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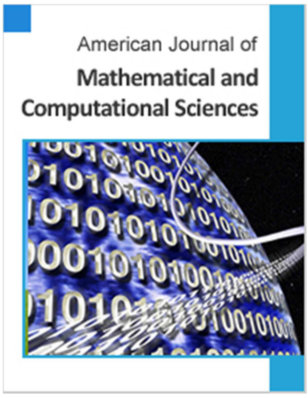

Quasi-HNN Groups and Length Functions

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Abstract
The concept of length functions on groups was first introduced by Lyndon [1]. This was used to give direct proofs of many other results in combinatorial group theory. Further work was done by many others such as, Chiswell [2], [3], Hoare [4], [5], Wilkins [6], etc. The aim of the paper is to investigate the nature of some particular elements of the Quasi-HNN groups, namely the Archimedean elements N and M which are introduced in chapter two. Length functions are used to prove the connection between the elements of the Quasi-HNN group and to achieve certain objectives, such as M is a subset of N and identify the conjugates of each set.

1. Introduction

In this paper we look at a construction given by G. Higman, B. H. Neumann and H. Neumann in 1949. This construction is called HNN extension which was generalized by Khanfar [7] and called Quasi-HNN extension. Subsequently, this was also studied by Meier [8].

We define a length function on Quasi-HNN extensions to get some further results concerning the structure of Quasi-HNN extensions, factor groups and some predefined important parts of this group. However, we have to formulate a normal form theorem for Quasi-HNN extensions and consider reduced forms of the elements of this group.

Two important sets called M and N satisfying some certain axioms of length functions were introduced by Lyndon [1].

The last section of this paper investigates the nature and the structures of the sets M and N in relation to the elements of Quasi-HNN group.

2. Length functions

Definition 2.1: A length function | | an a group G, is a function giving each element x of G a real number $|x|$, such that for all $x, y, z \in G$, the following axioms are satisfied

$A1'$ $|e| = 0, e$ is the identity elements of G.

$A2$ $|x^{-1}| = |x|$

$A4$ $d(x,y) < d(y,z) \Longrightarrow d(x,y) = d(x,z)$, where $d(x,y) = \frac{1}{2}(|x| + |y| - |xy^{-1}|$

Lyndon [1] showed that $A4$ is equivalent to $d(x,y) \geq \min\{d(y,z), d(x,z)\}$ and to

$$d(y,z), d(x,z) \geq m \Longrightarrow d(x,z) \geq m.$$

$A1', A2$ and $A4$ imply $|x| \geq d(x,y) = d(y,x) \geq 0$.

Assuming, $A2$ and $A4$ only, it is easy to show that:

i. $d(x,y) \geq |e|$

ii. $|x| \geq |e|$

iii. $d(x,y) \leq |x| - \frac{1}{2}|e|$, see [7].

The Axiom A3 states that $d(x,y) \geq 0$. This is deductible from $A1', A2$ and $A1'$ which is a weaker version of the axiom $A1$: $|x| = 0$ if and only if $x = 1$ in G.

The following propositions were introduced by Lyndon [1].

Proposition 2.1 $d(xy,y) + d(x,y^{-1}) = |y|$

Proposition 2.2 $d(x,y^{-1}) + d(y,z^{-1}) \leq |y|$ implies that $|x\,y\,z| \leq |x| - |y| + |z|$

Proposition 2.3 $d(x,y^{-1}) + d(y,z^{-1}) \leq |y|$ Implies$d(xy,z^{-1}) = d(y,z^{-1})$

Proposition 2.4 $d(x,y) + d(x^{-1},y^{-1}) \geq |x| = |y|$ Implies $|(xy^{-1})^2| \leq |xy^{-1}|)$

Proposition 2.1 implies that, for any $x, y \in G, d\,(x,y) = |y| - d\,(x\,y^{-1}, y^{-1}) \leq |y|$ by A3.

Since $d(x,y) = d\,(y,x)$, we get $d\,(x,y) \leq \min\{|x|,|y|\}$,

Axiom A5 states that: $d\,(x,y) + d\,(x^{-1},y^{-1}) > |x| = y \Rightarrow x = y$

Definition 2.1 A non-trivial element g of a group G is called Non-Archimedean if $|g^2| \leq |g|$

Definition 2.2 Let G be a group with length function an element $x \neq 1$ in g is called Archimedean if $|x| \leq |x^2|$.

The following Axioms and results have added by Lyndon and others

$$A0\; x \neq 1 \implies |x| < |x^2|$$

$C0\; d(x,y\,)$ is always an integer

$C1\; x \neq 1, |x^2| \leq |x|\; implies\; |x|$ is odd

$$C2\; For\; no\; x\; is\; |x^2| = |x| + 1$$

$$C3\; if\; |x|\; is\; odd\; then\; |x^2| \geq |x|$$

$$C1'\; if\; |x|\; is\; even\; and\; |x| \neq 0, then\; |x^2| > |x|$$

$$N0\; |x^2| \leq |x|\; implies\; x^2 = 1\; is\; x = x^{-1}$$

$$N1^*\; G\; is\; general\; by\; \{x \in G: |x| \leq 1\}$$

Definition 2.3 The set of all non Archimedean elements is G will be denoted by N, is $N = \{x \in G: |x^2| \leq |x|\}$

Lyndon [1] also gave the following $M = \{xy \in G: |xy| + |yx| < 2|x| = 2\,|y|\}$, and showed that $M \subseteq N$.

The nature of the elements of M and N will be investigated in the next section.

3. HNN Extension

We now introduce an important group constructed by G.Higman, B.H. Neumann and H. Neumann.

Definition 3.1 Let G be a group and let I be an index set let $\{A_i : i \in I\}$ and $\{B_i : i \in I\}$ be families of subgroup of G and$\{\emptyset_i : i \in I\}$ be a family of maps such that, each $\emptyset_i : A_i \to B_i$ be an isomorphism. Then the H.N.N extension with base $G_1$and stable lettes $t_i, i \in I$ and associated subgroups A_i and $B_i, i \in I$ is the group.

$G^* = < G, t_i; rel\; G, t_i^{-1} a_i t_i = \emptyset_i(a_i), a_i \in A_i >$, where $\langle G, rel\; G\rangle$ is a presentation of G.

To formulate a normal from theorem for H.N.N extensions, we shall consider the following:

Any element of G^* is equal to a product $g_0\, t_{i_1}^{\varepsilon_1} g_1 \ldots\; t_{i_n}^{\varepsilon_n} g_n, n \geq 0\; \varepsilon_i = \pm 1$

Note: Throughout this section g_i will denote an elements of G.

Definition 3.2 A sequence $g_0\, t_{i_1}^{\varepsilon_1} g_1 \ldots\; t_{i_n}^{\varepsilon_n} g_n, n \geq 0\; \varepsilon_i = \pm 1$ is said to be reduced if there is no consecutive subsequence $t_i^{-1} g_i\, t_i$ with $g_i\; \varepsilon\; A_i, or\; t_i g_i t_i^{-1}$ with $g_i \in B_i$ if w is a word in $G \cup \{t_i\} \cup \{t_i^{-1}\}$. Then we can get $t_i - reduction$ of w corresponding to the relations of G^* as follows:

1) Replace a subword of the from $t_i^{-1} g_i\, t_i$, by $\emptyset_i(g_i)$ whenever $g_i \in A_i$
2) Replace a subword of the from $t_i\; g_i\, t_i$, by $\emptyset_i(g_i)$ whenever $g_i \in\; B_i$

By consolidating and making all possible $t_i - reduction$ we get a reduced word defining the same element of G^* The products of the elements in two distinct reduced sequences may be equal in G^*. To get normal forms, once again we consider the coset representatives as follows:

Choose for each i a set of representatives of the right cosets of $A_i\; in\; G$ and a set of representatives of the right cosets of $B_i\; in\; G$. We shall assume that 1 is the representative of both cosets A_i and B_i

Definition 3.3 Given the sets of right coset representatives of A_i and B_i in G, then a normal form in G^* is a sequence of the form $g_0\, t_{i_1}^{\varepsilon_1} g_1 \ldots\; t_{i_n}^{\varepsilon_n} g_n, n \geq 0\; \varepsilon_r = \pm 1$, where

i) g_0 is an arbitrary element of G, except that $g_0 \neq 1$ if n=0
ii) $if\; \varepsilon_r = -1$ then g_r is a representative of a coset of $A_{i_r}\; in\; G$
iii) $if\; \varepsilon_r = +1$ then g_r is a representative of a coset of $B_{i_r}\; in\; G$ and
iv) There is no subsequence $t^{\varepsilon}\; 1\; t^{-\varepsilon}$ where $\varepsilon = \pm 1$

Because of the relations $t_i^{-1} a_i t_i = \emptyset_i(a_i)$ of G^*, we can replace $t_i^{-1} a_i$ by $\emptyset_i(a_i) t_i^{-1}$ without changing the corresponding element of G Similarly we can replace $t_i b_i$ by $\emptyset\; _i^{-1}(b_i) t_i^{-1}$, by working from right to left, we can show that every element of G^* is equal to a product $g_0\, t_{i_1}^{\varepsilon_1}\; \ldots\; t_{i_n}^{\varepsilon_n} g_n$where $g_0\, t_{i_1}^{\varepsilon_1}\; \ldots\; t_{i_n}^{\varepsilon_n} g_n$is a normal form.

Theorem 3.1 (Normal Form Theorem)

Let $G^* = < G, t_i; rel\; G, t_i^{-1} a_i t_i = \emptyset_i(a_i), a_i\; \varepsilon\; A_i, i\; \varepsilon\; I >$ be can H.N.N. extension then every element of G^* has a unique representation as a product $g_0\, t_{i_1}^{\varepsilon_1}\; \ldots\; t_{i_n}^{\varepsilon_n} g_n$ where $g_0\, t_{i_1}^{\varepsilon_1}\; \ldots\; t_{i_n}^{\varepsilon_n} g_n$is a normal form.

Proof See [9].

Theorem 3.2 (Higman, Neumann, Neumann)

Let $G^* = < G, t_i; rel\; G, t_i^{-1} a_i t_i = \emptyset_i(a_i), a_i\; \varepsilon\; A_i, i\; \varepsilon\; I >$ be can H.N.N. extension, then the group G is embedded in G^* by the map; $g \to g$.

Theorem 3.3 (Britton's Lemma) If $g_0\, t_{i_1}^{\varepsilon_1}\; \ldots\; t_{i_n}^{\varepsilon_n} g_n\; = 1$ in G^* where ≥ 1, then $g_0, t_{i_1}^{\varepsilon_1}\; \ldots\; t_{i_n}^{\varepsilon_n}, g_n$is not reduced.

Theorems 3.2 and 3.3 are equivalent to theorem 3.4 (proofs are in [6] and [10]).

Lemma 3.1 Let $G^* = < G, t_i; rel\ G, t_i^{-1} a_i t_i = \emptyset_i(a_i), a_i \in A_i, i \in I >$ be can H.N.N. extension. let $u = g_0\, t_{i_1}^{\varepsilon_1} \ldots t_{i_n}^{\varepsilon_n}\, g_n$ and $v = h_0\, t_{i_1}^{\delta_1} \ldots t_{i_m}^{\varepsilon_m}\, h_m$ be reduced words, and suppose that u = v in G^*. Then $m = n$ and $\varepsilon_i = \delta_i, i = 1, \ldots, n$.

Proof Since u = v in G^*, then

$1 = g_0\, t_{i_1}^{\varepsilon_1} \ldots t_{i_n}^{\varepsilon_n} g_n h_m^{-1} t^{-\delta_m} \ldots t^{-\delta_1} h_0^{-1}$

Since u and v are reduced, the only way the indicated sequence can fail to be reduced is that $\varepsilon_n = \delta_m$ and $g_n h_m^{-1}$ is in the appropriate sub-group A_i or B_i making successive t-reductions we see that each $\varepsilon_i = \delta_i$ and m = n.

The normal form theorem 2.4 for H.N.N. extension allows us to assign a well-defined length to each element of these extensions.

Definition 3.4 Let $G^* = < G, t_i; rel\ G, t_i^{-1} a_i t_i = \emptyset_i(a_i), a_i \in A_i, i \in I >$ be can H.N.N. extension, define the length of an element gG^* by:

$|g| = n, if\ g = g_0\, t_{i_1}^{\varepsilon_1} \ldots t_{i_n}^{\varepsilon_n}\, g_n, n \geq 0$ is in a reduced form, where $\varepsilon_1 = \pm 1$

Theorem 3.4 Let $G^* = < G, t_i; rel\ G, t_i^{-1} a_i t_i = \emptyset_i(a_i), a_i \in A_i, i \in I >$ be can H.N.N. extension and let $|g| = n, if\ g = g_0\, t_{i_1}^{\varepsilon_1} \ldots t_{i_n}^{\varepsilon_n}\, g_n, n \geq 0$ is in a reduced form, where $\varepsilon_1 = \pm 1$. Then | | is a length function on G^*.

It is proved in [1], that d (g,h) is always an integer, i.e C0 is satisfied in H.N.N. extensions.

The following two theorems are proved in [10].

Theorem 3.15 The elements of N are the conjugates of the elements of the base G, and are equivalent if and only if they are conjugates by the same elements of G^*.

Theorem 3.6 Let $G^* = < G, t_i; rel\ G, t_i^{-1} a_i t_i = \emptyset_i(a_i), a_i\ \varepsilon\ A_i, i\ \varepsilon\ I >$ be can H.N.N. extension. Then the elements of M are the conjugates of the elements of the associated subgroups.

4. Quasi-H.N.N Group

We introduce a construction given by Khanfar [7] called a quasi- H.N.N extension. We shall also consider a more general construction and use some results from [7] to give a normal from theorem. We will then be in a position to define a length function on quasi-H.N.N extensions in general.

Definition 4.1 Let G be a group for an index set I, let $\{A_i : i \in I\}$ be a family of subgroups of G. For each i, let $a_i : A_i \rightarrow A_i$ be an auto orphism of order 2, such that the inner auto orphism a_i^2 is determined by a_i', for some $a_i'\ \varepsilon\ A_i$ fixed by a_i. Then the quasi- H.N.N extension is given by:

$$G^* =< G, t_i | G, t_i^{-1} a_i t_i = a_i(a_i), t_i^2 = a_i'\ \varepsilon\ A_i, i \in I >$$

The group G is called the base of G^*, t_i are called the stable letters and $(A_i, A_i^{a_i})$ are called the associated pairs of subgroups. Khanfar [7] showed that G is embedded in G^*. He also considered a general situation given as follows:

Definition 4.2 Let G be a group containing three collections of subgroups A_i, B_i, for $i \in I$ and C_j, for $j \in J$, for each i, let $\emptyset_i : A_i \rightarrow B_i$ be an isomorphism. For each j, let $\alpha_j : C_j \rightarrow C_j$ be an automorphism of order 2, such that the inner automorphism α_j^2 is determined by c_j' fixed by α_j. Then the quasi- H.N.N extension is defined by

$$G^* =< G, t_i, t_j | rel\ G, t_i^{-1} a_i t_i = \emptyset_i(a_i), t_j^{-1} c_j t_j = a_j(c_j), t_j^2 = c_j', a_i \in A_i, c_j\ \varepsilon\ C_j, i\ \varepsilon\ I, j \in J >$$

If w is a word in the generators of G^* given in definition 2 then w can be written as

$w = g_0 t_{i_1}^{\varepsilon_1} g_1 \ldots t_{i_n}^{\varepsilon_n} g_n, n > 0\ \varepsilon_i = \pm 1$ and t_{i_k} is either in $\{t_i : i \in I\}$, or in $\{t_j : j \in J\}$

Throughout this paper g_i will denote an element of G.

Definition 4.3 A sequence $g_0 t_{i_1}^{\varepsilon_1} g_1 \ldots t_{i_n}^{\varepsilon_n} g_n, n > 0\ \varepsilon_i = \pm 1$ is said to be reduced if it contains no subword of the forms:

1) $t_i^{-1} a_i t_i, a_i \in A$ for some $i \in I$
2) $t_i b_i t_i^{-1}, b_i \in B_i$ for some $i \in I$
3) $t_j^{\varepsilon} c_j t_j^{\delta}, c_j \in C_j$ for $j \in J$, and $\varepsilon, \delta = \pm 1$
4) t_i^{-1} for $j \in J$
5) $g_0 \neq 1$ if n = 0

If w is a word $U\, t_i^{\pm 1}\, U\, t_j^{\pm 1}$, then we can get t_i, t_j - reduction of w corresponding to the relations of G^* as follows:

1) Replace a subword of the from $t_i^{-1} g_i\, t_i$ by $\emptyset_i(g_i)$ whenever $g_i \in A_i$
2) Replace a subword of the from $t_i g_i\, t_i^{-1}$ by $\emptyset_i^{-1}(g_i)$ whenever $g_i \in B_i$
3) Replace a subword of the from $t_j^{-1} c_j\, t_j$ by $\alpha_j(c_j)$ whenever $c_j \in C_j$
4) Replace t_j^2 by c_j

The resulting word defines the same element of G^*

The products of the elements in two distinct reduced sequences may be equal in G^*. To get normal forms, we consider the coset representatives as follows.

For each $i\ \varepsilon\ I$ choose a set of representatives of the right cosets of A_i in G, and a set of representatives of the right cosets of B_i in G. For each $j \in J$ choose a set of representatives of the right cosets of C_j in G. We shall assume that 1 is the representatives of all the cosets A_i, B_i and C_j

Definition 4.4 Given the sets of right coset representatives of A_i, B_i and C_i in G then a normal from in G^* is a sequence of the from $g_0 t_{i_1}^{\varepsilon_1} g_1 t_{i_2}^{\varepsilon_2} \ldots t_{i_n}^{\varepsilon_n} g_n, \varepsilon_i = \pm 1, n \geq 0$, where

1) g_0 is any element of G
2) If $\varepsilon_r = +1$ and $t_{i_r} \in \{t_i : i \in I\}$, then g_r is a representatives of a cosets of A_{i_r} in G
3) If $\varepsilon_r = +1$ and $t_{i_r} \in \{t_i : i \in I\}$, then g_r is a representatives of a cosets of B_{i_r} in G
4) If $\varepsilon_r = +1$ and $t_{i_r} \in \{t_i : i \in I\}$, then g_r is a representatives of a cosets of C_{i_r} in G
5) There is not a subword of the from $t^{\varepsilon}\, 1\, t^{\delta}$, where $\varepsilon, \delta = \pm 1$

Khanfar, [7] showed that the base group G is embedded in G^* and this result was the following.

Theorem 4.5 Let $G^* = < G, t_i, t_j | rel\ G, t_i^{-1} a_i t_i =$

$\emptyset_i,(a_i), t_j^{-1}\, c_j t_j = \alpha_j,(c_j)$

$t_j^2 = c_j', a_i \in A_i, c_j \in C_j, i \in I, j \in J >$. Then G is embedded in G^*.

Khanfar, [7] also introduced a version of Britton's Lemma for the quasi- H.N.N extension, and his general result was the following.

Theorem 4.6 Let G^* be the quasi-H.N.N extension of G given in definition 3.2. If $g_0 t_{i_1}^{\varepsilon_1} g_1 \dots t_{i_n}^{\varepsilon_n} g_n = 1$ in G^* where n ≥ 1, then the sequence $g_0, t_{i_1}^{\varepsilon_1}, \dots, t_{i_n}^{\varepsilon_n}, g_n$ is not reduced.

Therefore the word representing the identity element in the general quasi-H.N.N extension G^* is the empty word. So we have the normal from theorem, which is equivalent to theorem 4.5 and theorem 4.6.

Theorem 4.7 LetG^* be the quasi- H.N.N extension of G given in definition 4.2. Then every element of G^* has a unique normal from.

Lemma 4.8 Let $G^* =< G, t_i, t_j | rel\, G, t_i^{-1} a_i t_i = \emptyset_i,(a_i), t_j^{-1}\, c_j t_j = \alpha_j,(c_j), t_j^2 = c_j', a_i \in A_i, c_j \in C_j, i \in I, j \in J >$

Let $g = g_0 t_{i_1}^{\varepsilon_1} g_1 \dots t_{i_n}^{\varepsilon_n} g_n$ and $h = h_0 t_{i_1}^{\delta_1} g_1 \dots t_{i_m}^{\delta_m} h_m$ be reduced word, and suppose that $g = h$ in G^*. Then $m = n$ and $\varepsilon_i = \pm\delta_i, i = 1, \dots, n$

Proof Similar to proof of Lemma 3.7

Now, we can assign a well-defined length to each element of G^* in the following definition

Definition 4.6 Let $G^* =< G, t_i, t_j | rel\, G, t_i^{-1} a_i t_i = \emptyset_i,(a_i), t_j^{-1}\, c_j t_j = \alpha_j,(c_j)$

$t_j^2 = c_j', a_i\, \varepsilon\, A_i, c_j \in C_j, i \in I, j \in J >$, be a quasi- H.N.N extension, define $|\ \ |$ on elements of G^* by $|g\,| = n$ if $g = g_0\, t_{i_1}^{\varepsilon_1} \dots t_{i_n}^{\varepsilon_n} g_n, n \geq 0, \varepsilon = \pm 1$ in reduced from.

Theorem 4.8 $G^* =< G, t_i, t_j | rel\, G, t_i^{-1} a_i t_i = \emptyset_i,(a_i), t_j^{-1}\, c_j t_j = \alpha_j,(c_j)$

$t_j^2 = c_j', a_i \in A_i, c_j \in C_j, i \in I, j \in J >$ is a quasi- H.N.N extension. Define $|\ \ |$ on elements of G^* by $|g\,| = n$ if $g = g_0\, t_{i_1}^{\varepsilon_1} \dots t_{i_n}^{\varepsilon_n} g_n, n \geq 0, \varepsilon = \pm 1$ in reduced from. Then $|\ \ |$ is a length function on G*, by $|g\,| = n$ if

Proof $A1' = |1| = 0$

$A2\ |g| = |g^{-1}|\ g \in G^*$ is obvious as g^{-1} will be reduced if g is reduced.

Let $g, h, k \in G^*$

Suppose $d\,(g,h), d\,(h,t\,) \geq s$

Let $g = x_1 t_1^{\varepsilon_1} x_1^{-1} \dots . x_n t_n^{\varepsilon_n} x_n^{-1}, |g| = n \geq 1$ and $h = y_1 t_1^{\varepsilon_1} y_1^{-1} \dots . y_m t_m^{\varepsilon_m} y_m^{-1}, |h| = m \geq 1$ in reduced forms.

$gh^{-1} = x_1 t_1^{\varepsilon_1} x_1^{-1} \dots x_n t_n^{\varepsilon} x_n^{-1} t^{-1} y_m^{-1} \dots y_1 t_1^{\varepsilon_1} y_1^{-1}$

$x_n^{-1} y_m = 1 \in G$, then $t_m^{-1}\, 1\, t_m^{-1} = 1 \in G$

Suppose $gh^{-1} = x_1 t_1^{\varepsilon_1} x_1^{-1} \dots x_{n-s} a_s y_{m-s}^{-1} \dots . y_1^{-1}$

Let $k = z_1 t_1^{\varepsilon_1} z_1^{-1} \dots z_n t_n^{\varepsilon} z_n^{-1}$ and let $gk^{-1} = gh^{-1}\, hgk^{-1}$

$$hk^{-1} = y_1 t_1^{\varepsilon_1} y_1^{-1} \dots y_{n-s} b_s\, z_{u-s}^{-1} t_{m1}^{\varepsilon_{m-1}}\, z_{u-s} \dots z_1^{-1}$$

Therefore

$gh^{-1} = x_1 t_1^{\varepsilon_1} x_1^{-1} \dots x_{n-s} a_s y_{m-s}^{-1} \dots . y_1^{-1} k_u^{-1} t_u^{\varepsilon_n} k_u \dots z_1^{-1}$

As $d(g,h)$ and $(h,k) \geq s$, then $gh^{-1} = x_1 t_1^{\varepsilon_1} x_1^{-1} \dots a_{s+1} b_{s+1} z_{u-s} t_{n-1}^{\varepsilon_{n-1}}\, z_{u-s} \dots z_1^{-1}$

Therefore, $|gh^{-1}| \leq n + u - 2s$, ie $d(g,k) \geq s, So\ |\ \ |$ is a length function.

Theorem 4.9 $G^* =< G, t_i, t_j | rel\, G, t_i^{-1} a_i t_i = \emptyset_i,(a_i), t_j^{-1}\, c_j t_j = \alpha_j,(c_j)$

$t_j^2 = c_j', a_i \in A_i, c_j \in C_j, i \in I, j \in J >$ is a quasi- H.N.N extension, define $|\ \ |$ on elements of G^* by $|g\,| = n$ if $g = g_0\, t_{i_1}^{\varepsilon_1} \dots t_{i_n}^{\varepsilon_n} g_n, n \geq 0, \varepsilon = \pm 1$ in reduced from. Then the elements of N are conjugates of the elements of the base G.

Proof To Prove that if $\in N$, then $g = x\, a\, x^{-1}, x \in G^*$ and $a \in G$.

Suppose that $g \in N$ and $g = x_1 t_1^{\varepsilon_1} x_2 \dots x_{n-1} t_n^{\varepsilon_n} x_n$ is reduced. ie $|\,g\,| = n$

The result is trivial if n = 0 or 1

Now $|\,g^2\,| \leq |g|$, then

$g^2 = x_1 t_1^{\varepsilon_1} x_2 \dots x_{n-1} t_n^{\varepsilon_n} x_n x_1 t_1^{\varepsilon} x_2 \dots x_{n-1} t_n^{\varepsilon_n} x_n$ where $|x_s a_s x_{s+1}| \leq 2$, ie there is no further cancellation.

Therefore, $a_{s+1} \in G$. Further, if $x_{s-1} a_{s+1} x_{s-1}^{-1}$ is not $reduced, then\ g =$

$$g = \underbrace{(x_1 t_1^{\varepsilon_1} x_2 \dots x_{n-1} t_n^{\varepsilon_n} x_n)(x_1 t_1^{\varepsilon_1} x_2 \dots x_{s-1} t_s^{\varepsilon_s} x_s)}\,(x_1 t_1^{\epsilon_1} x_2 \dots x_{s-1} t_s^{\varepsilon_s} x_s\,)^{-1}$$

Therefore,

$g = (x_1 t_1^{\varepsilon_1} x_2 \dots x_{s-1} t_s^{\varepsilon_s} x_s)^{a_s} (x_1 t_1^{\varepsilon_1} x_2 \dots x_{s-1} t_s^{\varepsilon_s} x_s\,)^{-1}$

Therefore, g is a conjugate of an element a_s of G

Conversely, suppose

$g = (x_1 t_1^{\varepsilon_1} x_2 \dots x_{s-1} t_s^{\varepsilon_s} x_s) a_s (x_1 t_1^{\varepsilon_1} x_2 \dots x_{s-1} t_s^{\varepsilon_s} x_s\,)^{-1}$

If $x_s a_s x_s^{-1} \in G$ then put $x_s a_s x_s^{-1} = a_{s+1}$ this means $a_s \in G$

If $|a_{s+1}| = 0$, then $|a_{s+1}^2| = 0$ so $a_{s+1} \in G$.

If $|a_{s+1}| = 1$ then $a_{s+1}^2 = x_s a_s x_s^{-1}$ where $a_s^2 \in G$

Suppose $x_s a_s x_s^{-1}$ is reduced, ie $|a_{s+1}^2| = 2$

Therefore$a_s \in G$, which is a contradiction. So $x_s a_s x_s^{-1}$ is reduced, ie $|a_{s+1}| = |a_s\,|$

$(x_1 t_1^{\varepsilon_1} x_2 \dots x_{r-1} t_{r-1}^{\varepsilon_{r-1}} x_r) b_r (x_1 t_1^{\varepsilon_1} x_2 \dots x_{r-1} t_{r-1}^{\varepsilon_{r-1}} x_r)^{-1}$, where $b \in G$ and $x_r\, b_r x_r^{-1}$ is reduced

If $b \in G$, then $|g| = zr, b^2 \in G$ and $|g^2| = \left|(x_1 t_1^{\varepsilon_1} x_2 \dots x_s)(x_1 t_1^{\varepsilon_1} x_2 \dots x_s)^{-1}\right| \leq 2r$

So $g \in N$, $b \notin G, x_r b$ and $b x_r^{-1}$ are reduced implies $|g| = 2r + 1$

Since $|b^2| \leq 1$, then

$|g^2| = \left|(x_1 t_1^{\varepsilon_1} x_2 \dots x_s) b_s^2 (x_1 t_1^{\varepsilon_1} x_2 \dots x_s)^{-1}\right| \leq 2r + 1 = |g|$

Therefore $g \in N$

In case if $b \notin G$ and either $x_r b_r x_r^{-1}$ is not reduced, then $|g| = 2r$

Since $g \in N$, then $|b^2| \leq |b|$ so b^2 is not reduced

Now consider $g^2 = \left(x_1 t_1^{\varepsilon_1} x_2 \dots x_s\right) b_s^2 \left(x_1 t_1^{\varepsilon_1} x_2 \dots x_s\right)^{-1}$ and suppose $x_r b^2 x_r^{-1}$ is reduced, then $b_r x_r^{-1}$ is not reduced or $x_r b$ is not reduced

Therefore $x_r b^2 x_r^{-1}$ is not reduced ie $|x_r b^2 x_r^{-1}| \leq 2$. So $|g^2| = 2r = |g|$ ie $g \in N$

Theorem 4.10 $G^* = < G, t_i, t_j | rel\ G, t_i^{-1} a_i t_i = \emptyset_i, (a_i), t_j^{-1} c_j t_j = \alpha_j, (c_j)$

$t_j^2 = c_j', a_i \in A_i, c_j \in C_j, i \in I, j \in J >$ is a quasi- H.N.N extension, define $|\ \ |$ on elements of G^* by $|g| = n$ if $g = g_0 t_{i_1}^{\varepsilon_1} \dots t_{i_n}^{\varepsilon_n} g_n, n \geq 0, \varepsilon = \pm 1$ in reduced from. Then the elements of N are equivalent if and only if they are conjugates by the same element of G^*

Proof Suppose $g \sim h$ in N then $|gh^{-1}| \leq |g| = |h|$

Let $g = x_0 t_1^{\varepsilon_1} x_2 \dots x_{n-1} t_n^{\varepsilon_n} x_n$ and $h = y_0 t_1^{\varepsilon_1} y_2 \dots y_{n-1} t_n^{\varepsilon_n} y_n$ where $\epsilon_i = \mp 1$ be both reduced

1) The result is trivial if $|g| = |h| = 0,1$
2) If n>1

By theorem 2.4 $g = \left(x_0 t_1^{\varepsilon_1} \dots x_{n-s} t_n^{\varepsilon_n} x_s\right) a_s \left(x_0 t_1^{\varepsilon_1} \dots x_s\right)^{-1}$, where $a_s \in G$ and

$$h = \left(y_0 t_1^{\varepsilon_1} y_2 \dots y_s\right) b_s \left(y_0 t_1^{\varepsilon_1} \dots y_s\right) \quad, b_s \in G$$

$$gh^{-1}$$

$$= \left(x_0 t_1^{\varepsilon_1} x_1 \dots x_{s-1}\right)\left(t_{s-1} a_s\right) \left(x_0 t_1^{\varepsilon_1} \dots x_s\right)^{-1} \left(y_0 t_1^{\varepsilon_1} y_1 \dots y_s\right)\left(b_s^{-1} y_s^{-1}\right)$$

$$\left(y_0 t_1^{\varepsilon_1} y_1 \dots y_{s-1}\right)^{-1}$$

Since $|gh^{-1}| \leq n$ then $\left(x_0 t_1^{\epsilon_1} x_1 \dots x_s\right)^{-1} \left(y_1 t_1^{\varepsilon_1} \dots y_s\right) = G$

$$y_0 t_1^{\varepsilon_1} y_1 \dots y_s = \left(x_0 t_1^{\varepsilon_1} x_1 \dots x_s\right) a_s$$

Thus $h = \left(x_0 t_1^{\varepsilon_1} x_1 \dots x_s\right)\ a_s b_s a_s^{-1} \left(x_0 t_1^{\varepsilon_1} x_1 \dots x_s\right)^{-1}$ where $a_s b_s a_s^{-1} = a \in G$. Hence g,h are conjugate of $a \in G$.

Conversely suppose

$g = \left(x_0 t_1^{\varepsilon_1} x_1 \dots x_r\right)\ a_s \left(x_0 t_1^{\varepsilon_1} x_1 \dots x_s\right)^{-1}$ where $a_r \in G$ and $h = (y_0 t_1^{\varepsilon} y_1 \dots y_r)\ b_r \left(y_0 t_1^{\varepsilon_1} \dots y_r\right)^{-1}, b_r \in G$ where $a \sim b$

Similar argument shows that $x_r a_r x_r^{-1}$ is not reduced.

Since $a \sim b$, then either $a_r, b_r \in G$ then $a_r b_r^{-1} \in G$ and $|gh^{-1}| = \left|\left(x_0 t_1^{\varepsilon_1} x_2 \dots x_r\right)\left(x_0 t_1^{\varepsilon_1} x_2 \dots x_r\right)^{-1}\right| \leq 2r$

So $g \sim h$

Theorem 4.11 $G^* = < G, t_i, t_j | rel\ G, t_i^{-1} a_i t_i = \emptyset_i, (a_i), t_j^{-1} c_j t_j = \alpha_j, (c_j)$

$t_j^2 = c_j', a_i \in A_i, c_j \in C_j, i \in I, j \in J >$ is a quasi- H.N.N extension, the element of M are conjugates of the associated pairs of subgroups (provided we exclude the case when $G^* = \langle c, t\ | rel\ c, tct^{-1} = \alpha(c), t^2 = c' \in C \rangle$

Proof To prove that $g, h \in M \rightarrow gh = xax^{-1}$ where $x \in G$.

Let $g = x_0 t_1^{\varepsilon_1} x_1 \dots x_n, h = y_0 t_1\ y_1 \dots y_n$ be reduced.

Suppose $|gh\ \ | + |hg| < 2|h| = |g|$, Then $|g|, |h| \geq 1$.

$n = 1$ is trivial as, suppose $|y_1 x_1\ \ | = 0$, then $gh = x_1 y_1 x_1 x_1^{-1}$= conjugate of $(y_1 x_1)$

Similarly if $|y_1 x_1\ \ | = 0$, then hg is conjugate of $x_1 y_1$

So let $n \geq 2$ and let $gh = \varepsilon\left(x_0 t_1^{\varepsilon_1} x_2 \dots x_s\right) a_s (y_{s+1} \dots y_n)$ (1) where $s \leq n$ and s is maximum

Then (1) is reduced in which case, $|gh\ \ | = 2n - 2s + 1$ or $S_s \in G$ and $x_{n-s} a_s y_{s+1}$ is reduced, in which case $|gh\ \ | = 2n - 2s$, where $a_s = x_{n-s+1} \dots t_n^{\varepsilon_1} x_n\ y_1 t_1^{\varepsilon_1} y_1 \dots y_s$

Similarly $hg = y_0 t_1^{\varepsilon_1} y_1 \dots y_{n-r} b_r x_{r+1} t_{r+2}^{\varepsilon_{r+2}} \dots x_n$ (2)

Then either (2) is reduced so $|hg\ \ | = 2n - 2r + 1$ or $b_r \in G$ and $y_{n-r} b_r x_{r+1}$ is not reduced so $|hg\ \ | = 2n - 2r$, where $b_r = y_{m-r+1} t_{m-r+1}^{\varepsilon_i} \dots\ y_1\ x_1 t_1^{\varepsilon_1} y_1 \dots x_r$

Then $2n - 2s + 1 + 2n - 2r + 1 < 2n$

$$2n - 2r - 2s + 2 < 0 \Rrightarrow r + s > n - 1$$

$r > n - s + 1$ and $s > n - r + 1$

Then $b_{n-s+1} \in G$ and since $a_{s-1} \in G$, then $b_{n-s+1} a_{s-1} \in G$ or gh is conjugate of an element in G.

5. Conclusions

This paper has proved that the Quasi-HNN groups possess a length function as defined in section three. It has also proved that the elements of N are conjugates of the elements of the base group G. The elements of N are conjugates if and only if they are conjugated by the same elements of G^*.

Finally, it is proved that the elements of the set M are conjugates of the associated subgroups.

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