Mathematical Model of the Pipeline Connected to the Ends of an Area with Dampers of Pressure

Mamadaliev X. A., Khujaev I. Q.

Laboratory "Modeling of Complex Systems", Center for Development of Software Products and Hardware-Software Complexes, Tashkent, Uzbekistan

Email address
husniddin_m1@bk.ru (Mamadaliev X. A.), i_k_hujayev@mail.ru (Khujaev I. Q.)

Abstract
The analytical decision of the linearized problems about distribution of indignations of a stream speed in the pipeline with application of a method of division of variables is received. In the equation of preservation of an impulse, forces of a friction and gravitation, a local component of inertial force are considered. The equation of preservation of weight is constructed taking into account deformation of a thin wall and compressibility of a liquid only by the influence of a consolidation wave. Initial conditions are formulated according to the constant speed of a stream before indignations. Boundary conditions in each end of linear area of the pipeline reflect variants of the task of speed and gradient of a stream speed, and also connection of an air chamber. Numerical results, which prove reduction of a change interval of static pressure at the expense of connection of the damper, are presented.

1. Introduction
Repeated jumps of mechanical stress on certain areas of the pipeline, caused by the distribution and reflection of consolidation waves, lead to fatigue failure of the pipeline. To eliminate such undesirable phenomena, dampers of static pressure in the form of an air chamber, a receiver and other devices are applied.

Researches have shown that at partial or full braking of a liquid in a pipe consolidation waves are formed. At a positive bias of the pipeline excessive stress of the pipeline caused by increase of static pressure, is expected in the starting area, and at negative biases – in the end of an area [1]. In connection with it, for elimination of concentration of high stress air chambers can be connected in the beginning or in the end of area respectively.

2. Statement of the Problem
The pipeline functioned till \( t = 0 \) in a stationary mode and speed of a stream in it made up \( w_0 \). From the moment of time \( t = 0 \) in the entering to the area \( x = 0 \) the expense \( Q_0(t) \) and in the exit from area – the expense \( Q_1(t) \) were set. It is required to find the decision of a problem on an area condition, which ends are connected with air chambers. Values of coefficient of resistance of a friction and a line bias – are constants.
3. Mathematical Model of a Problem

Influence of air chambers on a dynamic condition of the pipeline can be studied within the limits of N. E. Zhukovskiy’s model with the additional account of gravitation force according to which the equations look like [2]

\[
\begin{align*}
-\frac{\partial p}{\partial x} &= \rho \left( \frac{\partial w}{\partial t} + 2aw + g \sin \alpha \right), \\
-\frac{\partial p}{\partial t} &= \rho c^2 \frac{\partial w}{\partial x}.
\end{align*}
\]

(1)

Here, \( p(x,t) \), \( w(x,t) \) – averages on section of value \( x \) of pressure and speed of a stream at the moment of time \( t \); \( g \) – acceleration of gravitation force; \( a = \lambda w, \) / \( (4D) \); \( \sin \alpha \) – a line bias; \( w \) – characteristic speed of a stream; \( \lambda \) – factor of resistance of a friction; \( \rho \) – density of environment; \( c = \left( \frac{\rho_0}{k} + \frac{D p_0}{E \delta} \right)^{1/2} \) – speed of distribution of indignations of pressure in system of a liquid-pipe, where \( \rho_0 \) and \( D \) – density of a liquid and diameter of the pipeline in indignant condition; \( k \), \( E \) – modules of elasticity of water and a pipe material; \( \delta \) – a thickness of a pipe [3].

As till the moment of time \( t = 0 \) the pipeline worked in a stationary mode so entry conditions of the problem will be

\[ w(x,0) = w_0, \quad \frac{\partial w(x,0)}{\partial t} = 0. \]

The case of connection of an air chamber in the end of area is considered in [2] and the corresponding condition is made in a kind:

\[
\frac{\partial w(l,t)}{\partial x} = \frac{1}{f} \left[ Q_0(t) - \beta_1 w(l,t) \right],
\]

where \( \beta_1 = \frac{\rho c^2 V_1}{p_1} \); \( V_1, p_1 \) volume of an air chamber and pressure in it till disturbance.

The similar condition is made for a chamber connected to an input of the pipeline

\[
-\frac{\partial w(0,t)}{\partial x} = \frac{1}{f} \left[ Q_0(t) - \beta_0 w(0,t) \right],
\]

where \( \beta_0 = \frac{\rho c^2 V_0}{p_0} \).

Concerning functions \( Q_0(t) \) and \( Q_1(t) \) it is required, that they had derivatives of the first and second order, and also integral on \( t \).

To expand a class of solved problems coefficients \( \alpha_0, \alpha_1 \) are entered and boundary conditions are represented in the form:

\[
-\alpha_0 f \frac{\partial w(0,t)}{\partial x} = Q_0(t) - \beta_0 w(0,t),
\]

\[
\alpha_1 f \frac{\partial w(l,t)}{\partial x} = Q_1(t) - \beta_1 w(l,t),
\]

The case \( \alpha_0 = 0 \) corresponds to the set expense on an input at \( \beta_0 = f \). If \( \beta_0 = 0 \) and \( \alpha_0 = 1 \), then on an input the gradient of stream speed is set. At both \( \alpha_0 = 1 \) and \( \beta_0 \neq 0 \) the entrance expense is set \( \beta_0 \neq 0 \) and receiver presence is considered.

Similar conditions are provided and for the second end of a site.

From system of the equations the equation rather is worked out \( w \)

\[
\frac{\partial^2 w}{\partial t^2} + 2a \frac{\partial w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2},
\]

(2)

For which initial and boundary conditions are formulated above.

4. The Solution of the Problem

To solve the problem a method of division of variables [4] is applied. At first replacement is entered

\[
u(x,t) = w(x,t) + (a_0 x + b_0)Q_0(t) + (a_1 x + b_1)Q_1(t).
\]

(3)

At values of unknowns

\[
a_0 = \frac{\beta_1}{\alpha_0 f \beta_1 + \alpha_1 f \beta_0 + \beta_0 \beta_1},
\]

\[
b_0 = -\frac{\alpha_0 f + \beta_1}{\alpha_0 f \beta_1 + \alpha_1 f \beta_0 + \beta_0 \beta_1},
\]

\[
a_1 = -\frac{\beta_0}{\alpha_0 f \beta_1 + \alpha_1 f \beta_0 + \beta_0 \beta_1},
\]

\[
b_1 = -\frac{\alpha_0 f}{\alpha_0 f \beta_1 + \alpha_1 f \beta_0 + \beta_0 \beta_1}.
\]

Boundary conditions take a traditional homogeneous form:

\[
\alpha_0 f \frac{\partial u(0,t)}{\partial x} = \beta_0 u(0,t), \quad \alpha_1 f \frac{\partial u(l,t)}{\partial x} = -\beta_0 u(l,t).
\]

Thus the equation takes an inhomogeneous form
\[
\frac{\partial^2 u}{\partial t^2} + 2a \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = R(t)x + S(t),
\]
where, \( R(t) = a_0[Q_0'(t) + 2aQ_0'(t)] + a_1[Q_1'(t) + 2aQ_1'(t)] \)
\( S(t) = b_0[Q_0'(t) + 2aQ_0'(t)] + b_1[Q_1'(t) + 2aQ_1'(t)]. \)

The decision is searched in the form of the sum of the common decision \( U(x,t) \) of the homogeneous equation and the private decision \( V(x,t) \) of the inhomogeneous equation. Initial conditions are realized after the estimation of decision \( w(x,t). \)

At first we find the general equation solutions
\[
\frac{\partial^2 U}{\partial t^2} + 2a \frac{\partial U}{\partial t} - c^2 \frac{\partial^2 U}{\partial x^2} = 0.
\]

Under boundary conditions \( \alpha_0 f \frac{\partial U(0,t)}{\partial x} = \beta_0 U(0,t) \),
\( \alpha_0 \frac{\partial U(l,t)}{\partial x} = -\beta_0 U(l,t). \)

Search of the decision of a problem by a method of division of variables leads to the independent equations
\[
\frac{T''(t) + 2aT'(t)}{c^2T(t)} = \frac{X''(x)}{X(x)} = \frac{-\lambda^2}{X(x)} < 0.
\]

The decision \( X(x) \) is searched in the form
\[
X(x) = A \sin \lambda x + B \cos \lambda x.
\]

Realization of boundary conditions leads to the century equation:
\[
tg\lambda l = -\frac{\alpha_0 \beta_1 + \alpha_1 \beta_0}{\beta_0 \beta_1 - \alpha_0 \alpha_1 f^2 \lambda^2}.
\]

Function in the left part of the century equation – known periodic function. In the right part of the equation – is odd fractional-rational function with positive coefficients. As a whole, it is decreasing, has ruptures of the second sort at
\[
\lambda_s = \pm \frac{1}{f} \sqrt{\frac{\beta_1 \beta_0}{\alpha_0 \alpha_1}}. \text{ At } \lambda \to \pm \infty \text{ aspires to zero. Has zero value at } \lambda = 0. \text{ At } \lambda_s > \frac{\pi}{2} \text{ the decision of equation }
\lambda_s < \frac{\pi}{2} \text{ make monotonously decreasing range: from 0 to } \infty. \text{ Roots are located in intervals }
\lambda_s \in (n - 1)\pi; (n - 0.5)\pi .
\]

When \( \lambda_s > \frac{\pi}{2} \), at \( \lambda < \lambda_s \) roots belong to intervals \( \lambda_s \in (n); (n + 0.5)\pi \) and at \( \lambda < \lambda_s \) - to intervals \( \lambda_s \in (n - 0.5); n\pi \).

A range of special cases takes place. Without a receiver it is \( \lambda_s = \frac{\pi n}{l} \) and \( X(x) = \sin \lambda x \). In case of receiver connection on an input the century equation takes a form, \( t g\lambda l = \frac{\beta_0}{\alpha_0 f \lambda} \) in case of receiver connection on an exit from a site- \( t g\lambda l = \frac{\beta_1}{\alpha_1 f \lambda} \). Thus, if a complex is \( \frac{\beta}{\alpha f} > 1 \), then \( \lambda_s \in (n); (n + 0.5)\pi \) otherwise \( \lambda_s \in (n - 1); (n - 0.5)\pi . \)

As the special case of degeneration of the equation at \( \alpha_0 f^2 \lambda^2 = \beta_0 \beta_1 \) we have \( X_n x = \cos \lambda_n x \) and \( \lambda_s = (n - 0.5)\frac{\pi}{l} . \)

In the numerical solution of the century equation it is expedient to use a method of division of a piece half-and-half [5].

Taking into account possible variants of boundary conditions and realization of the first boundary condition eigenfunction of a problem looks like
\[
X_n(x) = \beta_0 \sin \lambda_n x + \alpha_0 f \lambda_n \cos \lambda_n x .
\]

Orthogonality of eigenfunction is proved and the norm square is found:
\[
\|X_n(x)\|^2 = \frac{\beta_0^2}{2} + \frac{\alpha_0^2 f^2 \lambda_n^2}{2} l + \frac{-\beta_1^2}{2} + \frac{\alpha_1^2 f^2 \lambda_n^2}{2} l \sin \lambda_n l \cos \lambda_n l + \alpha_0 \beta_0 f \sin^2 \lambda_n l .
\]

At found eigenvalues regarding \( T_n(t) \) we will receive:
\[
T''(t) + 2aT'(t) + c^2 \lambda_n^2 T_n(t) = 0 .
\]

Let \( T_n(t) = e^{\lambda_n t} \), then the characteristic equation will be worked out
\[
s^2 + 2as_n + c^2 \lambda_n^2 = 0 .
\]

At a designation of the solution \( D_n = a^2 - c^2 \lambda_n^2 \) of the given equation make \( (s_n)_{1,2} = -a \pm \sqrt{D_n} \), according to \( T_n(t) \) we will receive the solution:
We pass to the solution of the inhomogeneous equation

$$\frac{\partial^2 W}{\partial t^2} + 2a \frac{\partial W}{\partial t} - c^2 \frac{\partial^2 W}{\partial x^2} = R(t)x + S(t).$$  \hspace{1cm} (12)$$

Required $W(x,t)$ and right part of the equation we will spread out abreast on eigenfunction of a problem on $x$ (that satisfies boundary conditions for $W(x,t)$) \[6\]:

$$W(x,t) = \sum_{n=1}^{\infty} Y_n(t)X_n(x),$$

where

$$Y_n^{''} t + 2a Y_n^{'} t + c^2 \lambda_n^2 Y_n t = C_n R(t) + D_n S(t).$$

Having substituted decomposition in the inhomogeneous equation and having equated coefficients of eigenfunction, we will receive

$$Y_n^{''} t + 2a Y_n^{'} t + c^2 \lambda_n^2 Y_n t = C_n R(t) + D_n S(t).$$

Let’s put, that the solution of this equation \[7\] is received (at $C_n = 0$, $D_n = 0$ takes place $Y_n(t) \equiv 0$.)

Received solutions $U x, t$ and $W x, t$ allow carrying out return transition by the speed

$$w x, t = -a_0 x + b_0 Q_0 t - a_1 x + b_1 Q_1 t +$$

$$+ \sum_{n=1}^{\infty} \begin{cases} 
  e^{-at} A_n ch \sqrt{D_n} t + B_n sh \sqrt{D_n} t & \text{when } D_n > 0 \\
  e^{-at} A_n + B_n t & \text{when } D_n = 0 \\
  e^{-at} A_n \cos \sqrt{D_n} t + B_n \sin \sqrt{D_n} t & \text{when } D_n < 0
\end{cases} + Y_n^{'} t X_n x. \hspace{1cm} (13)$$

Realization of entry conditions gives

$$A_n = -\gamma_n Y_n^{'} 0 + R C_n + S D_n,$$
$$B_n = \frac{1}{\gamma_n} \left[ a A_n - Y_n^{'} 0 + R C_n + S D_n \right],$$

where

$$\gamma_n = \begin{cases} 
  \sqrt{D_n} & \text{when } D_n > 0 \\
  1 & \text{when } D_n = 0 \\
  \sqrt{|D_n|} & \text{when } D_n < 0
\end{cases}$$

$$R_1 = a_0 Q_0 0 + a_1 Q_1 0 , S_1 = w_0 + b_0 Q_0 0 + b_1 Q_1 0 ,$$
$$R_2 = a_0 Q_0 0 + a_1 Q_1 0 , S_2 = b_0 Q_0 0 + b_1 Q_1 0 .$$

The problem is solved concerning the mass expense of a liquid.

Boundary conditions of a problem are difficult enough to work out the cut down telegraph equation, which has been constructed for speed, and boundary conditions corresponding to them. Therefore it is directly addressed to the equations of initial system, considering in it known value of speed stream $w x, t$ \[7\].

Integration of the second equation of system \[1\] gives

$$p x, t = p x, 0 - \rho c^2 \int_0^t \frac{\partial w x, \tau}{\partial x} d\tau. \hspace{1cm} (14)$$

Initial distribution of pressure on the pipeline, at value of pressure $p_{w0}$ for $x = 0$ and $t = 0$, is defined in the form

$$p x, 0 = p_{w0} - 2a_{w0} x + \rho g \sin \alpha x. \hspace{1cm} (15)$$

Let’s calculate a derivative $\frac{\partial w x, \tau}{\partial x}$ and we pass to integration of its parts. Believing that, members out of the
sum are integrated in a $$-a_0 \int_{0}^{t} Q_0 \, \tau \, d\tau - a_i \int_{0}^{t} Q_i \, \tau \, d\tau$$

$$T_0^0 \, t = \begin{cases} \frac{1}{c\lambda^2} & \left( -A_0 a - B_0 \sqrt{D:a} \, e^{-\alpha a} ch \sqrt{D:a} \, t - 1 \right) \\ + & A_0 \sqrt{D:a} - B_0 a \, e^{-\alpha a} sh \sqrt{D:a} \, t \right) \text{when } D:a > 0, \\
- \frac{1}{c\lambda^2} & \left( \frac{1 - e^{-\alpha a}}{a^2} \right) \left( -A_0 - B_0 \sqrt{D:a} \, e^{-\alpha a} cos \sqrt{D:a} \, t - 1 \right) \\ + & A_0 \sqrt{D:a} - B_0 a \, e^{-\alpha a} sin \sqrt{D:a} \, t \right) \text{when } D:a < 0. \end{cases}$$ (16)

At limiting process $$a \to 0$$, replacements are made

$$\frac{1 - e^{-\alpha a}}{a} \to t$$,  $$- \frac{te^{-\alpha a}}{a} + \frac{1 - e^{-\alpha a}}{a^2} \to t^2$$.

Consider that integrals $$Y_n^0 \, t = \int_{0}^{t} Y_n \, \tau \, d\tau$$ are calculated too.

As a result the solution of a problem concerning pressure becomes

\[
p \, x, \, 0 - a_0 \int_{0}^{t} Q_0 \, \tau \, d\tau - a_i \int_{0}^{t} Q_i \, \tau \, d\tau + \sum_{n=1}^{\infty} T_n^0 \, t + Y_n^0 \, t \bigg| \beta_0 \lambda_n \cos \lambda_n x - \alpha_n \lambda_n \sin \lambda_n x.
\] (17)

It is possible to note following features of the received solution.

On the given method, solving the problem with any initial distribution of speed on length of an area will not make difficulties.

5. Discussion

Entrance and target expenses $$Q_0 \, t$$ and $$Q_i \, t$$ should be coordinated at the task of boundary conditions within the limits of volumes $$V_0$$ and $$V_i$$ of receivers. Other variants result to rupture of environment (or solutions) at $$t \to \infty$$, and then it is necessary to be limited with small values of time $$t$$ [8].

Variants of indignations in conformity to frequencies of "thermal" waves $$D:a > 0$$, the «quasi-resonant» phenomenon $$D:a = 0$$ and consolidation waves $$D:a > 0$$ are separated. The solution works both for long $$D:a > 0$$, and for short pipelines.

Choosing parameters of boundary conditions in appropriate way, it is possible to use solutions for first form, we will enter designations of integrals under the sum

$$\alpha_0 = \alpha_i = 0$$, second $$\beta_0 = \beta_i = 0$$ and third types of boundary conditions [9].

The program of calculation for a case when at an input to the area, value of pressure remains constant: $$p(0, t) = p_{00}$$ (thus the first boundary condition becomes $$\frac{\partial u(0, t)}{\partial x} = 0$$).

Calculations were made according to data from [1]:

$$D = 0.200 \, m$$,  $$\lambda = 0.018$$,  $$c = 1200.0 \, m/s$$,

$$w_{h} = w_{i} = 12.0 \, m/s$$. The step on time has made $$t / (4c)$$.

The number of members of decomposition has made 500.

Results have shown that at the distances exceeding 10 km, the result represents typical curves of heat conductivity. Reflections of the wave formed by braking of a liquid in the end of the pipeline practically are not observed. For such lengths it is possible to use decisions of the equation without inertia of a liquid. With reduction of length of area, influence of inertial force becomes notable, and repeated reflections of waves of indignation are appeared (fig. 1).

At positive, zero and small negative biases of a line the greatest value of pressure is expected at achievement of a wave of indignation of the beginning of area. At considerably negative biases, in a post-skimming mode, the greatest values of pressure are observed in the bottom end of the area. In such cases the impulse of braking and potential energy of a liquid jointly can form substantial increase of pressure.

In the next figure, changes of pressure for a case are given, when without indignations the stream represents a current post-skimming mode of flow. The post-skimming mode is characterized by that potential energy of gravitation of a liquid on value exceeds volume of the work spent for overcoming of frictional force. Excess energy of a liquid is collected in the form of the raised pressure of a liquid. At the indignations of a stream caused by braking of a liquid, in a discussed case entrance pressure of a liquid is the least for all time pieces, which have calculated under the program. The greatest pressure as have already noted, is observed at the first reflection of a stream.

At braking of the stream, which is directed vertically top, it is observed that the greatest pressure at the moment of achievement of reflected wave the bottom part of the area. Connection of an air chamber results to reduction of this value. With increase in volume of air chamber $V_0 = 0.001, 0.01, 0.1, 1.0, 10^3$ the interval of pressure change decreases. In a following figure the graphs of the pressure, received by the calculation of braking of vertically top directed stream are shown [10].

The presented results of calculations concern only to one of nine variants of realization of boundary conditions for the received solution of a problem. As boundary conditions constant values of speed on borders of an area taking considering connection of an air chamber to its end have been accepted. Besides, on an input speed of a stream remained without change, and in the end of area it became zero. It is possible to realize other variants of functions $Q_0(t), Q_1(t)$. The solution can be applied also to calculation of cases of time change of entrance and target pressure of a liquid.

6. Conclusion

Thus, the received solution and the program, which were made on its basis allow to study dynamics of the stream formed by change of entrance and target expenses of a liquid in an inclined area of the pipeline and a fluid drive; to pick up volume of an air chamber; to reach a certain interval of pressure change in a dynamic mode.

References


