\} \) in the semilattice \( Y \), and \( a \in D^{(\alpha)}_\beta, b \in D^{(\alpha)}_\gamma \). We are not difficult to see that \( \alpha \geq \beta \).

Similarly, \( \alpha \leq \beta \). Hence \( aD^{(\alpha)}b \).

Lemma 7 Let \( S \) be a strongly \( C \)-wpp semigroup whose set of idempotents is a semilattice \( Y \). Then \( (ab)^+ = a^+(ab)^+b^+ \) for all \( a, b \in S \).

Lemma 8 Let \( S \) be a strongly \( C \)-wpp semigroup whose set of idempotents is a semilattice \( Y \), and \( a, b \in S \). If \( a^+ \geq (ab)^+ \), then the following statements are hold:

1. \( L^{(\alpha)}_\beta \) is a left- \( R \) cancellative monoid;
2. If \( \lambda, \mu \in Y \), and \( \lambda \geq \mu \), then the mapping \( \Phi_{\lambda, \mu} \):

\[
L^{(\alpha)}_\beta \to L^{(\alpha)}_\mu, x \mapsto x\mu
\]

is a semigroup homomorphism. Moreover, with respect to the following multiplication \( " \cdot " \): \( a \cdot b = a\Phi_{\lambda, \mu}a\Phi_{\gamma, \mu}a \).

\[
S \text{ form a } C \text{-wpp semigroup, where } a\Phi_{\lambda, \mu}a\Phi_{\gamma, \mu}a \text{ is the product in } L^{(\alpha)}_{\mu, \gamma}.
\]

(3). a \cdot b = a\Phi_{\lambda, \mu}a\Phi_{\gamma, \mu}a \Phi_{\gamma, \mu}a \Phi_{\gamma, \mu}a \Phi_{\gamma, \gamma}a, \text{ where } ab \text{ is the product of } a \text{ and } b \text{ in } S.

Proof (1) Let \( x, y \in L^{(\alpha)}_\beta \). Notice that there is exactly one idempotent in \( L^{(\alpha)}_\beta \), we have \( x^+ = y^+ \). By the fact that \( S \) being a strongly \( C \)-wpp semigroup, we have \( x^+y^+ = y^+x^+ \). Since \( L^{(\alpha)}_\beta \) is a right congruence, we know that \( yxL^{(\alpha)}_\beta Y = Y \). Hence \( L^{(\alpha)}_\beta \) is a subsemigroup of \( S \). Notice that \( yx = yx \), it follows that \( x^+ = yx \) is the identity of \( L^{(\alpha)}_\beta \). Now put \( u, v \in S \), and \( xuRyv \). Then \( x^+uRv \). Thus \( uRv \), that is, \( L^{(\alpha)}_\beta \) is a left- \( R \) cancellative monoid.

(2) Let \( x \in L^{(\alpha)}_\beta \). Since \( L^{(\alpha)}_\beta \) is a right congruence, we have \( x\muL^{(\alpha)}_\beta Y = Y \), that is, \( x\mu \in L^{(\alpha)}_\beta \). By (1), we know that \( L^{(\alpha)}_\beta \) is a left- \( R \) cancellative monoid with identity \( \mu \).

Consequently, for all \( y \in L^{(\alpha)}_\beta \), we have

\[
(xy)\Phi_{\lambda, \mu} = x\mu y = x(\mu y) = x\mu y\mu = x\Phi_{\lambda, \mu}y\Phi_{\lambda, \mu}.
\]
\(b^*(ab)^* = (ab)^*\). It means that \(b(ab)^* \in L_{(ab)}^{(s)}\). Because \((ab)^*\) is the identity of \(L_{(ab)}^{(s)}\), we have

\[ab = a(ab)^* b(ab)^* = a\Phi_{b}a (ab)^* b\Phi_{a}b(ab)^* \tag{9}\]

Lemma 9 Let \(S\) be a semigroup satisfying the conditions in Lemma 8. Then every regular element of \(S\) is completely regular, that is, a regular element is \(H\)-related to an idempotent element.

Proof Let \(a\) be a regular element of \(S\). Then there exists \(b \in S\) such that \(ab = a\), so \(b\) is an idempotent. Hence \(b\Phi_{b}a \in L_{(b)a}^{(s)}\). As it is argued in Lemma 3.5, \(a = a\Phi_{b}a \in L_{(b)a}^{(s)}\). Hence \(a\) is a regular element of \(L_{(b)a}^{(s)}\). Then there exists only one idempotent in \(L_{(b)a}^{(s)}\), so \(a^* = ab\Phi_{b}a\). Thus \(aha\) is a completely regular element.

Lemma 10 Let \(S\) be a strongly semigroup whose set of idempotents is a band. Then every regular element of \(S\) is a completely regular element.

Proof Since \(E(S)\) is a band, \(Reg(S)\) is an orthodox semigroup. Since \(S\) is a strongly wrpp semigroup, we can easily see that \(Reg(S)\) is a strongly wrpp semigroup. Hence \(Reg(S)\) is a strongly wrpp semigroup, where \(Reg(S)\) is the smallest inverse semigroup congruence on \(Reg(S)\). According to Lemma 9, we can follow that \(Reg(S)\) is a Clifford semigroup. Let \(a \in Reg(S)\). Then there exists \(e, f \in E(S)\) such that \(eLaRf\). It follows that \(e\gamma f\) is a Clifford band, \(ef = f\). On the other hand, since \(\gamma_{E(S)}^{(s)} = E(S)^{(s)}\), we have \(eD^{(s)}f\) and hence \(eLeRf\). Therefore \(efHa\), that is, \(a\) is a completely regular element.

As an application of above results, we now give some conditions which lead to a C-wrpp semigroup \(S/\xi\) for some congruence \(\xi\) defined on a right C-wrpp semigroup \(S\). In fact, all we need to find a congruence \(\xi\) on \(S\) so that \(\xi\) preserves the \(L^{(s)}\)-classes of \(S\).

For convenience, we denote the rectangular band \(B_{a}\) by \(E(a^+)\) if the idempotent \(a^+\) is in \(B_{a}\). Also, we write \(B_{a} \leq B_{b}\) if \(B_{a} \subseteq B_{b}\).

We now characterize right C-wrpp semigroups.

Theorem 2 The following conditions are equivalent for a strongly wrpp semigroup \(S\):

1. \(S\) is a right C-wrpp semigroup;
2. \(E(S)\) is a regular band and \(D^{(s)}\) is a semilattice congruence on \(S\);
3. The relation \(\xi = \{(x, y)| (\exists f \in E(y^-)) x = yf\}\) is a congruence on \(S\) such that \(S/\xi\) is a C-wrpp semigroup.

Proof (1) \(\Rightarrow\) (2) This part is an immediate consequence of Lemma 6 and Theorem 1.
\( \text{ayef}(ay^*) \, v = axef = axef, \) that is, \((axef, ayef) \in R \). So we also have \((bxef, byef) \in R \). Therefore, by the definition of \( R \), there exists \( k, l \in S \) such that \( bxefk = byef \) and \( byefl = bxef \). On the other hand, since \((a, b) \in L^*(\gamma)\), we have \((ax, bx) \in L^*(\gamma)\) and hence we deduce that \( E((ax)^*) = E((bx)^*) \). Similarly, \( E((by)^*) = E((by)^*) \).

Since \( E((ax)^*) = E((ay)^*) \), we have \( E((ax)^*) = E((by)^*) \) and hence \( ef \in E((by)^*) \). This leads to \((bx)^*(efk) \xi = (by)^* \). Similarly, \((by)^*(efl) \xi = (bx)^* \xi \). Thus, we have \((bx)^*(by)^* \xi \in R(S / \xi) \). From this relation and its dual, we conclude that \((a^*(b^*), b^*(a^*)) \in L^*(S / \xi) \). This shows that the relation \((a, b) \in L^*(S / \xi) \) on \( S \) is preserved in the quotient semigroup \( S / \xi \), and hence \( S / \xi \) is a wrpp semigroup.

Finally, we show that the idempotents of \( S / \xi \) are central.

It suffices to show that \((ea, ae) \xi \in S \) for all \( e \in E(S) \) and \( a \in S \). Since by Theorem 1 (4), \( E(a)^* \), \( E(e) \xi \in E((ea)^*) = E((ea)^*) \), it is clear that \( e(ae)^* \xi \in E((ea)^*) \). Thus, by \( ae = (ae)^* e(ae)^* = e(ae)^* e(ae)^* = e(ae)^* \), we obtain \((ea, ae) \xi \in S \). This shows that \( S / \xi \) is a semilattice of the \( S / \xi \)-simple semigroups.

(3) \( \Rightarrow \) (1). Suppose that \( \xi \) is a congruence on \( S \) such that \( S / \xi \) is a C-wrpp semigroup. We can easily see that \( \xi \subseteq (S / \xi) \) and hence \( E(S / \xi) = E(S / \xi) \) is a semilattice. Hence \( S \) is a semilattice congruence on \( S \), and so \( E(S) \) is a right regular band. Now let \( E(S) = \bigcup_{a \in S} A_a \) be the semilattice decomposition of \( E(S) \) into right zero bands \( A_a \). Clearly, \( Y \) is isomorphic to \( E(S / \xi) = E(S / \xi) / R \). We identify \( Y \) with \( E(S / \xi) = E(S / \xi) / R \). By \( S / \xi \) is a C-wrpp semigroup, we let \( S / \xi = \bigcup_{a \in S} A_a \) be the semilattice decomposition of the \( S / \xi \)-simple semigroup \( S / \xi \) into left-R cancellative monoids \( A_a \).

Put \( T = \bigcup_{a \in S} A_a \). Then we define \( \varphi : S \to T \) by \( x \to (x^*, x^*) \). Clearly, \( \varphi \) is well defined, and we deduce that
\[(xy) \varphi = ((xy)^* \xi, (xy)^*) = ((x^*y^*) \xi, x^*y^*) = ((x^*y^*) \xi, y^* \xi) \] (10).

Thus \( \varphi \) is a semigroup homomorphism.

Now we prove that \( \varphi \) is a semigroup isomorphism. For all \((t, \lambda) \in T \), we have \( x, \xi \in S \) such that \( x^* = t \) and \( \lambda \xi \). It follows that \((x^* \lambda)^* \xi = \lambda^* \xi \). On the other hand, since \( x^* \in S \), we have \( x^* \lambda \in S \). But \( E(S) \) is a right regular band, we know that each \( L \)-class of \( E(S) \) contains precisely one element, and hence \((x^* \lambda)^* = \lambda^* \). Consequently, \((x^* \lambda)^* \xi = (\lambda \xi) \). This means that \( \varphi \) is an epimorphism. To prove \( \varphi \) is a monomorphism, now let \( x, y \in S \) and \( x^* = y^* \) and \( x^* = y^* \). By using the latter formula, we see that there exists \( f \in E((y^*)) \) such that \( x = yf \) and furthermore, \( x = xf \). This shows that \( \varphi \) is a monomorphism. On the other hand, \( T \) is a semilattice of direct products \( M_a \times \Lambda_a \) and hence \( S \) is a right C-wrpp semigroup.

Now we define a new relation \( \bar{R} \) on a strongly wrpp \( S \) as follows:
\[ aRb \Leftrightarrow a^*Rb^*. \] (11)

It is easy to verify that \( \bar{R} \) is an equivalence relation, and \( \bar{R} \subseteq D^{(\gamma)} \subseteq J^{(\gamma)} \).

Theorem 3 Let \( S \) be a strongly wrpp semigroup. Then \( S \) is a right C-wrpp semigroup if and only if \( \bar{R} \) is a semilattice congruence and \( E(S) \) is a right regular band.

Proof Assume that \( S \) is a right C-wrpp semigroup. By Lemma 6 (1), we only need to prove that \( \bar{R} \) is a semilattice congruence. For this purpose, we let \( S \) be a semilattice of the direct products \( M_a \times \Lambda_a \) for \( a \in Y \), where \( M_a \) is a left-R cancellative monoid and \( \Lambda_a \) is a right zero band. We can easily check that \((a, i) = (1, i) \) for any \((a, i) \in M_a \times \Lambda_a \), where \( 1_a \) is the identity of \( M_a \). Hence it is difficult to verify that identical formula \( \bar{R} = \bigcup_{a \in Y} (M_a \times \Lambda_a) / (M_a \times \Lambda_a) \). It follows that \( \bar{R} \) is a semilattice congruence.

Suppose that \( \bar{R} \) is a semilattice congruence on \( S \) and \( E(S) \) is a right regular band. Since \( \bar{R} \) is a semilattice congruence on \( S \), \( S \) is a semilattice of some \( \bar{R} \)-classes. But \( a^* \in \bar{R} \), each \( \bar{R} \)-class of \( S \) is a strongly wrpp semigroup, therefore it is \( \bar{R} \)-simple. Next we shall show that each \( \bar{R} \)-simple semigroup is also \( D^{(\gamma)} \)-simple semigroup.

For this purpose, we only need to prove \( D^{(\gamma)} = \bar{R} \vee L^{(\gamma)} \). Let \((a, b) \in D^{(\gamma)} \). Then \( a^*b^* \). Hence there exists \( c \in S \) such that \( a^* c \in L^{(\gamma)} \). By \( a^* \), we can see that \( c \) is a regular element, and by Lemma 10, \( c \) is completely regular. Hence, we can follow that \( a^* c \). This means that \((a, b) \in \bar{R} \vee L^{(\gamma)} \), so \( D^{(\gamma)} \subseteq \bar{R} \vee L^{(\gamma)} \). Conversely, if \((a, b) \in \bar{R} \vee L^{(\gamma)} \), then there exist \( x_1, x_2, \ldots, y_2, y_3, \ldots, y_n, \in S \) with \( a = x_1, b = y_n \) such that \( x_1 \bar{R} _1 \bar{L} \cdots \bar{L} x_2 \bar{R} \cdots \bar{L} y_2, \ldots, \bar{R} _k \bar{L} \cdots \bar{L} y_n \). From the above, we have
\[ x_1 \bar{L} \cdots \bar{L} x_2 \bar{R} \cdots \bar{L} x_3 \bar{R} \cdots \bar{L} y_2, \ldots, \bar{R} _k \bar{L} \cdots \bar{L} y_n \]. (12)

This shows that \( aD^{(\gamma)} \). Hence \( \bar{R} \vee L^{(\gamma)} \subseteq D^{(\gamma)} \). This shows that \( D^{(\gamma)} = \bar{R} \vee L^{(\gamma)} \). Thus, each \( \bar{R} \)-simple semigroup is also \( D^{(\gamma)} \)-simple semigroup, it deduces that \( S \) is a semilattice of \( D^{(\gamma)} \)-simple strongly wrpp semigroups.

Also, Since \( E(S) \) is a right regular band, by the proof of
(2) ⇒ (1) in Theorem 2, we know that \( D_{\text{reg}} = R_{\text{reg}} \).
Therefore, \( S \) is a right C-wrpp semigroup.

**Theorem 4** Let \( S \) be a strongly wrpp semigroup whose set of idempotents forms a right regular band. Then the following statements are equivalent:

1. \( S \) is a right wrpp semigroup;
2. \( \tilde{R} = J^* \);
3. \( D^* = J^* \).

**Proof** (1) ⇒ (3). By the Corollary 2, clearly.

(3) ⇒ (2). Let \( D^* = J^* \). Since \( E(S) \) is a right regular band, we have \( D_{\text{reg}} = R_{\text{reg}} \) (see the proof of Theorem 2). Let \( a,b \in S \) and \( aJ^*b \). Then \( aD^*b \) and hence \( a^*D^*b^* \).
This leads to \( a^*Db^* \) by Lemma 4. Thus \( a^*Rb^* \), that is, \( a^\#Rb^\# \). Consequently, \( J^* \subseteq \tilde{R} \) and so \( \tilde{R} = J^* \).

(2) ⇒ (1). Assume that \( \tilde{R} = J^* \). By Theorem 4, we only need to verify that \( \tilde{R} \) is a semilattice congruence on \( S \). For this purpose, we prove that \( J^* \) is a semilattice congruence on \( S \). Let \( a \in S \). Since \( L^* \) is a right congruence, we have \( a^*L^*a = a \). Hence \( a^*J^*a \), this means that \( J^*(a^*) = J^*(a) \).
Thus, for any \( b,c \in S \), we have \( J^*(bc) = J^*(b)J^*(c) = J^*(b)J^*(c) \). Similarly, we have \( J^*(cb) \subseteq J^*(bc) \) and so \( J^*(cb) = J^*(bc) \). Now we let \( a,b,u \in S \) with \( aJ^*b \). Because \( L^* \) is a right congruence, we have \( auL^*a'J^*u'a'*a' \). Similarly, \( buJ^*u'J^*b' \). According to \( E(S) \) being a right regular band, we can follow that \( u'a'D^*u'b' \), thus \( auL^*bu \). Therefore, \( J^* \) is a semilattice congruence, that is, \( \tilde{R} \) is a semilattice congruence. Consequently, \( S \) is a right C-wrpp.

**4. Conclusions**

In this paper, we show that a right C-wrpp semigroup can be described as a semilattice of the direct product of left-R cancellative monoids and zero bands, our results further generalizes both results of Shum-Ren on right C-rpp semigroups and Guo on a notes on right dual of left C-rpp semigroups. Last, the characterizations of a C-wrpp semigroup are given, that is, \( S \) is a right wrpp semigroup if and only if the relations \( \tilde{R} = J^* \) or \( D^* = J^* \).

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**References**