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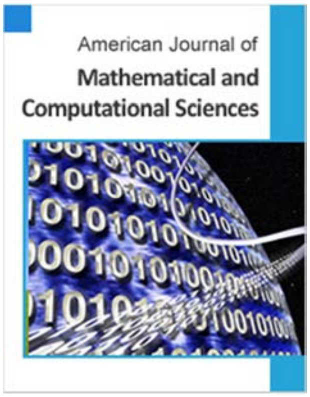

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Classifications of Some Groups with Length Functions

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Abstract
The concept of length functions on groups was first introduced by Lyndon [1]. Subsequently many other researchers worked in this field which resulted in adding further axioms in connection to the original axioms set by Lyndon. These new axioms additional axioms and conditions enabled the other researchers to achieve remarkable new results or to give direct proofs of many other cases. This procedure has given motivation to the researchers to enter new domains within the field of Combinatorial Group Theory.

1. Introduction

As a result of development in the construction of Length Functions in Groups, direct proofs of many other results in combinatorial group theory were established. Further work was done by many others such as, Cheswell [2], [3], [4], Hoare [5], [6], Wilkins [7], etc.

We start this paper by giving a presentation for groups generated by elements of length zero and one. We investigate the situation of applying Length Functions on various groups and their subgroups and decide if these results can be obtained and proved by using the length functions. We use our presentation to prove sufficient conditions for such a group to be a free, free product, amalgamated free product, H.N.N. extension or a quasi-H.N.N. extension. These groups were classified by using classical procedures, however, we have adapted length functions to obtain either similar results in much shorter procedures or to get new constructions.

2. Length Functions

Definition 2.1: A length function | | an a group G, is a function giving each element x of G a real number $|x|$, such that for all $x, y, z \in G$, the following axioms are satisfied.

$$A1'\ |e| = 0, e \text{ is the identity elements of G.} \tag{1}$$

$$A2\ |x^{-1}| = |x| \tag{2}$$

$$A4\ d(x,y) < d(y,z) \Longrightarrow d(x,y) = d(x,z), \text{ where} d(x,y) = \tfrac{1}{2}\left(|x| + |y| - |xy^{-1}| \right. \tag{3}$$

Lyndon [8] showed that $A4$ is equivalent to $d(x,y) \geq \min\{ d(y,z), d(x,z)\}$ and to

$$d(y,z), d(x,z) \geq m \Longrightarrow d(x,z) \geq m. \tag{4}$$

$$A1', A2 \text{ and } A4 \text{ imply } |x| \geq d(x,y) = d(y,x) \geq 0. \tag{5}$$

Assuming, $A2$ and $A4$ only, it is easy to show that:

$$d(x,y) \geq |e| \quad (6)$$

$$|x| \geq |e| \quad (7)$$

$$d(x,y) \leq |x| - \tfrac{1}{2}|e|, \quad (8)$$

The Axiom $A3$ states that $d(x,y) \geq 0$ is deductible from $A1', A2$ and $A1'$ is a weaker version of the axiom:

$$A1 |x| = 0 \text{ if and only if } x = 1 \text{ in G.} \quad (9)$$

The following propositions were introduced by Wilkens [7].

Definition 2.2 A non-trivial element g of a group G is called Non-Archimedean if $|g^2| \leq |g|$

Definition 2.3 Let G be a group with length function an element $x \neq 1$ in g is called Archimedean if

$$|x| \leq |x^2|. \quad (10)$$

The following Axioms and results have added by Lyndon and others

$$A0\; x \neq 1 \implies |x| < |x^2| \quad (11)$$

$$C0\; d(x,y) \text{ is always an integer.} \quad (12)$$

$$C1\; x \neq 1, |x^2| \leq |x| \; implies\; |x| \text{ is odd} \quad (13)$$

$$C2\; For\; no\; x\; is\; |x^2| = |x| + 1 \quad (14)$$

$$C3\; if\; |x|\; is\; odd\; then\; |x^2| \geq |x| \quad (15)$$

$$C1'\; if\; |x|\; is\; even\; and\; |x| \neq 0, then\; |x^2| > |x| \quad (16)$$

$$N0\; |x^2| \leq |x|\; implies\; x^2 = 1\; is\; x = x^{-1} \quad (17)$$

$$N1^*\; G\; is\; general\; by\; \{x \in G: |x| \leq 1\} \quad (18)$$

Definition 2.4 The set of all Non-Archimedean elements in G will be denoted by N, where

$$N = \{x \in G: |x^2| \leq |x|\} \quad (19)$$

Lyndon [1] also gave the following $M = \{xy \in G: |xy| + |yx| < 2|x| = 2|y|\}$, and showed that $M \subseteq N$. The nature of the elements of M and N will be investigated in the next section.

3. Presentation of Groups

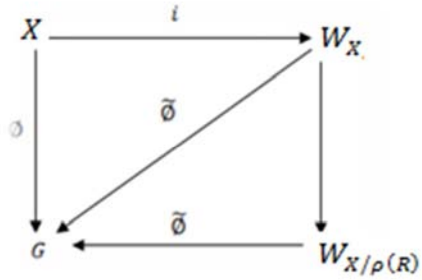
Let G be a group with length function $|\;\;|$ satisfying $N1^*, let\; X = \{x: |x| = 0,1\}, R = \{(x_i, x_j, x_k): |x_i x_j| = 0\; or\; 1\; and\; x_i x_j = x_k, where\; x_i, x_j, x_k \in X\}$ and let ρ(R) be the equivalence relation generated by R.

Since G is generated by elements of length zero and one, then every element $g \,\varepsilon\, G, |g| = n \geq 1$, can be expressed as $g = x_1, \dots, x_n$, where $|x_i x_{i+1}| = 2, for\; 1 \leq i \leq n-1$ and $|x_j| = 1$ for $1 \leq j \leq n$. That is $|g| = n, if\; n > 1$ and $|g| = |x_1|\; if\; g = x_1 \in X$

Let W_X be the set of words in X with usual binary operation. Let $\widetilde{\emptyset}: W_X \to G$ be defined by:

$$x_1, x_2, \dots, x_n \xrightarrow{\widetilde{\emptyset}} x_1 x_2 \dots x_n, \text{ for all x in X.}$$

$\widetilde{\emptyset}$ is the unique extension of the map $\emptyset: X \to G$, such that $\widetilde{\emptyset}(x) = \emptyset(x)$ and $\emptyset(x) = x \in G$, for all $x \;\varepsilon X$.



By $N1^*, \widetilde{\emptyset}(x)$ generates G and $\widetilde{\emptyset}(x_i, x_j) = \widetilde{\emptyset}(x_\ell)$, whenever $x_i x_j = x_\ell$, all $x_i, x_j, x_\ell \in X$.

Hence $\exists \widetilde{\emptyset}: w_x/\rho(R) \xrightarrow{onto} G$ given by: $[x_1, \dots, x_n]\rho \to x_1 \dots x_n$. Henceforth $\widetilde{\emptyset}$ will denote this map.

Lemma 3.1 $\widetilde{\emptyset}$ is one-to – one, for $\widetilde{\emptyset}(w) = 1 \Longrightarrow w =$ identity in $w_x/\rho(R)$, i.e. $[w]_\rho = [e]_\rho$ for $w \,\varepsilon\, W_x$ and e is the empty word in W_x.

Proof Suppose $\widetilde{\emptyset}([x_1, \dots, x_n]) = 1$ and suppose that $x_1, \dots, x_n$ is not p-equivalent to a shorter sequence.

$\widetilde{\emptyset}([w]) = \widetilde{\emptyset}([x_1, \dots, x_n]) = x_1 x_2 \dots x_n = 1\; in\; G$. If n = 1, then $\widetilde{\emptyset}([x_1]) = x_1 = 1$.

Suppose n>1, then $x_i x_{i+1} = x_j$ for some $i, 1 \leq i \leq n-1$ where $|x_j| = 0,1$ otherwise $|x_1, \dots, x_n| = n > 1$, so a contradiction to $\widetilde{\emptyset}([w]) = 1$ and $x_1, \dots, x_n \neq e$

Thus w contains a subword of the form $x_i x_{i+1}$, where $|x_i x_{i+1}| = 0\; or\; 1$, and $x_i x_{i+1} = x_j$, a contradiction to the assumption. Thus $n \ngtr 1$.

Hence $[w]_p = [e]_p$, so $\widetilde{\emptyset}$ is one – to –one.

Therefore $G =< X/R >= w_x/p\,(R)$ is a presentation for G. (20)

We now use this result to give direct proofs for some embedding theorems, which have been proved by Lyndon [8], Chiswell [3], [4] and Khanfar [9].

Theorem 3.1 The group G is free, if G has a length function $|\;\;|$ satisfying C0, A0 and $N1^*$.

Proof Since $|\;\;|$ satisfies $N1^*$, then G has a presentation $G \cong < X|R >$, where $X = \{x: |x| = 0,1\}$ and $R = \{(x_i, x_j, x_k): |x_i x_j| = 0{,}1 \text{ and } x_i x_j = x_k, \text{where } x_i, x_j\; and\; x_k \varepsilon X\}$

Let $(x_i, x_j, x_k) \in R$, then $|x_i x_j| = 0$ or 1 and $x_i x_j = x_k$ and $2d\,(x_i, x_j^{-1}) = |x_i| + |x_j| - |x_i x_j|$.

If $|x_i| + |x_j| = 1$, then by C0, $|x_i x_j| = |x_k| = 0$.

If $x_k \neq e$ then by A0, $|x_k| < |x_k^2|$, but $|x_k| = 0$ and also $|x_k^2| = 0$, so a contradiction.

Therefore $x_i = x_j^{-1}$, and (x_i, x_j, x_k) is a relation of the from $(x.x^{-1}, e)$ in R.

If $|x_i| = 1$, $|x_j| = 0$, then by C0, $|x_i x_j| = 1$, and as above $x_j = e \in G$.

Thus (x_i, x_j, x_k) is of from $(x.e, x)$.

Similarly for $|x_i| = 0$, $|x_j| = 1$ and the corresponding relation $(e.x, x)$ in R.

If $|x_i| = |x_j| = 0$, then by A0, $x_i = x_j = e = x_k$ and (x_i, x_j, x_k) is of the from (e.e,e).

Therefore G is free.

Theorem 3.2 If G has a length function | | satisfying A1, A2, A4, C2 and $N1^*$, then G is a free product of groups and the length function is the same as that derived from the free product decomposition.

Proof Let $G = < X; R >$ as in (1). By A1, $|x_i| = 1$ for all $x_i \in X \setminus e$.

Therefore G is generated by elements of length one.

Define $x_i \sim x_j$ if and only if $|x_i x_j^{-1}| \leq 1$, for all $x_i, x_j \in X \setminus e$.

(1) $|x_i x_j^{-1}| = 0, so\ x_i \sim x_i$

(2) Suppose $x_i \sim x_j$, i.e. $|x_i x_j^{-1}| \leq 1$, then by A2, $|(x_i x_j^{-1})^{-1}| = |x_i x_j^{-1}| \leq 1$,so $x_j \sim x_i$.

(3) Suppose $x_i \sim x_j$ and $x_j \sim x_k$, i.e. $|x_i x_j^{-1}| \leq 1$ and $|x_j x_k^{-1}| \leq 1$; then $2d(x_i, x_j) = |x_i| + |x_j| - |x_i x_j^{-1}|$ $= 1 + 1 - |x_i x_j^{-1}|, so 2d(x_i, x_j) \geq 1$, hence$d(x_i, x_j) \geq \frac{1}{2}$

Similarly $d(x_i, x_k) \geq \frac{1}{2}$

By A4,$d(x_i, x_k) \geq \frac{1}{2}$

$$2d(x_i, x_k) = 1 + 1 - |x_i x_k^{-1}| \geq 1 \implies |x_i x_k^{-1}| \leq 1 \implies x_i \sim x_k.$$

Hence $\sim$ is an equivalence relation on elements of $G \setminus e$. This can also be proved by observing that by C2, elements of length 1 are non-Archimedean, so $\sim$ is an equivalence relation.

Put $[x_i] = \{x_j : x_i \sim x_j, x_j \in X\}$

$[x_i]$is equivalence class containingx_i.

Therefore $x = \cup \{[x_i]: x_i\ \varepsilon\ X\}, U\{e\}$

Now suppose $|x_i| = 1$. By $C2\ |x_i^2| \neq |x_i| + 1 i.e. |x_i^2| \neq 2$

Then $|x_i^2| \leq 1$,i.e. $x_i \sim x_i^{-1}$ for any $x_j\ \varepsilon\ X$.

Now suppose $(x_i, x_j, x_k) \in R$, then $|x_i x_j| = |x_k| \leq 1$, then $x_i \sim x_j^{-1} \sim x_j$, i.e. x_i, x_j are in the class; moreover $|x_i x_j^{-1}| = |x_i x_j x_j^{-1}| = |x_i| \leq 1$, then x_i, x_j, x_k are in the same class, i.e. $x_i \sim x_j \sim x_k$.

On the other hand, if x_i, x_j are in the same class, then$x_i \sim x_j \sim x_j^{-1}$, and $|x_i x_j| \leq 1$.

Therefore, $|x_i x_j| \leq 1$, if and only if x_i, x_j and $x_i x_j$ are in the same class, i.e. $(x_i, x_j, x_k) \in R$ if and only if x_i, x_j, x_k are in the same class, and $x_i x_j = x_k$.

Let R_i br the set of elements of R, which are defined on the elements of the class $[x_i]$.

Then $G = < [x_1] \cup [x_2] \cup \ldots |R_1, R_2, \ldots >$

Therefore G is a free product of the groups $< [x_i]|R_i >, < [x_2]|R_2 >, \ldots$ (21)

The group G in the following theorem is a free product of groups with amalgamation as defined in [8] and [10].

Theorem 3.3 If G a length function satisfying C2 and $N1^*$, then G is a free product with amalgamation and the length function is the same as that derived from the amalgamated free product decomposition.

Proof since | | satisfies $N1^*$, then G has a presentation $G \cong \langle X; R \rangle$, where $x = \{x: |x| = 0,1\}$ and $R = \{(x_i.x_j, x_k): |x_i x_j|\} = 0,1$ and $x_i x_j = x_k$, where $x_i, x_j, x_k \in X$.

Let $A = \{x \in X: |x| = 0\}, then\ A \leq G$.

Define $x_i \sim x_j \Leftrightarrow |x_i x_j^{-1}| \leq 1$, for $x_i, x_j \in X \setminus A$.

$\sim$is an equivalence relation on $X \setminus A$

Denote the classes of $X \setminus A$ by $[x_1], [x_2] \ldots$.

Put $X \setminus A = U\{[x_i]: x_i \in X\}$, also by the proof of theorem 4.2, $|x_i x_j| = 2 \Leftrightarrow x_i$ and x_j are in different classes, and $|x_i x_j| \leq 1$, whenever x_i, x_j and $x_i x_j$ are in the same class, also $x_i \sim x_i^{-1}$.

Let $X_i = A\ U\ \{[x_i]: x_i \in [x_i]\}$ and let R_i be defined by $(x_i.x_j, x_k) \in R_i \Leftrightarrow |x_\ell x_j| = |x_k| = 0$ and $x_\ell, x_j \in A\ or\ x_\ell = x_j^{-1}$, or $|x_\ell x_j| = |x_k| = 1$ and at least one of $x_\ell$or $x_j \in X_i \setminus A$, where $x_k = x_\ell x_j$.

It is clear from definition of the equivalence relation that X_i is a subgroup of G containing A, and moreover that R_i is the complete multiplication table for X_i . Therefore if $G_i \cong \langle x_i | R_i \rangle$ we have A is a subgroup of G_i.

Consider ${}^*_A G_i$ - free product of the group G_i amalgamating the subgroup A.

Define $\emptyset$: ${}^*_A G_i \to G$ by $\emptyset(x_1, \ldots, x_n) = x_1 x_2 \ldots x_n = 1$, for ≥ 1.

If n = 1, then $\emptyset\ (x_1) = x_1 = 1$.

Suppose n > 1 then, there exists some I for which $|x_i x_{i+1}| \leq 1$ in the product $x_1, \ldots, x_n$, i.e. $x_i, x_{i+1}^{-1}$are in the same class.

Then $x_i x_{i+1}$ are in the same factor, contradiction to the normal from. Hence $\emptyset$ is one – to –one

Since X generates G, then $\emptyset$ is onto. Therefore G is a free product with amalgamation.

The group G in the following theorem is an HNN extension as defined in [9]

Theorem 3.4 Let G be a group with length function | | satisfying C0, C3 and $N1^*$. Then G is an H.N.N. extension, with length function.

Proof Since G has a length function satisfying $N1^*$, then G can be presented by:

$G_i \cong \langle x_i | R_i \rangle$ where $x = \{x: |x| = 0,1\}$ and $R = \{(x_i.x_j, x_k): |x_i x_j| = 0,1 and x_i x_j = x_k, where\ x_i, x_j, x_k \in X\}$.

Let $A = \{x \in X: |x| = 0\}$ and $B = \{x \in X: |x| = 1\}$. Then $A \leq G$.

Consider the double cosets A x A,$|x| = 1$, of A in G, then B is the union of these double cosets (for if $b \in B, then\ A\ b\ A \subseteq B$).

Let I be an index set of these double cosets and choose a representative $x_i, i\ \varepsilon\ I$ for each double coset. By C3 we have $A \times A \neq A\, x^{-1}\, A\ for\ x\ \varepsilon\ A$.

So we choose the double coset representative x_i such that x_i^{-1} is also a double coset representative.

Suppose $x_i \in B$ such that $x_i^2 \in A$, then $|x_i^2| = 0$, a contradiction to C3.

Hence $x_i^2 \notin A$ for all $i \varepsilon I$.

For each $i \varepsilon I$ let $A_i = \{a \varepsilon A: |x_i^{-1}\, a\, x_i| = 0\}$ N.B. $|x_i^{-1}\, a\, x_i| = 0$ or 2 by C0.

Suppose $a, a' \varepsilon A, then\ |x_i^{-1}\, a\, x_i| = 0\ and\ |x_i^{-1} a' x_i| = 0$, so also

$\left|x_i^{-1} {a'}^{-1} x_i\right| = 0, and \left|x_i^{-1}\, a\, {a'}^{-1} x_i\right| = 0$, so $A_i \le A$.

Let $A_i' = \{x_i^{-1}\, a\, x_i : a \in A_i\} \le A$

Define $\emptyset_i: A_i \to A_i'$, by $\emptyset_i(a) = x_i^{-1}\, a\, x_i, \forall\, a \in A_i$

Then $\emptyset_i$ is isomorphism ($A_i \cong A_i'$).

Thus, for each $x_i, i\, \varepsilon\, I, \exists\, A_i \le A$ such that $x_i^{-1} A_i x_i = A_i'$ and $A_i' \le A$.

If, $b \in A$, then $|ab| = 0, ab = c \in A$.

Let $G^* = < A, x_i;\ x_i^{-1}. a_i. x_i = \emptyset_i(a_i), a_i\, \varepsilon\, A_i, i\, \varepsilon\, I >$.

Let $\lambda: X \to G^*$ be given by

$\lambda: x \to a. x_1. b\ in\ G^*$ where $a, b\, \varepsilon\, A, x_i$ is a doublecoset representative, and $x = a\, x_i\, b\ in\ G, x \notin A$.

$\lambda: x \to x$, for $x \in A$.

We show that λ define a homomorphism of G into G^*

Suppose $(x. y, z) \in R$ then $|xy| = |z| \le 1$ and $xy = z$.

Case 1 If $|xy| = 0$ then either

(i) $x, y \in A$ and $xy = z \in A$, hence $\lambda(x)\, \lambda(y) = xy = \lambda(z)$ in G^*

or

(ii) $|x| = |y| = 1$ then let $x = a_1 x_k b_1$, $y = a_2\, x_\ell\, b_2$ in G where x_k, x_ℓ are double coset representatives, and $a_1, b_1, a_2, b_2 \in A$.

$|a_1 x_k b_1 a_2 x_\ell b_2| = 0 \implies x_k$ and x_ℓ^{-1} are in the same double coset, therefore by choice of double of double coset representatives $x_k = x_\ell^{-1}$.

$x_\ell^{-1}(b_1 a_2) x_\ell = a_1^{-1}\, z\, b_2^{-1} = \emptyset_\ell(b_1 a_2) \in A_\ell'$ by the definition of A_ℓ'.

Since $x_\ell^{-1}. b_1 a_2. x_\ell = a_1^{-1}\, z\, b_2^{-1}\ in\ G^*$, therefore

$$\lambda(x)\, \lambda(y) = a_1. x_\ell^{-1}. b_1. a_2. x_\ell\ . b_2\ in\ G^*$$

$$= a_1. x_\ell^{-1}. (b_1 a_2). x_\ell\ . b_2\ in\ G^*$$

$$= a_1. \emptyset_\ell\ (b_1 a_2). b_2\ in\ G^*$$

$$= a_1 \emptyset_\ell\ (b_1 a_2) b_2\ in\ G^*$$

$$= z = \lambda(z)$$

Case2 If $|xy| = 1$, then by C0 either $|x|$ or $|y| = 0, and\ |z| = 1$ where xy=z.

Assume that $|x| = 0$.

Let $x = a, y = a_1 x_k b_1, z = a a_1 x_k b_1$

$$\lambda(x)\, \lambda(y) = a. a_1. x_k. b_1\ in\ G^*$$

$$= a a_1. x_k. b_1\ in\ G^*$$

$$= \emptyset(z)$$

Therefore λ is a homomorphism.

Similarly, let $\psi: G^* \to G$ defined by

$$\psi(a) = a\ in\ G, and\ \psi(x_i) = x_i\ in\ G.$$

It can easily be show that $\psi(a. b) = \psi(a)\, \psi(b)$ and $\psi\, (x_i^{-1}. a_i. x_i) = \psi(x_i)^{-1}\, \psi\, (a_i)\, \psi(x_i) = \psi\big(\emptyset_i(a_i)\big)$.

Clearly $\psi\ o\ \lambda$ and $\lambda\ o\psi$ are the identity maps on G and G^* respectively so G has the presentation above. Moreover λ and ψ preserve the length.

The group G in the following theorem is a Quasi- HNN extension as defined in [9] and [11].

Theorem 3.5 If G is group with length function | | satisfying C0 and NI^*, then G is a quasi-H.N.N. extension.

Proof Once again G can be presented by:

$G \cong \langle X; R\rangle$ where $X = \{x: |x| = 0{,}1\}$ and $R = \{(x_i. x_j. x_k): |x_i x_j| = 0\ or\ 1\}$ and $x_i x_j = x_k$ where $x_i, x_j, x_k \in X$.

Let $A = \{x\, \varepsilon\, X: |x| = 0\}$and $B = \{x\, \varepsilon\, X: |x| = 1\}$. Then A is a subgroup of G.

Now consider the double casets $A \times A$ of A in G where $|x| = 1$, then B is a union of double cosets.

Let I be an index set of those double cosets $A\, x\, A \ne A\, x^{-1}\, A$, and let J be an index set of the double cosets for which $A \times A = A\, x^{-1}\, A$.

Let $\{x_i: i \in I\}$, be a set of double coset representative for the coset $A\, x\, A$ for which $A\, xA \ne A\, x^{-1}\, A$, chosen as before so that x_i^{-1} is a double coset representative if x_i is.

For each $i \in I$ let $A_i = \{a \in A: |x_i^{-1}\, a\, x_i| = 0\}$. Then A_i is a subgroup of A for each j.

Let $A_i' = \{x_i^{-1}\, a\, x_i : a \in A_i\} \le A. i\, \varepsilon\, I$.

Define $\emptyset_i(a): A_i \to A_i'$ by:

$$\emptyset_i(a) = x_i^{-1}\, a\, x_i, \forall\, a \in A_i.$$

$\emptyset$ is an isomorphism ($A_i \cong A_i'$).

Therefore, for each $x_i, i\, \varepsilon\, I, \exists\, A_i \le A$, such that $x_i^{-1} A_i\, x_i = A_i', A_i' \le A$.

For each $\in J$, define $\alpha_j: A_j \to A_j$ by $\alpha_j(a_j) = x_j a_j\, x_j^{-1}$, by the definition of $A_j, \alpha_j(a_j) \in A_j$ where $a_j\, \varepsilon\, A_j, x_j \notin A_j$ and $x_j^2\, \varepsilon\, A_j$.

Thus for each $x_j, j\, \varepsilon\, J, \exists\, A_j \le A$, and an automorphism $\alpha_j: A_j \to A_j$ of order t most two such that the inner-automoorphism α_j^2 is determined by $x_j^2 \in A_j$ fixed by α_j.

Let $G^* = < A, X_i, X_j | rel\ A, x_j, a_j,\ x_j^{-1} = \alpha_j(a_j), a_j \in A_j,\ x_j^2 = a_j' \in A_j$ for all $j \in J$, $x_j^{-1} a_i x_i = \emptyset_i(a_i)$, for all $a_i \in A_i, i\, \varepsilon\, I >$.

Define $\lambda: X \to G^*$ by $\lambda(x) = a. x_k. b$ in G^* where $a, b \in A$ $x_k$is any double coset representative and $x = a. x_k. b \notin A$ and $\lambda(x) = x$for $x \in A$.

As in the previous theorem it is easily show that λ defines a homomorphism from G into G^*.

It can also be show that if we define $\psi: G^* \to G$ by:

$\psi(a) = a\ in\ G\ and\ \psi(x) = x\ in\ G$ where $x \in \{x_i: i \in I\} \cup \{x_j: j \in J\}$. Then ψ is a homomorphism, Moreover as before $\psi\ o\ \lambda$ and $\lambda\ o\psi$ are identity maps on G and G^* respectively, and ψ and λ preserve the length.

4. Conclusion

This paper proved that groups with length functions are classified as follows:

(1) The group is free if it satisfies C0, A0 and $N1^*$.
(2) The group is free product of its components if it satisfies A1, A2, A4, C2 and $N1^*$.
(3) The group is Free product with amalgamation if it satisfies C2 and $N1^*$.
(4) The group is an HNN extension if it satisfies C0, C3, $N1^*$.
(5) The group is a Quasi-HNN extension if it satisfies C0 and $N1^*$.

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