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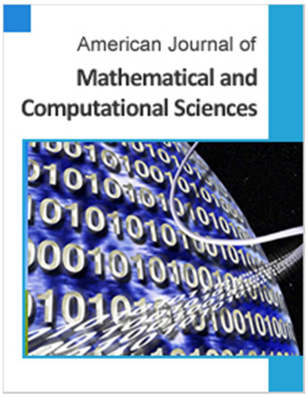

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Equivalent Characterizations and Structure Theorem of Right C-wrpp Semigroups

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Abstract
The aim of this paper is to study the right dual of left C-wrpp semigroup, that is, the strongly wrpp semigroup whose set of idempotents forms a right regular band and the relation $L^{(+)} \vee R$ is a congruence. We call this kind of strongly wrpp semigroups right C-wrpp semigroups. This paper generalizes results of right C-rpp semigroups. Some properties and characterizations of right C-wrpp semigroups are investigated.

1. Introduction

A semigroup S is an rpp semigroup if all of its principal right ideals aS^1 regarded as right S^1-systems, are projective (see [3]-[5]). These classes of semigroups and its special subclasses have been studied by Fountain. He also defined a generalized Green's relation L^* on a semigroup S by aL^*b if and only if the elements a,b of S are related by the Green's relation L in some oversemigroup of S in [5]. In particular, if $(e,f)\in L^*$ for some idempotent elements $e,f\in S$, then $(e,f)\in L$. Next, he showed that a monoid S is rpp if and only if each L^*-class of S contains an idempotent. Thus, a semigroup S is an rpp semigroup if each L^*-class of S contains at least one idempotent. An rpp semigroup having all of its idempotents lying in its center is called a C-rpp semigroup. It is well known that a semigroup S is a C-rpp semigroup if and only if S is a strongly semilattice of left cancellative monoids. Thus, C-rpp semigroups are natural generalizations of Clifford semigroups.

For rpp semigroups, the concept of strongly rpp semigroups was first introduced by Guo-Shum-Zhu, that is, an rpp semigroup in which each L_a^* contains a unique idempotent $a^+\in L_a^*\cap E(S)$ such that $a^+a=a$, where $E(S)$ is set of idempotents of S. Strongly rpp semigroups and their special cases have been extensively studied by many authors (see [6]-[10], [14]-[17]). In particular, we call a strongly rpp semigroup S left C-rpp semigroup if L^* is a congruence and $eS\subseteq Se$ for all $e\in E(S)$. For such semigroups, Guo has proved that an rpp semigroup is a left C-rpp semigroup if and only if S is a semilattce of direct products of left cancellative monoids and left zero bands. Thus, a left C-rpp semigroup is clearly a generalized C-rpp semigroup and left C-semigroup.

On the other hand, we call a strongly an rpp semigroup S a right C-rpp semigroup if $L^*\vee R$ is a congruence and $Se\subseteq eS$ for all $e\in E(S)$. It was observed that the concept of right C-rpp semigroups is not a dual of left C-rpp semigroups by Guo [9]. The structure of right C-rpp semigroups has been recently described by Shum and Ren in [17]. It is noteworthy that many properties of left C-rpp semigroups are not dual to those of right C-rpp semigroups (see [9], [17]).

Tang [18] has introduced a new set of Green's two stars relations on a semigroup S by modifying Green's one star relations on semigroups. Let R be a Green's relation on a semigroup S. A relation L^{**} on S by: for some $a,b\in S$, $(a,b)\in L^{**}$ if and only if $(ax,ay)\in R \Leftrightarrow (bx,by)\in R$ for all $x,y\in S^1$. In view of this new Green's two stars relation, he defined the concept of wrpp semigroups, that is, a wrpp semigroup in which each L^{**}-class contains at least one idempotent. A wrpp semigroup is a C-wrpp semigroup if all the idempotents of S are central. We have known that a C-wrpp semigroup can be expressed as a strongly semilattice of left R-cancellative monoids. Recently, C-wrpp semigroups have been extended to left C-wrpp semigroups by Du-Shum [2]. A wrpp semigroups S is called a left C-wrpp semigroup if S satisfies the following conditions: (i)for all $e\in E(L_a^{**}), a=ae,$ where $E(L_a^{**})$ is set of idempotents in L_a^{**}; (ii)for all $a\in S$, there exists a unique idempotent a^+ satisfying $aL^{**}a^+$ and $a=a^+a$; (iii) for all $a\in S$, $aS\subseteq L^{**}(a)$, where $L^{**}(a)$ is the smallest left **-ideal of S generated by a. if the condition (i) only holds, then S is called quasi strong wrpp semigroup, if conditions (i) and (ii) hold, then S is called strong wrpp semigroup. For left C-wrpp semigroups, Du-Shum [3] have obtained a description of curler structure. In fact, left C-wrpp semigroups are indeed a common generalizations on left C-semigroups and left C-rpp semigroups.

Naturally, one would ask whether we can describe the kind of wrpp semigroup as an analogy of right C-rpp semigroups. In this paper, we generalize right C-rpp semigroups to right C-wrpp semigroups. We first introduce the concept of right C-wrpp semigroups, and some properties on right C-wrpp semigroups are given. We shall show that a right C-wrpp semigroup can be described as a semilattice of the direct product of left-R cancellative monoids and left zero bands, but the task is not simple as we need to consider the semilattice congruence on abundant wrpp semigroups. Anyway, our results clearly further generalizes both results of Shum-Ren on right C-rpp semigroups and Guo on a notes on right dual of left C-rpp semigroups. Last, the characterizations of a C-wrpp semigroup are given.

For the notations and terminology not given in this paper, the reader is referred to [2], [13], [18].

2. Basic Definitions and Results

We first introduce some definitions and results that are useful in the sequel.

In order to describe wrpp semigroups, Du-Shum introduced the following (+)-Green's relations. For elements a,b in a semigroup S, we define:

$$L^{(+)}=L^{**},\ R^{(+)}=R, \tag{1}$$

$$D^{(+)}=L^{(+)}\vee R^{(+)}, \tag{2}$$

$$H^{(+)}=L^{(+)}\wedge R^{(+)}, \tag{3}$$

$$J^{(+)} \Longleftrightarrow J^{(+)}(a)=J^{(+)}(b) \tag{4}$$

where $J^{(+)}(a)$ is the smallest ideal of S containing a such that $J^{(+)}(a)$ is the union of some $L^{(+)}$-classes or some $R^{(+)}$-classes. For the sake of simplicity, we denote the $L^{(+)}$-class (resp., $R^{(+)}$-class, $H^{(+)}$-class, $D^{(+)}$-class and $J^{(+)}$-class) of S containing a by $L_a^{(+)}$ (resp., $R_a^{(+)}, H_a^{(+)}, D_a^{(+)}$ and $J_a^{(+)}$). The Green's "egg box" diagram for Green's relation still holds for these Green's (+) relations. We have

Lemma 1 [2] The equalities $L^{(+)}\circ R^{(+)}=R^{(+)}\circ L^{(+)}$ and $D^{(+)}=L^{(+)}\circ R^{(+)}$ hold on a semigroup.

On the other hand, the Green's (+) relations on S are also similar to the Green's relations on S, for instance, we have $D^{(+)}\subseteq J^{(+)}$. Moreover, we have the following lemma:

Lemma 2 [2] Let S be a semigroup, $a,b\in S$. Then $b\in J^{+}(a)$ if and only if there exists $a_0,a_1,\cdots,a_n\in S$ with $a=a_0, b=a_n$ and $x_1,x_2,\cdots,x_n;y_1,y_2,\cdots y_n\in S^1$ such that $a_iL^{(+)}x_ia_{i-1}y_i$, for all $i=1,2,\cdots,n$.

A $L^{(+)}$-class may contain more than one regular L-class. This is because if $eL^{(+)}a$ for some idempotent $e\in E(S)$ and $a\in S$, then the relation $a=ae$ need not always hold. For example, if $S=M(G;I,\Lambda;P)$ is a Rees matrix semigroup over a group G with $|\Lambda|\geq 2$, then it is not difficult to see that S is an $L^{(+)}$-simple semigroup containing $|\Lambda|$ regular L-classes. But if S is a quasi strongly wrpp semigroup, then we have

Lemma 3 Let S be a quasi strongly wrpp semigroup and $a,b\in \text{Reg}S$. Then $aL^{(+)}b$ if and only if aLb.

Proof Necessity. We only need verify that a^+Lb^+. If $aL^{(+)}b$, then $a^+L^{(+)}b^+$. By the quasi strong wrpp property of S, we have $a^+=a^+b^+, b^+=b^+a^+$, so a^+Lb^+. Thus aLb.

Sufficiency. It is clear and we omit the proof.

Lemma 4 Let S be a quasi strongly wrpp semigroup. Then each $D^{(+)}$-class contains at most one regular D-class.

Proof According to Lemma 1, we have $D^{(+)}=R^{(+)}\circ L^{(+)}=R\circ L^{(+)}$. Let $a,b\in \text{Reg}S.$ If $aD^{(+)}b,$ then there exists c such that $aRcL^{(+)}b$. By regularity of a, we obtain that c is a regular element of S. According to Lemma 3, we know that cLb. Hence aDb, so $D^{(+)}|_{\text{Reg}S}=D|_{\text{Reg}S}$.

Definiton 1 A wrpp semigroup S is called a right C-wrpp semigroup, if S satisfies the following conditions:

(1). S is a quasi strong wrpp semigroup;

(2). $D^{(+)}$ is a congruence on S;

(3). $(\forall e\in E(S))$ $Se\subseteq eS$.

We call a band a right regular band if it satisfies the identity $ef = fef$. We now cite the following lemma:

Lemma 5 [15] The following statements are equivalent on a band B:

(1). B is a right regular band;

(2). R is a congruence;

(3). B is a semilattice of right zero bands.

An immediate result of this lemma is:

Corollary 1 If B is a right regular band, then each L-class of B contains precisely one idempotent.

3. Characterizations of Right C-wrpp Semigroups

In this section, we shall describe some characterizations of right C-wrpp semigroups and hence generalize the main results of right C-rpp semigroups obtained by Guo in [9]. The results obtained in [9] will be amplified and strengthened.

Lemma 6 Let S be a right C-wrpp semigroup. Then the following hold:

(1). $E(S)$ is a right regular band;

(2). Reg S is a right C-semigroup.

Proof (1) Let $e,f\in E(S)$. Since $Se\subseteq eS$, then there exists x such that $fe=ex$, so $efe=eex=ex=fe$. Hence $(ef)^2=ef$, it implies that $E(S)$ is a band and a right regular band.

(2) According to (1), $E(S)$ is a band, so Reg S is a regular subsemigroup of S. And for all $e\in E(S)$, $(\text{Reg}\,S)e\subseteq eS\cap$ Reg $S=e($ Reg S), consequently, Reg S is a right C-semigroup (see [20]).

Theorem 1 The following statements are equivalent:

(1). S is a right C-wrpp semigroup;

(2). S is a strong wrpp semigroup such that $D^{(+)}$ is a semilattice congruence, and $D|_{\text{Reg}S}=R|_{\text{Reg}S}$;

(3). S is a semilattice of $D^{(+)}$-simple strong wrpp semigroups, and $D|_{\text{Reg}S}=R|_{\text{Reg}S}$;

(4). S is a semilattice of $S_\alpha=M_\alpha\times\Lambda_\alpha$ for $\alpha\in Y$, where M_α is a left $-R$ cancellative monoid, Λ_α is a right zero band.

Proof (1)$\Rightarrow$(2). Let S be a right wrpp semigroup. Then $D^{(+)}$ is a congruence of S. Let $a,b\in S, e,f\in E(S)$, and $aL^{(+)}e, bL^{(+)}f$. Then clearly $aD^{(+)}e, bD^{(+)}f$. But $D^{(+)}$ is a congruence, we have $a^2D^{(+)}e$, so $a^2D^{(+)}a$. Notice that $E(S)$ is a right regular band, it leads to $abD^{(+)}ef = fefDfe$ $D^{(+)}ba$. Consequently, $D^{(+)}$ is a semilattice congruence.

According to Lemma 6, Reg S is a right C-semigroup. Therefore, Reg S/R is a semilattice, and $D^{\text{Reg}S}=R^{\text{Reg}S}$ by Lemma 6, We easily prove that $D|_{\text{Reg}S}=R|_{\text{Reg}S}$. By quasi wrpp property of S, and Lemma 4, we know that each $D^{(+)}$-class exactly contains one regular D-class, and each $D^{(+)}$-class exactly contains one regular R-class, it means that each $L^{(+)}$-class in each $D^{(+)}$-class contains a unique idempotent which is a left identity of this $L^{(+)}$-class. Again, $Se\subseteq eS$ for all $e\in E(S)$, so this unique idempotent is also a right identity of above $L^{(+)}$-class. Hence S is a strongly wrpp semigroup.

(2)$\Rightarrow$(3). Let $S=\cup_{\alpha\in Y}S_\alpha$ be a semilattice decomposition corresponding to the semilattice congruence $D^{(+)}$. Obviously, for an arbitrary subsemigroup T of S, we have $L^{(+)S}|_T\subseteq L^{(+)T}$. Hence the elements of S_α having $L^{(+)}$ relation in S also have $L^{(+)}$ relation in S_α. By $D|_{\text{Reg}S}=R|_{\text{Reg}S}$, and Lemma 4, it implies that each S_α only contains one regular R-class. Therefore, the elements of S_α having R relation in S and also have R relation in S_α, so each $D^{(+)}$-class S_α is $D^{(+)}$-simple, and is a strongly wrpp semigroup.

(3) $\Rightarrow$ (4). Let S be a semilattice decomposition $S=\cup_{\alpha\in Y}S_\alpha$, where S_α is a $D^{(+)}$-simple strong wrpp semigroup. Let $\Lambda_\alpha=E(S_\alpha)$. According to $D|_{\text{Reg}S}=R|_{\text{Reg}S}$, we know that each $L^{(+)}$-class $L_e^{(+)}$ of S_α contains a unique idempotent e, and Λ_α is a right zero band. Next we shall verify that $S_\alpha e=L_e^{(+)}$. Let $a\in S_\alpha$. Then $aD^{(+)}e$, so a^+De. Since $D|_{\text{Reg}S}=R|_{\text{Reg}S}$, it leads to a^+Re. Hence $e=a^+eL^{(+)}ae$, so $S_\alpha e\subseteq L_e^{(+)}$. Conversely, if $aL^{(+)}e$ ($e\in E(\Lambda_\alpha)$), then $a=ae$, and $aD^{(+)}e$, it is easily observed that $a\in S_\alpha$, it means $L_e^+\subseteq S_\alpha e$. Thus $L_e^+=S_\alpha e$. By strong wrpp property of S_α, we have $L_e^+=S_\alpha e=eS_\alpha e$, which is a monoid with identity element e for all $e\in E(\Lambda_\alpha)$. We claim that $S_\alpha e$ is left-R cancellative. In fact, for all $eae,ebe,ece\in eS_\alpha e=S_\alpha e$, if $((ece)(eae),(ece)(ebe))\in R$, notice that $eceL^{(+)}e$, then $(eae,ebe)\in R$. Now define a mapping:

$$\Phi: S_\alpha e_0\times\Lambda_\alpha\to S_\alpha, \Phi(a,e)=ae \tag{5}$$

for any fixed $e_0\in\Lambda_\alpha$. Then we deduce that $\Phi[(a,e)(b,f)]=\Phi(ab,f)=abf=aebf \quad =\Phi(a,e)\Phi(b,f)$. Thus, Φ is a semigroup homomorphism.

We now show that Φ is a semigroup isomorphism. By virtue of the strongly wrpp property of S_α, for all $x\in S_\alpha$, there exists $e\in\Lambda_\alpha$ such that $x=xe=xe_0e$. By the definition of Φ, this means that $\Phi(xe_0,e)=x$, and hence Φ is an epimorphism. To prove Φ is a monomorphism, we assume that $\Phi(a,e)=\Phi(b,f)$. Then we have $ae=bf$. Since Λ_α is a right zero band, we have $ae_0=aee_0=bfe_0=be_0$. This implies that $a=b$ for all $a,b\in S_\alpha e_0$. Invoking the strongly wrpp property of S_α, we obtain that $e=f$. This shows that Φ is a monomorphism as well. Thus $S_\alpha \cong S_\alpha e_0 \times \Lambda_\alpha$. The proof is completed.

1em $(4)\Rightarrow(1)$. Let S is a semilattice of $S_\alpha = M_\alpha \times \Lambda_\alpha$ for $\alpha\in Y$, where M_α is a left-R cancellative monoid, Λ_α is a right zero band. Then $E(M_\alpha\times\Lambda_\alpha)=E_\alpha=\{(1_\alpha,i_\alpha)\mid i_\alpha\in\Lambda_\alpha\}$, where 1_α is unique identity of left-R monoid M_α. We now show that $S_\alpha=M_\alpha\times\Lambda_\alpha$ is a $D^{(+)}$-class. Let $a\in S_\alpha, b\in S_\beta$, and $aD^{(+)}b$. Then there exists $c\in S_\gamma$ such that $aL^{(+)}cRb$. Since cR^Sb if and only if $\beta=\gamma$, so $c\in S_\beta$. Hence we have $cRc(1_\beta,j_\beta)$ for any $(1_\beta,j_\beta)\in E(S_\beta\times\Lambda_\beta)$, it implies that $aRa(1_\beta,j_\beta)$. This means that $\alpha\le\beta$. Similarly, we can verify that $\beta\le\alpha$. Hence we conclude that $\alpha=\beta$. Because S_α is just a $D^{(+)}$-class of S, $D^{(+)}$ must be a semilattice congruence on S.

1emNext, we need verify that $Se\subseteq eS$. Let $(1_\alpha,i_\alpha)\in E_\alpha,(1_\beta,j_\beta)\in E_\beta$ and $\alpha,\beta\in Y$ with $\alpha\ge\beta$. Then $(1_\alpha,i_\alpha)(1_\beta,j_\beta)\in E_\beta$. In fact, $(1_\alpha,i_\alpha)\ (1_\beta,j_\beta)\in S_{\alpha\beta}=S_\beta$, then $(1_\beta,j_\beta)(1_\alpha,i_\alpha)(1_\beta,j_\beta)=(1_\alpha,i_\alpha)(1_\beta,j_\beta)$, so

$$
\begin{aligned}
&(1_\alpha,i_\alpha)(1_\beta,j_\beta)(1_\alpha,i_\alpha)(1_\beta,j_\beta)\\
&=(1_\alpha,i_\alpha)[(1_\beta,j_\beta)(1_\alpha,i_\alpha)(1_\beta,j_\beta)]\\
&=(1_\alpha,i_\alpha)(1_\alpha,i_\alpha)(1_\beta,j_\beta)=(1_\alpha,i_\alpha)(1_\beta,j_\beta),
\end{aligned}
$$

that is, $(1_\alpha,i_\alpha)(1_\beta,j_\beta)\in E_\beta$. Clearly, $\beta\ge\alpha\beta$, by using above analogous methods, we obtain that $(1_\beta,j_\beta)(1_{\alpha\beta},k_{\alpha\beta})\in E_{\alpha\beta}$ for any $k_{\alpha\beta}\in\Lambda_{\alpha\beta}$. Hence, we have

$$
\begin{aligned}
&(1_\beta,j_\beta)(1_\alpha,i_\alpha)(1_\beta,j_\beta)\\
&=(1_\beta,j_\beta)(1_{\alpha\beta},k_{\alpha\beta})(1_\alpha,i_\alpha)(1_\beta,j_\beta)\\
&=(1_\alpha,i_\alpha)(1_\beta,j_\beta).
\end{aligned}
$$

This means that $E(S)$ is a right regular band. Now let $(a,i_\alpha)\in S_\alpha$, then

$$
\begin{aligned}
(a,i_\alpha)(1_\beta,j_\beta)&=(1_{\alpha\beta},k_{\alpha\beta})(a,i_\alpha)(1_\beta,j_\beta)\\
&=(1_\beta,j_\beta)[(1_{\alpha\beta},k_{\alpha\beta})(a,i_\alpha)(1_\beta,j_\beta)]\\
&=(1_\beta,j_\beta)(a,i_\alpha)(1_\beta,j_\beta).
\end{aligned}
$$

This verifies that $Se\subseteq eS$.

Summing up the above results, then S is a right C-wrpp semigroup.

Corollary 2 Let S is a right C-wrpp semigroup. Then $D^{(+)}=J^{(+)}$.

Proof Because $D^{(+)}\subseteq J^{(+)}$, we only need to prove that $J^{(+)}\subseteq D^{(+)}$. Suppose that $aJ^{(+)}b$. Then $b\in J^{(+)}(a)$. By Lemma 2, there exists $a_0,a_1,\cdots,a_n\in S$ with $a=a_0, b=a_n$ and $x_1,x_2,\cdots,x_n;y_1,y_2,\cdots y_n\in S^1$ such that $a_iL^{(+)}x_ia_{i-1}y_i$, for all $i=1,2,\cdots,n$. Since $L^{(+)}\subseteq D^{(+)}$ and $D^{(+)}$ is a congruence, we have $bD^{(+)}x_nx_{n-1}\cdots x_1a\ \ y_1y_2\cdots y_n$. By Theorem 1, we know that $D^{(+)}$ is a semilattice congruence. We denote semilattice $S/D^{(+)}$ by Y. Index $D^{(+)}$-class in virtue of the elements $\{\alpha,\beta,\cdots\}$ in the semilattice Y, and let $a\in D^{(+)}_\alpha, b\in D^{(+)}_\beta$. We are not difficult to see that $\alpha\ge\beta$. Similarly, $\alpha\le\beta$. Hence $aD^{(+)}b$.

Lemma 7 Let S be a strongly wrpp semigroup whose set of idempotents is a semilattice Y. Then $(ab)^+=a^+(ab)^+b^+$ for all $a,b\in S$.

Lemma 8 Let S be a strongly wrpp semigroup whose set of idempotents is a semilattice Y, and $a,b\in S$. If $a^+\ge(ab)^+$, then the following statements are hold:

(1). $L^{(+)}_a$ is a left-R cancellative monoid;

(2).If $\lambda,\mu\in Y$, and $\lambda\ge\mu$, then the mapping $\Phi_{\lambda,\mu}$:

$$L^{(+)}_\lambda\to L^{(+)}_\mu, x\mapsto x\mu \tag{6}$$

is a semigroup homomorphism. Moreover, with respect to the following multiplication "$\circ$":

$$a\circ b=a\Phi_{a^+,a^+b^+}b\Phi_{b^+,a^+b^+}, \tag{7}$$

S form a C-wrpp semigroup, where $a\Phi_{a^+,a^+b^+}b\Phi_{b^+,a^+b^+}$ is the product in $L^{(+)}_{a^+b^+}$;

(3). $ab=a\Phi_{a^+,(ab)^+}b\Phi_{b^+,(ab)^+}$, where ab is the product of a and b in S.

Proof (1) Let $x,y\in L^{(+)}_a$. Notice that there is exactly one idempotent in $L^{(+)}_a$, we have $x^+=y^+$. By the fact that S being a strongly wrpp semigroup, we have $x^+y=y^+y=y$. Since $L^{(+)}$ is a right congruence, we know that $xyL^{(+)}x^+y=y$. Hence $L^{(+)}_a$ is a subsemigroup of S. Notice that $yx^+=yy^+=y$, it follows that x^+ is the identity of $L^{(+)}_a$. Now put $u,v\in S$, and $xuRxv$. Then x^+uRx^+v. Thus uRv, that is, $L^{(+)}_a$ is a left-R cancellative monoid.

(2) Let $x\in L^{(+)}_\lambda$. Since $L^{(+)}$ is a right congruence, we have $x\mu L^{(+)}\lambda\mu=\mu\mu=\mu$, that is, $x\mu\in L^{(+)}_\mu$. By (1), we know that $L^{(+)}_\mu$ is a left-R cancellative monoid with identity μ. Consequently, for all $y\in L^{(+)}_\lambda$, we have

$$(xy)\Phi_{\lambda,\mu}=xy\mu=x(y\mu)=x\mu y\mu=x\Phi_{\lambda,\mu}y\Phi_{\lambda,\mu}. \tag{8}$$

Thus $\Phi_{\lambda,\mu}$ is a semigroup homomorphism. It is not difficult to verify that $\Phi_{\lambda,\mu}$ is a strongly semilattice structure homomorphism on $L^{(+)}_\lambda(\lambda\in Y)$. Therefore, S is a C-wrpp semigroup.

(3) Since $L^{(+)}$ is a right congruence, we have $b(ab)^+L^{(+)}b^+(ab)^+$. By Lemma 7, we obtain that

$b^+(ab)^+=(ab)^+$. It means that $b(ab)^+\in L^{(+)}_{(ab)^+}$. Because $(ab)^+$ is the identity of $L^{(+)}_{(ab)^+}$, we have

$$ab=a(ab)^+b(ab)^+=a\Phi_{a^+,(ab)^+}b\Phi_{b^+,(ab)^+}. \quad (9)$$

Lemma 9 Let S be a semigroup satisfying the conditions in Lemma 8. Then every regular element of S is completely regular, that is, a regular element is H-related to an idempotent element.

Proof Let a be a regular element of S. Then there exists $b\in S$ such that $aba=a$, so $b^+\geq a^+$. Hence $b\Phi_{b^+,a^+}\in L^{(+)}_{a^+}$. As it is argued in Lemma 3.5, $a=a\Phi_{a^+,a^+}b\Phi_{b^+,a^+}a\Phi_{a^+,a^+}=ab\Phi_{b^+,a^+}a$. Hence a is a regular element of $L^{(+)}_{a^+}$ and $ab\Phi_{b^+,a^+}$ is an idempotent of $L^{(+)}_{a^+}$. But there is only one idempotent in $L^{(+)}_{a^+}$, so $a^+=ab\Phi_{b^+,a^+}$. Thus aHa^+, that is, a is a completely regular element.

Lemma 10 Let S be a strongly semigroup whose set of idempotents is a band. Then every regular element of S is a completely regular element

Proof Since $E(S)$ is a band, Reg S is a orthodox semigroups. Since S is a strongly wrpp semigroup, we can easily see that Reg S is a strongly wrpp semigroup. Hence Reg S/γ is a strongly wrpp semigroup, where γ is the smallest inverse semigroup congruence on Reg S. According to Lemma 9, we can follow that Reg S/γ is a Clofford semigroup. Let $a\in \mathrm{Reg}S$. Then there exists $e,f\in E(S)$ such that $eLaRf$. It follows that $e\gamma La\gamma Rf\gamma$. By Reg S/γ being a Clifford semigroup, $e\gamma=f\gamma$. On the other hand, since $\gamma|_{E(S)}=D^{E(S)}$, we have $eD^{E(S)}f$ and hence $eLefRf$. Therefore $efHa$, that is, a is a completely regular element.

As an application of above results, we now give some conditions which lead to a C-wrpp semigroup S/ξ for some congruence ξ defined on a right C-wrpp semigroup S. In fact, all we need to find a congruence ξ on S so that ξ preserves the $L^{(+)}$-classes of S.

For convenience, we denote the rectangular band B_α by $E(a^+)$ if the idempotent a^+ is in B_α. Also, we write $B_\alpha\leq B_\beta$ if $B_\alpha B_\beta\subseteq B_\alpha$.

We now characterize right C-wrpp semigroups.

Theorem 2 The following conditions are equivalent for a strongly wrpp semigroup S:

(1). S is a right C-wrpp semigroup;

(2). $E(S)$ is a right regular band and $D^{(+)}$ is a semilattice congruence on S;

(3).The relation $\xi=\{(x,y)\mid(\exists f\in E(y^+))x=yf\}$ is a congruence on S such that S/ξ is a C-wrpp semigroup.

Proof (1) $\Rightarrow$ (2). This part is an immediate consequence of Lemma 6 and Theorem 1.

(2) $\Rightarrow$ (1). Let $E(S)$ be a right regular band and suppose that $D^{(+)}$ is a congruence on S. To show that S is a right C-wrpp semigroup, we only need to verify that $D|_{\mathrm{Reg}S}=R|_{\mathrm{Reg}S}$. By Lemma 5, we have $D^{E(S)}=R^{E(S)}$. On the other hand, for all $a.b\in \mathrm{Reg}S$ with aDb, by Lemma 10, there exists $e,f\in E(S)$ such that aHe,bHf. Clearly, eDf, and then there exists some $c\in\mathrm{Reg}S$ such that $eLcRf$. Again by Lemma 10, there exists $g\in E(S)$ such that cHg. Thus, $eLgRf$, that is, $eD^{E(S)}f$. Consequently, $eR^{E(S)}f$ and aRb. Thus, we have proved that $D|_{\mathrm{Reg}S}=R|_{\mathrm{Reg}S}$. The proof is completed.

(2) $\Rightarrow$ (3). We can assume that S is a right C-wrpp semigroup. Then we have $Se\subseteq eS$ for every $e\in E(S)$ and because S is a strongly wrpp semigroup, $aL^{(+)}a^+$ for $a^+\in E(a^+)$. Thus, $a=aa^+$. This means that ξ is reflexive. To see that ξ is symmetric, we let $a\xi b$. Then, by the definition of ξ, we have $a=bf$ for $f\in E(b^+)$. Since $bL^{(+)}b^+$, we also have $bfL^{(+)}b^+f$. Consequently, we get $a^+L^{(+)}b^+f$ and $E(a^+)=E(b^+f)=E(b^+)$. Thus $b^+\in E(a^+)$. From $a=bf$, we immediately get $ab^+=b$. This shows that ξ is symmetric. To see that ξ is transitive, we let $a\xi b$ and $b\xi c$. Then there exists $g\in E(c^+)$ such that $b=cg$. By repeating the arguments given above, we have $E(b^+)=E(c^+)=E(a^+)$. This leads to $gf\in E(c^+)$. By $a=cgf$, we have $a\xi c$. Hence ξ is indeed an equivalent relation on S.

1emTo see that ξ is a congruence on S, we let $a\xi b$. Then, by the definition of ξ, there exists $f\in E(b^+)$ such that $a=bf$. Hence $ca=cbf=cb(cb)^+f$. By invoking Theorem 1 (4), we have $c^+f\in E((cb)^+)$, that is, $E(b^+)\geq E((cb)^+)$. This leads to $(cb)^+f\in E((cb)^+)$. In other words, we have $ca\xi cb$ and hence ξ is left compatible. Similarly, we can verify that ξ is right compatible. Thus ξ is indeed a congruence on S.

We still need to show that ξ preserve the $L^{(+)}$-classes of S. For this purpose, we let $(a,b)\in L^{(+)}$ for some $a,b\in S$. If there are $x,y\in S^1$ such that $((ax)\xi,(ay)\xi)\in R(S/\xi)$, then there exists $u,v\in S^1$ such that $(ax)\xi(u\xi)=(ay)\xi$ and $(ay)\xi(v\xi)=(ax)\xi$. Hence, we can find $e\in E((ay)^+), f\in E((ax)^+)$ such that $axu=(ay)e, ayv=(ax)f$. By $D^{(+)}$ being a semilattice congruence, we can deduce that $E((ax)^+)\geq E((ay)^+)$ and similarly, $E((ay)^+)\geq E((ax)^+)$. This leads to $E((ax)^+)=E((ay)^+)$. Clearly, $ef\in E((ax)^+)$ and consequently, $axef(ax)^+uf=ayef$ and

$ayef(ay)^+v = axfef = axef$, that is, $(axef, ayef) \in R$. So we also have $(bxef, byef) \in R$. Therefore, by the definition of R , there exists $k, l \in S^1$ such that $bxefk = byef$ and $byefl = bxef$. On the other hand, since $(a,b) \in L^{(+)}$, we have $(ax, bx) \in L^{(+)}$ and hence we deduce that $E((ax)^+) = E((bx)^+)$. Similarly, $E((ay)^+) = E((by)^+)$.

Since $E((ax)^+) = E((ay)^+)$, we have $E((ax)^+) = E((by)^+)$ and hence $ef \in E((by)^+)$. This leads to $(bx)\xi(efk)\xi = (by)\xi$. Similarly, $(by)\xi(efl)\xi = (bx)\xi$. Thus, we have $((bx)\xi, (by)\xi \in R(S/\xi)$. From this relation and its dual, we conclude that $(a\xi, b\xi) \in L^{(+)}(S/\xi)$. This shows that the relation $(a,b) \in L^{(+)}(S/\xi)$ on S is preserved in the quotient semigroup S/ξ , and hence S/ξ is a wrpp semigroup.

Finally, we show that the idempotents of S/ξ are central. It suffices to show that $(ea, ae) \in \xi$ for all $e \in E(S)$ and $a \in S$. Since by Theorem 1 (4), $E(a^+)E(e) \subseteq E((ae)^+) = E((ea)^+)$, it is clear that $e(ae)^+ \in E((ae)^+)$. Thus, by $ae = (ae)^+e(ae)^+(ae)(ae)^+ = e(ae)^+ae(ae)^+ = eae(ae)^+$, we obtain that $(ea, ae) \in \xi$. This shows that S/ξ is a C-wrpp semigroup.

(3) $\Rightarrow$ (1). Suppose that ξ is a congruence on S such that S/ξ is a C-wrpp semigroup. we can easily see that $\xi|_{E(S)} = R_{E(S)}$ and hence $E(S/\xi) = E(S)/\xi = E(S)/R$ is a semilattice. Hence R is a semilattice congruence on S, and so $E(S)$ is a right regular band. Now let $E(S) = \cup_{\alpha\in Y}\Lambda_\alpha$ be the semilattcie decomposition of $E(S)$ into right zero bands Λ_α . Clearly, Y is isomorphic to $E(S/\xi) = E(S)/R$. We identify Y with $E(S/\xi) = E(S)/R$. By S/ξ is a C-wrpp semigroup, we let $S/\xi = \cup_{\alpha\in Y}M_\alpha$ be the semilattice decomposition of the C-wrpp semigroup S/ξ into left- R cancellative monoids M_α.

Put $T = \cup_{\alpha\in Y}M_\alpha \times \Lambda_\alpha$. Then we define $\varphi: S \to T$ by $x \to (x\xi, x^+)$. Clearly, φ is well defined, and we deduce that

$$(xy)\varphi = ((xy)\xi, (xy)^+) = (x\xi y\xi, x^+y^+) = (x\xi, x^+)(y\xi, y^+) . \quad (10)$$

Thus φ is a semigroup homomorphism.

Now we prove that φ is a semigroup isomorphism. For all $(t,\lambda) \in T$, we have $x \in S$ such that $x\xi = t$ and λRx^+ . It follows that $(x\lambda)\xi = x\xi = t$. On the other hand, since $xL^{(+)}x^+$, we have $x\lambda L^{(+)}x^+\lambda = \lambda$. But $E(S)$ is a right regular band, we know that each L-class of $E(S)$ contains precisely one element, and thus $(x\lambda)^+ = \lambda$. Consequently, $(x\lambda)\varphi = (t,\lambda)$. This means that φ is an epimorphism. To prove φ is a monomorphism, now let $x, y \in S$ and $x\varphi = y\varphi$. Then $x^+ = y^+$ and $x\xi = y\xi$. By using the latter formula, we see that there exists $f \in E(y^+)$ such that $x = yf$, and furthermore, $x = xx^+ = yfy^+ = yy^+ = y$. This shows that φ is also a monomorphism. On the other hand, T is a semilattice of direct products $M_\alpha \times \Lambda_\alpha$ and hence S is a right C-wrpp semigroup.

Now we define a new relation $\tilde{R}$ on a strongly wrpp S as follows:

$$a\tilde{R}b \Leftrightarrow a^+Rb^+. \quad (11)$$

It is easy to verify that $\tilde{R}$ is a equivalent relation, and $\tilde{R} \subseteq D^{(+)} \subseteq J^{(+)}$.

Theorem 3 Let S be a strongly wrpp semigroup. Then S is a right C-wrpp semigroup if and only if $\tilde{R}$ is a semilattice congruence S and $E(S)$ is a right regular band.

Proof Assume that S is a right C-wrpp semigroup. By Lemma 6 (1), we only need to prove that $\tilde{R}$ is a semilattice congruence. For this purpose, we let S is a semilattice of the direct products $M_\alpha \times \Lambda_\alpha$ for $a \in Y$, where M_α is a left- R cancellative monoid and Λ_α is a right zero band. We can easily check that $(a,i) = (1_\alpha, i)$ for any $(a,i) \in M_\alpha \times \Lambda_\alpha$, where 1_α is the identity of M_α. Hence it is difficult to verify that identical formula $\tilde{R} = \cup_{\alpha\in Y}(M_\alpha \times \Lambda_\alpha) \times (M_\alpha \times \Lambda_\alpha)$. It follows that $\tilde{R}$ is a semilattice congruence.

Suppose that $\tilde{R}$ is a semilattice congruence on S and $E(S)$ is a right regular band. Since $\tilde{R}$ is a semilattice congruence on S, S is a semilattice of some $\tilde{R}$-classes. But $a^+ \in \tilde{R}_a$, each $\tilde{R}$ -class of S is a strongly wrpp semigroup, therefore it is $\tilde{R}$ -simple. Next we shall show that each $\tilde{R}$ -simple semigroup is also $D^{(+)}$-simple semigroup. For this purpose, we only need to prove $D^{(+)} = \tilde{R} \vee L^{(+)}$. Let $(a,b) \in D^{(+)}$. Then $a^+D^{(+)}b^+$. Hence there exists $c \in S$ such that $a^+RcL^{(+)}b$. By a^+Rc , we can see that c is a regular element, and by Lemma 10, c is completely regular. Hence, we can follow that $a\tilde{R}c$. This means that $(a,b) \in \tilde{R} \vee L^{(+)}$, so $D^{(+)} \subseteq \tilde{R} \vee L^{(+)}$. Conversely, if $(a,b) \in \tilde{R} \vee L^{(+)}$, then there exist $x_1, x_2, \cdots, x_{2n}, y_1, y_2, \cdots, y_{2n} \in S$ with $a = x_1, b = y_{2n}$ such that $x_1\tilde{R}y_1L^{(+)}x_2\tilde{R}y_2L^{(+)}\cdots.\ \tilde{R}x_{2n}L^{(+)}y_{2n}$. From the above, we have

$$x_1L^{(+)}x_1^+Ry_1^+L^{(+)}x_2Rx_2^+Ry_2^+L^{(+)}, y_2L^{(+)}\cdots L^{(+)}x_{2n}L^{(+)}y_{2n}, \quad (12)$$

This shows that $aD^{(+)}b$. Hence $\tilde{R} \vee L^{(+)} \subseteq D^{(+)}$. This shows that $D^{(+)} = \tilde{R} \vee L^{(+)}$. Thus, each $\tilde{R}$ -simple semigroup is also $D^{(+)}$-simple semigroup, it deduces that S is a semilattice of $D^{(+)}$-simple strongly wrpp semigroups.

Also, Since $E(S)$ is a right regular band, by the proof of

$(2)\Rightarrow(1)$ in Theorem 2, we know that $D|_{\mathrm{Reg}S}=R|_{\mathrm{Reg}S}$. Therefore, S is a right C-wrpp semigroup.

Theorem 4 Let S be a strongly wrpp semigroup whose set of idempotents forms a right regular band. Then the following statements are equivalent:

(1). S is a right wrpp semigroup;

(2). $\tilde{R}=J^{(+)}$;

(3). $D^{(+)}=J^{(+)}$.

Proof $(1)\Rightarrow(3)$. By the Corollary 2, clearly.

$(3)\Rightarrow(2)$. Let $D^{(+)}=J^{(+)}$. Since $E(S)$ is a right regular band, we have $D_{\mathrm{Reg}S}=R_{\mathrm{Reg}S}$ (see the proof of Theorem 2). Let $a,b\in S$ and $aJ^{(+)}b$. Then $aD^{(+)}b$ and hence $a^+D^{(+)}b^+$. This leads to a^+Db^+ by Lemma 4. Thus a^+Rb^+, that is, $a^+\tilde{R}b^+$. Consequently, $J^{(+)}\subseteq\tilde{R}$ and so $\tilde{R}=J^{(+)}$.

$(2)\Rightarrow(1)$. Assume that $\tilde{R}=J^{(+)}$. By Theorem 4, we only need to verify that $\tilde{R}$ is a semilattice congruence on S. For this purpose, we prove that $J^{(+)}$ is a semilattice congruence on S. Let $a\in S$. Since $L^{(+)}$ is a right congruence, we have $a^2L^{(+)}a^+a=a$. Hence $a^2J^{(+)}a$, this means that $J^{(+)}(a^2)=J^{(+)}(a)$. Thus, for any $b,c\in S$, we have $J^{(+)}(bc)=J^{(+)}((bc)^2)=J^{(+)}(b(cb)c)\subseteq J^{(+)}(cb)$. Similarly, we have $J^{(+)}(cb)\subseteq J^{(+)}(bc)$ and so $J^{(+)}(cb)=J^{(+)}(bc)$. Now we let $a,b,u\in S$ with $aJ^{(+)}b$. Because $L^{(+)}$ is a right congruence, we have $auL^{(+)}a^+uJ^{(+)}ua^+L^{(+)}u^+a^+$. Similarly, $buJ^{(+)}u^+b^+$. According to $E(S)$ being a right regular band, we can follow that $u^+a^+D^{(+)}u^+b^+$, thus $auJ^{(+)}bu$. Therefore, $J^{(+)}$ is a semilattice congruence, that is, $\tilde{R}$ is a semilattice congruence. Consequently, S is a right C-wrpp.

4. Conclusions

In this paper, we show that a right C-wrpp semigroup can be described as a semilattice of the direct product of left-R cancellative monoids and left zero bands, our results further generalizes both results of Shum-Ren on right C-rpp semigroups and Guo on a notes on right dual of left C-rpp semigroups. Last, the characterizations of a C-wrpp semigroup are given, that is, S is a right wrpp semigroup if and only if the relatons $\tilde{R}=J^{(+)}$ or $D^{(+)}=J^{(+)}$.

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