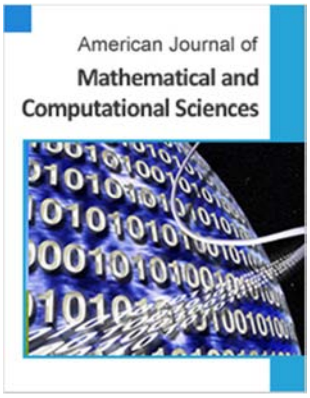
American Journal of Mathematical and Computational Sciences
2017; 2(1): 1-9
http://www.aascit.org/journal/ajmcs

American Association for Science and Technology



Keywords
Burgers Equation,
Fractional Calculus,
Exp-function Method

Received: March 25, 2017
Accepted: April 18, 2017
Published: July 5, 2017

A Novel Technique to Obtain Exact Solutions for Burgers Equation

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Citation
Qazi Mahmood Ul Hassan, Ayesha Sidiuqa, Kamran Ayub, Madiha Afzal, Muhammad Hashim. A Novel Technique to Obtain Exact Solutions for Burgers Equation. *American Journal of Mathematical and Computational Sciences*. Vol. 2, No. 1, 2017, pp. 1-9.

Abstract
In this paper, we use the fractional derivatives in Caputo's sense to construct exact solutions for Burgers equation of fractional order. A generalized fractional complex transform is appropriately used to convert this equation to ordinary differential equation which subsequently resulted into number of exact solutions.

1. Introduction

The class of fractional calculus is one of the most convenient classes of fractional differential equation which viewed as generalized differential equations [1]. In the sense that, much of the theory and, hence, applications of differential equation can be extended smoothly to fractional differential equations with the same flavor and spirit of the realm of differential equation. The seeds of fractional calculus (that is, the theory of integrals and derivatives of any arbitrary real or complex order) were planted over 300 years ago. Since then, many researchers have contributed to this field. Recently, it has turned out those differential equations involving derivatives of non-integer [2]. For example, the nonlinear oscillation of earthquakes can be modeled with fractional derivatives [3]. There has been some attempt to solve linear problems with multiple fractional derivatives (the so -called multi-term equations) [3, 4]. Not much work has been done on nonlinear problems and only a few numerical schemes have been proposed for solving nonlinear fractional differential equations. More recently, applications have included classes of nonlinear equation with multi-order fractional derivatives. We apply a generalized fractional complex transform [5-9] to convert fractional order differential equation to ordinary differential equation. Finally, we obtain exact solutions for it by using a novel technique [10, 11] called exp-function method, to obtain generalized solitary solutions and periodic solutions. Mohyud-Din [12-15] extended the same for nonlinear physical problems including higher-order BVPs; Oziz [16] tried this novel approach for Fisher's equation; Wu et. al. [17, 18] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu [19] for MBBN equations, Zhang [20] for high-dimensional nonlinear evolutions; Zhu [21, 22] for the Hybrid-Lattice system and discrete m KdV lattice; Kudryashov [23] for exact soliton solutions of the generalized evolution equation of wave dynamics; Momani [24] for an explicit and numerical solutions of the fractional KdV equation. Most scientific problems and phenomena in different fields of sciences and engineering occur nonlinearly. This method has been effectively and accurately shown to solve a large class of nonlinear problems. The solution procedure of this method, with the aid of Maple, is of utter simplicity and this method can easily extended to other kinds of nonlinear evolution equations. In engineering and science, scientific phenomena give a variety of solutions that are characterized by distinct features. Traveling waves appear in many distinct physical structures in solitary wave theory [25, 26] such as solitons, kinks, peakons, cuspons, and compactons and many others. Solitons are localized traveling waves which are asymptotically zero at large distances. In other words, solitons are localized wave packets with exponential wings or tails. Solitons are generated from arobust balance between nonlinearity and dispersion. Solitons exhibit properties typically associated with particles. Kink waves [26, 27] are solitons that rise or descend from one asymptotic state to another, and hence another type of traveling waves as in the case of the Burgers hierarchy. Peakons, that are peaked solitary wave solutions, are another type of travelling waves as in the case of Camassa-Holm equation. For peakons, the traveling wave solutions are smooth except for a peak at a cornerof its crest. Peakons are the points at which spatial derivative changes sign so that peakons have a finite jump in 1st derivative of the solution. Cuspons are other forms of solitons where solution exhibits cusps at their crests. Unlike peakons where the derivatives at the peak differ only by a sign, the derivatives at the jump of a cuspon diverge. The compactons, which are solitons with compact spatial support such that each compacton is a soliton confined to a finite core or a soliton without exponential tails or wings. Other types of travelling waves arise in science such as negatons, positons and complexitons. In this research, we use the Exp-function method along with generalized fractional complex transform to obtain new solitary waves solutions for the [28].

2. Preliminaries and Notation

In this section, we give some basic definitions and properties of the fractional calculus theory which will be used further in this work. For the finite derivative in $[a,b]$ we define the following fractional integral and derivatives.

Definition 1 A real function $f(x), x>0,$ is said to be in the space $C\mu, \mu \in R,$ If there exists a real number $(p>\mu)$ such that $f(x)=x^p f_1(x),$ where $f_1(x)=C(0,\infty)$ and it is said to be in the space $C_\mu^m \mu$ if $f^m \in C\mu, m\in N.$

Definition 2 The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f\in C\mu, \mu\geq -1,$ is defined as

$$J^\alpha(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, \alpha>0, x>0, J^0(x)=f(x). \quad (1)$$

Properties of the operator J^a can be found in [1]; we mention only the following:

For $f\in C\mu, \mu\geq -1,, \alpha,\beta \geq 0$ and $\gamma \geq -1$

$$\begin{aligned} J^\alpha J^\beta f(x) &= J^{\alpha+B} f(x),\\ J^\alpha J^B f(x) &= J^B J^\alpha f(x),\\ J^\alpha x^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}. \end{aligned} \quad (2)$$

The Riemann--Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator proposed by M. Caputo in his work on the theory of viscoelasticity [2].

Definition 3 For m to be the smallest integer that exceeds, α the Caputo time fractional derivative operator of order $\alpha>0$ and defined as

$$D_t^\alpha f(x)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\dfrac{1}{\Gamma(m-a)}\displaystyle\int_0^x (x-t)^{m-\alpha-1} f(t)dt, -1 << m, m\varepsilon N\\ \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}, \alpha=m\end{cases} \quad (3)$$

3. Chain Rule for Fractional Calculus and Fractional Complex Transform

In [3-6], the authors used the following chain rule $\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial u}{\partial s}\frac{\partial^\alpha s}{\partial t^\alpha}$ to convert a fractional differential equation with Jumarie's modification of Riemann-Liouville derivative into its classical differential partner. In [8], the authors showed that this chain rule is invalid and show following relation [8].

$$D_t^a u=\sigma_t'\frac{du}{d\eta}D_t^a\eta \text{ and } D_x^a u=\sigma_x'\frac{du}{d\eta}D_x^a\eta,$$

To determine σ_s we consider a special case as follows

$s=t^\alpha$ and $u=s^m$ and we have

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\Gamma(1+m\alpha)t^{m\alpha-\alpha}}{\Gamma(1+m\alpha-\alpha)}=\sigma.\frac{\partial u}{\partial s}=\sigma m t^{m\alpha-\alpha}.$$

Thus we can calculate σ_s as

$$\sigma_s=\frac{\Gamma(1+m\alpha)}{\Gamma(1+m\alpha-\alpha)}.$$

Other fractional indexes $\left(\sigma_x^{'}, \sigma_y^{'}, \sigma_z^{'}\right)$ can determine in similar way. Li and He[2-8] proposed following fractional complex transform for converting fractional differential equations into ordinary differential equations, so that all analytical methods for advanced calculus can be easily applied to fractional calculus.

$$u(x,t) = u(\eta), \eta = \frac{kx^{\beta}}{\Gamma(1+\beta)} + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} + \frac{Mx^{\gamma}}{\Gamma(1+\gamma)}, \quad (4)$$

Where k, ω and M are constants.

4. Exp-function Method

We consider the general nonlinear FPDE of the type

$$P(u, u_t, u_x, u_{xx}\, u_{xxx}, \ldots, D_t^{\alpha}u, D_x^{\alpha}u, D_{xx}^{\alpha}u, \ldots) = 0, \quad 0 < \alpha \le 1, \quad (5)$$

Where $D_t^{\alpha}u, D_x^{\alpha}u, D_{xx}^{\alpha}u$ are the modified Riemann-Liouville derivative of u with respect to t, x, xx respectively.

Using a transformation

$$\eta = kx + my + nz + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} + \eta_0 \quad (6)$$

Here k, m, n, ω, η_0 are all constants with $k, \omega \neq 0$

We can rewrite equation (5) in the following nonlinear ODE

$$Q\left(u, u', u'', u''', u^{iv}, \ldots\right) \quad (7)$$

Where the prime denotes derivative with respect to η. According to Exp-function method, we assume that the wave solution can be expressed in the following form

According to modified exp-function method, the solution will be

$$u(\eta) = \frac{\sum_{n=0}^{2M} a_n \exp[n\eta]}{\sum_{n=0}^{2M} b_n \exp[n\eta]}, \quad (8)$$

Where M is a positive integer which is known to be determine, a_n and b_n are unknown constants. To determine the value of M, we balance the linear term of highest order of equation (7) with the highest order nonlinear term [29-32].

5. Solution Procedure

Consider the Burgers equation of fractional order

$$D_t^{\alpha} + kuu_x - \beta u_{xx} = 0 \quad (9)$$

Where β is arbitrary constant.

Using transformation

$$\eta = kx + my + nz + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} + \eta_0 \quad (10)$$

Here k, m, n, ω, η_0 are all constants with $k, \omega \neq 0$

We can rewrite equation (9) in the following nonlinear ODE

$$\omega u' + kuu' - k^2 \beta u'' = 0 \quad (11)$$

Integrate once time, we get

$$\omega u + \frac{k}{2}u^2 - k^2 \beta u' = 0. \quad (12)$$

Balancing the u' and u^2 by using homogenous principal, we have

$$M + 1 = 2M$$

$$M = 1$$

Then the trail solution is

$$u(\eta) = \frac{a_0 + a_1 e^{\eta} + a_2 e^{2\eta}}{b_0 + b_1 e^{\eta} + b_2 e^{2\eta}} \quad (13)$$

Substituting equation (13) in to equation (12), we have

$$\frac{1}{2A}\left[c_0 + c_1 \exp(\eta) + c_2 \exp(2\eta) + c_3 \exp(3\eta) + c_4 \exp(4\eta)\right] = 0 \quad (14)$$

$$A = \left(b_0 + b_1 \exp(\eta) + b_2 \exp(2\eta)\right)^2$$

Where $c_i\ (i = 0,1,2,3,4)$ are constants obtained by Maple 17.

Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

$$\left(c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0\right) \quad (15)$$

We have following solution sets satisfy the given equation,
1st Solution set:

$$\left\{k = \frac{1}{2}\frac{a_0}{\beta b_0}, \omega = -\frac{1}{4}\frac{a_0^2}{\beta b_2^2}, a_0 = a_0, a_1 = a_1, a_2 = 0, b_0 = b_0, b_1 = \frac{b_2 a_0^2 + a_1^2 b_0}{a_0 a_1}, b_2 = b_2\right\}$$

We, therefore, obtained the following generalized solitary solution

$$u(x,t) = \frac{a_0 + a_1 e^{\frac{1}{2}\frac{x a_0}{\beta b_0} - \frac{1}{4}\frac{a_0^2 t^{\alpha}}{\beta b_0^2 \Gamma(1+\alpha)}}}{b_0 + \frac{\left(b_2 a_0^2 + a_1^2 b_0\right) e^{\frac{1}{2}\frac{x a_0}{\beta b_0} - \frac{1}{4}\frac{a_0^2 t^{\alpha}}{\beta b_0^2 \Gamma(1+\alpha)}}}{a_0 a_1} + b_2 e^{\frac{x a_0}{\beta b_0} - \frac{1}{2}\frac{a_0^2 t^{\alpha}}{\beta b_0^2 \Gamma(1+\alpha)}}}$$

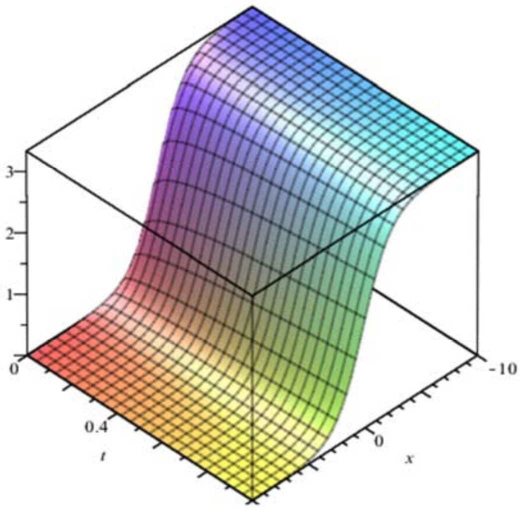

Figure 1. *1st solution set for* $\alpha = .25$.

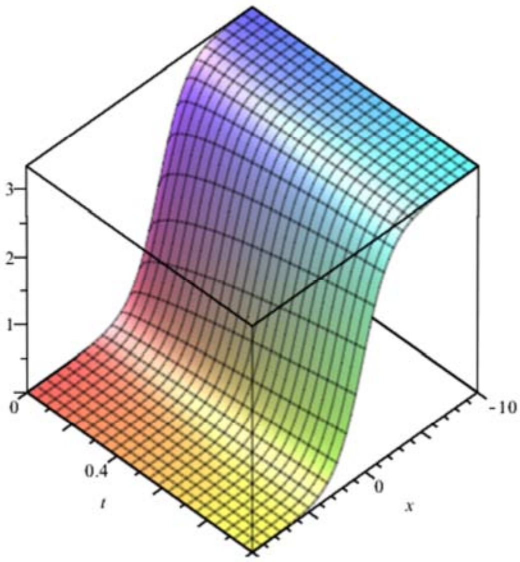

Figure 2. *1st solution set for* $\alpha = .50$.

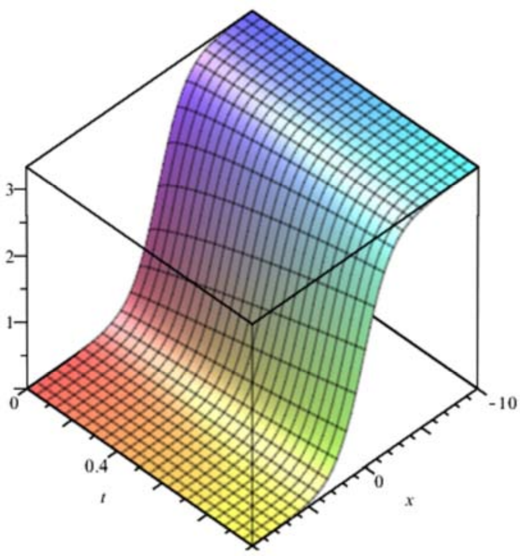

Figure 3. *1st solution set for* $\alpha = .75$.

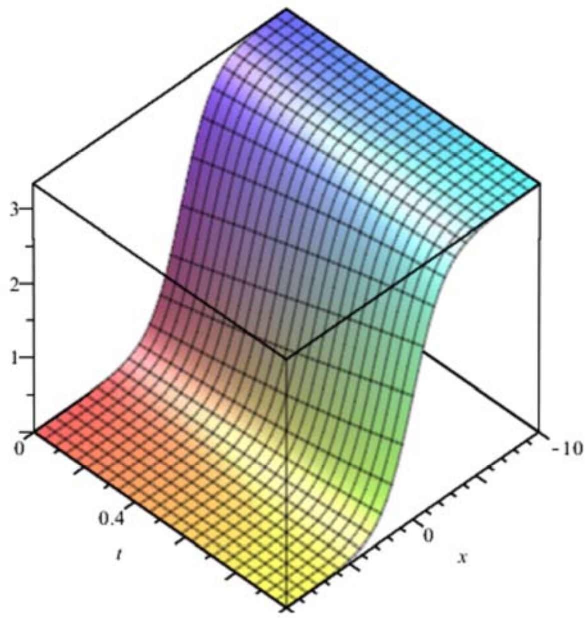

Figure 4. *1st solution set for* $\alpha = 1$.

2nd Solution set:

$$\left\{k = \frac{1}{2}\frac{a_0}{\beta b_0}, \omega = -\frac{1}{4}\frac{a_0^2}{\beta b_0^2}, a_0 = a_0, a_1 = 0, b_0 = b_0, b_1 = b_1, b_2 = 0\right\}$$

We, therefore, obtained the following generalized solitary solution

$$u(x,t) = \frac{a_0}{b_0 + b_1 e^{\frac{1}{2}\frac{x a_0}{\beta b_0} - \frac{1}{4}\frac{a_0^2 t^\alpha}{\beta b_0^2 \Gamma(1+\alpha)}}}$$

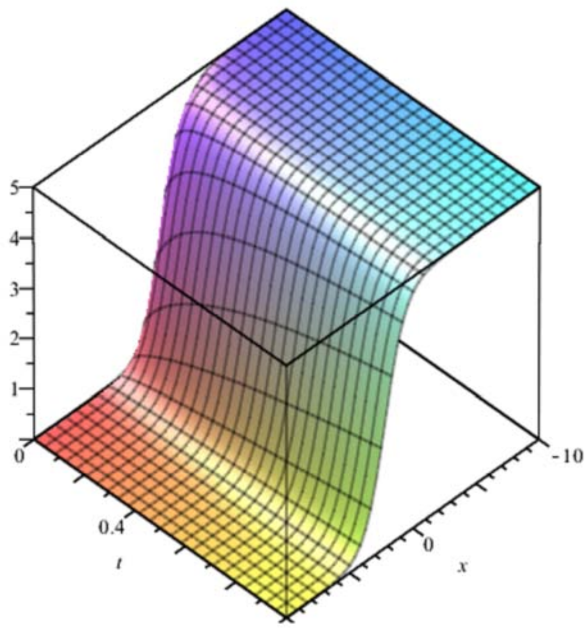

Figure 5. *2nd solution set for* $\alpha = .25$.

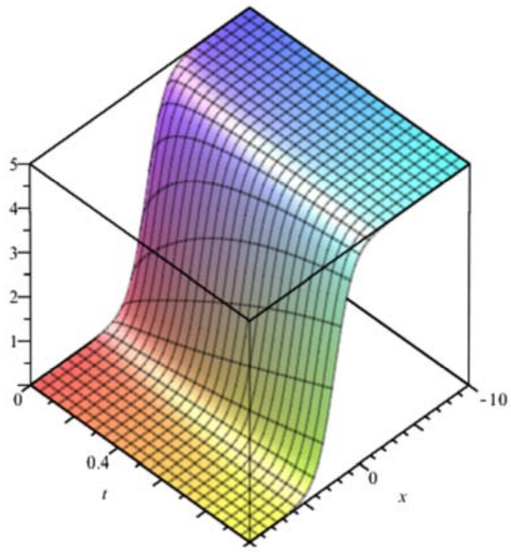

Figure 6. *2nd solution set for* $\alpha = .50$.

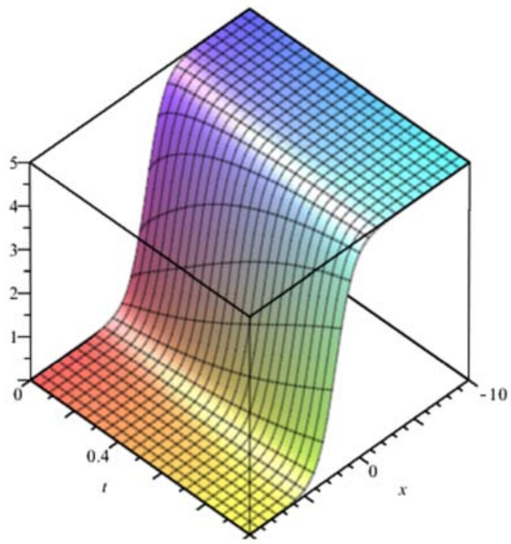

Figure 7. *2nd solution set for* $\alpha = .75$.

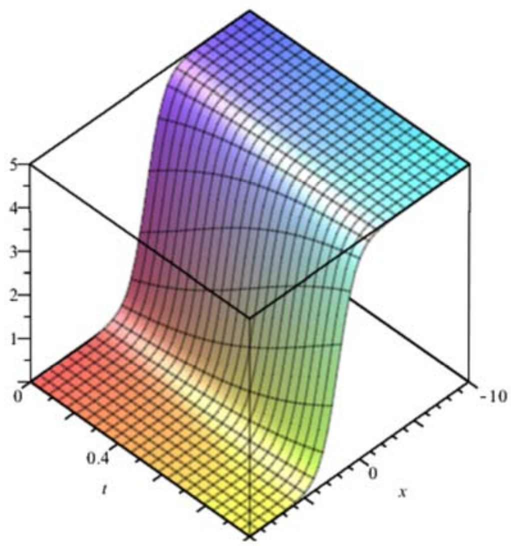

Figure 8. *2nd solution set for* $\alpha = 1$.

3rd Solution set:

$$\left\{k = -\frac{2\omega b_2}{a_2}, \omega = \omega, a_0 = 0, a_1 = a_1, a_2 = a_2, b_0 = 0, b_1 = \frac{a_1 b_2}{a_2}, b_2 = b_2\right\}$$

We, therefore, obtained the following generalized solitary solution

$$u(x,t) = \frac{a_1 e^{-\frac{2x\omega b_2}{a_2} + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}} + a_2 e^{-\frac{4x\omega b_2}{a_2} + \frac{2\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{\frac{a_1 b_2 e^{-\frac{2x\omega b_2}{a_2} + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{a_0 a_1} + b_2 e^{-\frac{4x\omega b_2}{a_2} + \frac{2\omega t^{\alpha}}{\Gamma(1+\alpha)}}}$$

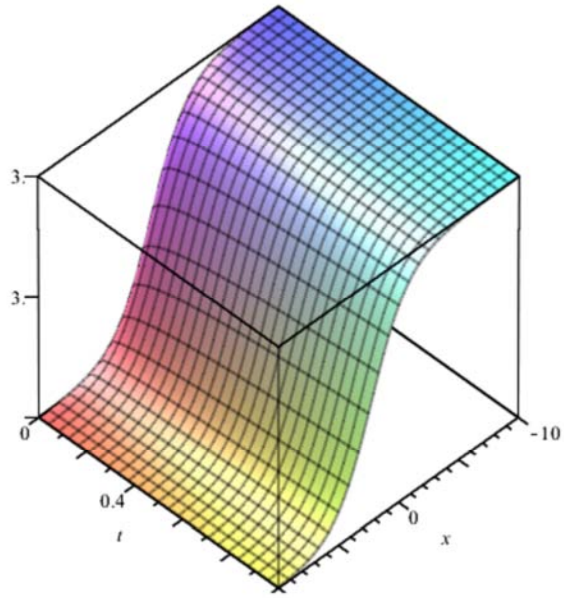

Figure 9. *3rd solution set for* $\alpha = .25$.

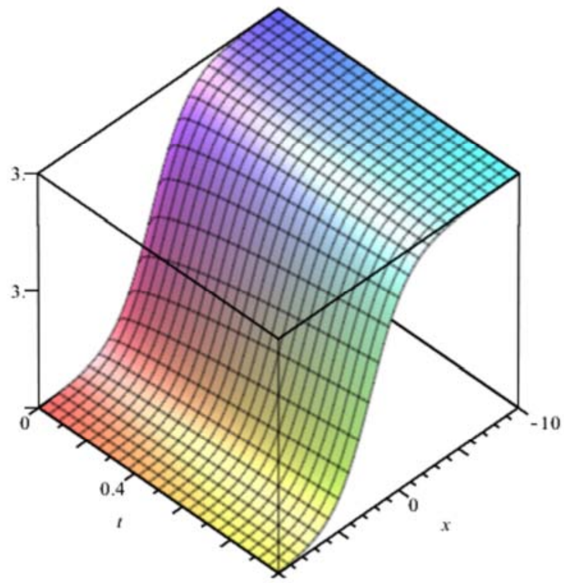

Figure 10. *3rd solution set for* $\alpha = .50$.

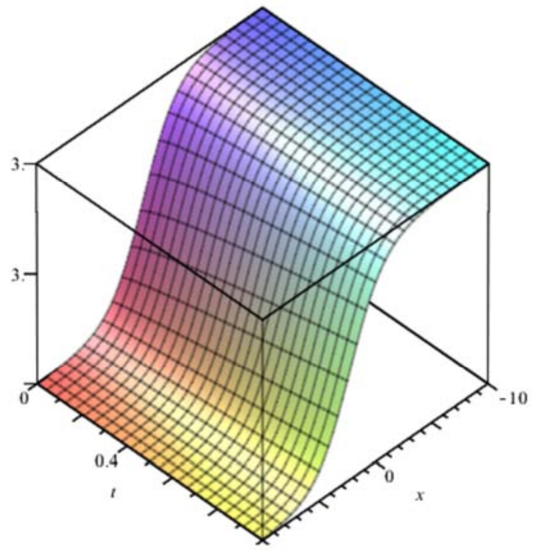

Figure 11. *3rd solution set for* $\alpha = .75$.

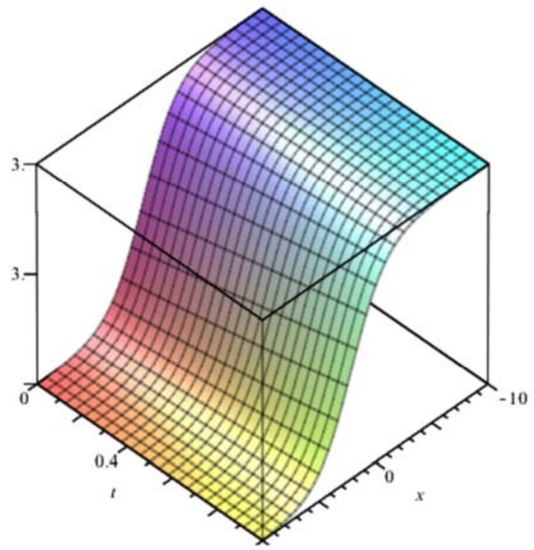

Figure 12. *3rd solution set for* $\alpha = 1$

4st Solution set:

$$\left\{k = -\frac{2\omega b_1}{a_1}, \omega = \omega, a_0 = a_0, a_1 = a_1, a_2 = a_2, b_1 = b_1, b_0 = \frac{a_0 b_1}{a_1}, b_2 = \frac{a_2 b_1}{a_1}\right\}$$

We, therefore, obtained the following generalized solitary solution

$$u(x,t) = \frac{a_0 + a_1 e^{-\frac{2x\omega b_1}{a_1} + \frac{\omega t^\alpha}{\Gamma(1+\alpha)}} + a_2 e^{-\frac{4x\omega b_1}{a_1} + \frac{2\omega t^\alpha}{\Gamma(1+\alpha)}}}{\frac{a_0 b_1}{a_1} + b_1 e^{-\frac{2x\omega b_1}{a_1} + \frac{\omega t^\alpha}{\Gamma(1+\alpha)}} + \frac{a_2 b_1 e^{-\frac{4\omega b_1}{a_1} + \frac{2\omega t^\alpha}{\Gamma(1+\alpha)}}}{a_1}}$$

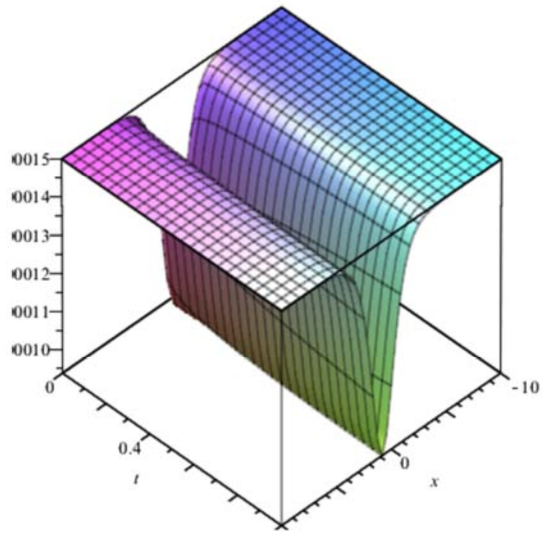

Figure 13. *4th solution set for* $\alpha = .25$.

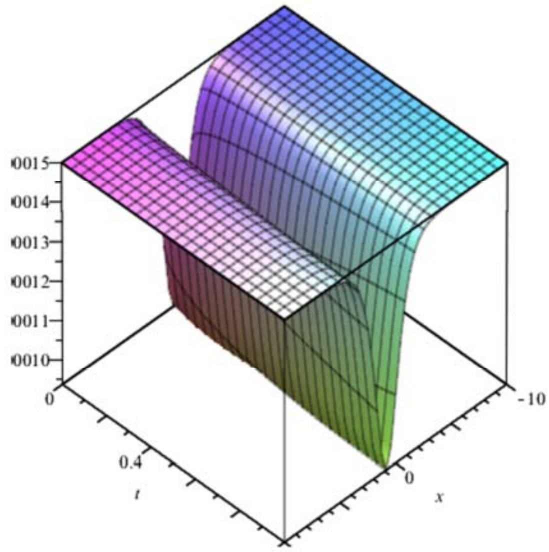

Figure 14. *4th solution set for* $\alpha = .50$.

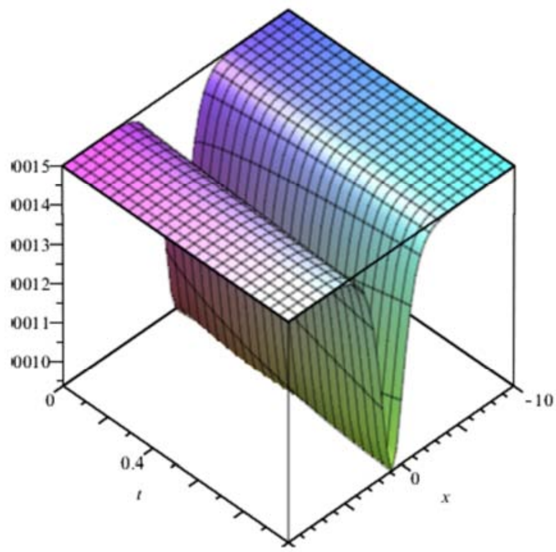

Figure 15. *4th solution set for* $\alpha = .75$.

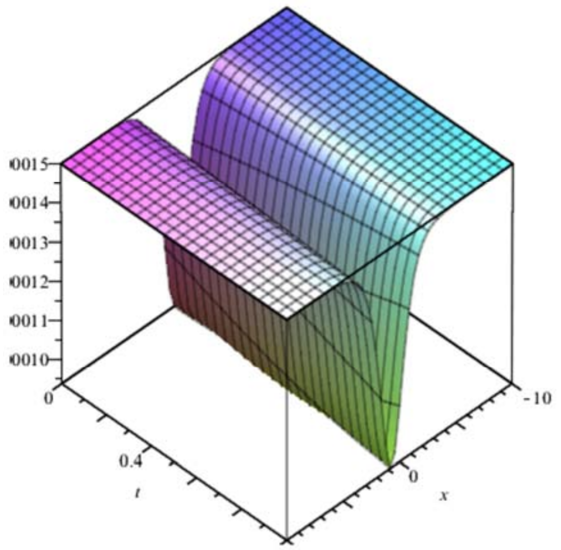

Figure 16. *4th solution set for* $\alpha = 1$.

5th Solution set:

$$\left\{ k = -\frac{2\omega b_0}{a_0}, \omega = \omega, a_0 = a_0, a_1 = a_1, a_2 = 0, b_0 = b_0, b_1 = \frac{a_1 b_0}{a_0}, b_2 = 0 \right\}$$

We, therefore, obtained the following generalized solitary solution

$$u(x,t) = \frac{a_0 + a_1 e^{-\frac{2x\omega b_0}{a_0} + \frac{\omega t^\alpha}{\Gamma(1+\alpha)}}}{b_0 + \frac{a_1 b_0 e^{-\frac{2x\omega b_0}{a_0} + \frac{\omega t^\alpha}{\Gamma(1+\alpha)}}}{a_0}}$$

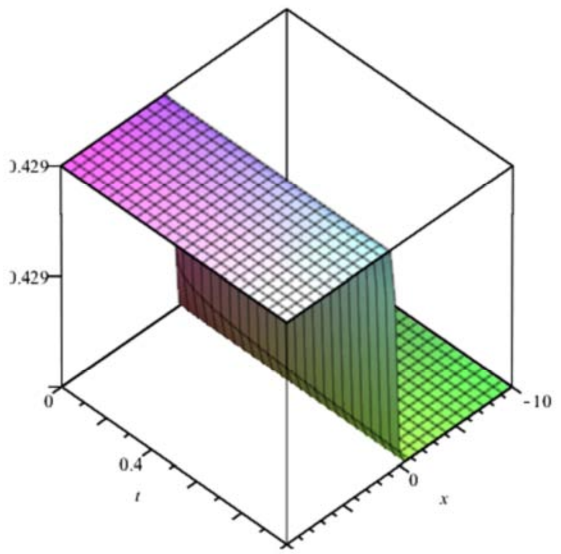

Figure 17. *5th solution set for* $\alpha = .25$

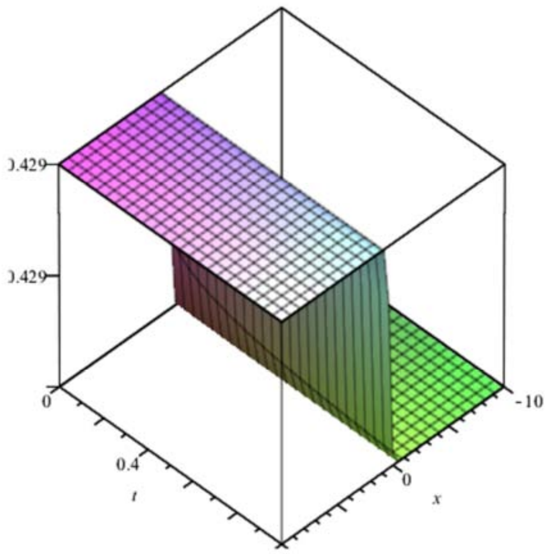

Figure 18. *5th solution set for* $\alpha = .50$

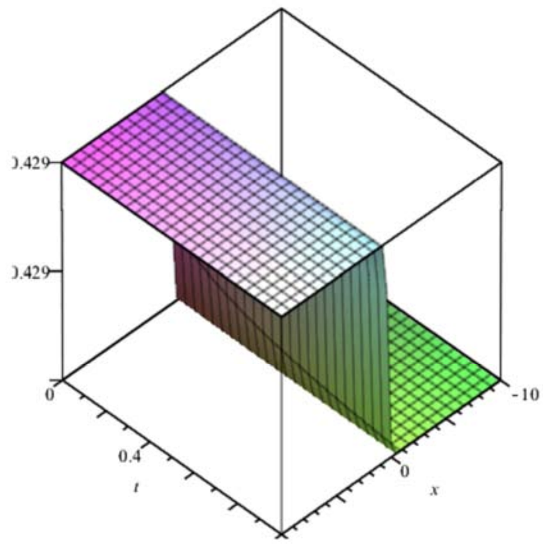

Figure 19. *5th solution set for* $\alpha = .75$

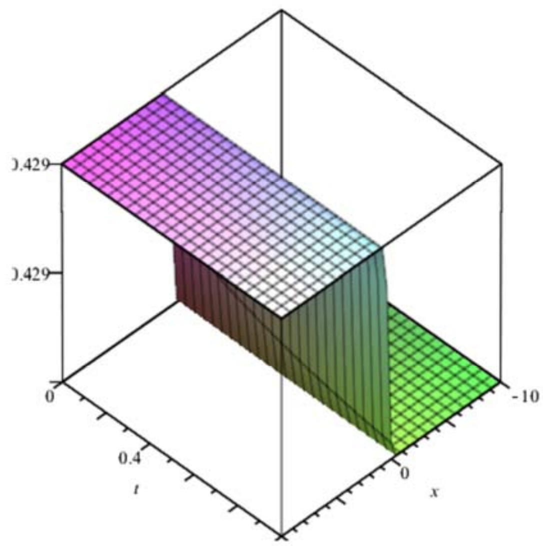

Figure 20. *5th solution set for* $\alpha = 1$

6th Solution set:

$$\left\{k=-\frac{2\omega b_0}{a_0},\omega=\omega,a_0=a_0,a_1=0,a_2=a_2,b_0=b_0,b_1=0,b_2=\frac{a_2 b_0}{a_0}\right\}$$

We, therefore, obtained the following generalized solitary solution

$$u(x,t)=\frac{a_0+a_2 e^{2xK+\frac{2\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{b_0+\frac{a_2 b_0 e^{2xK+\frac{2\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{a_0}}$$

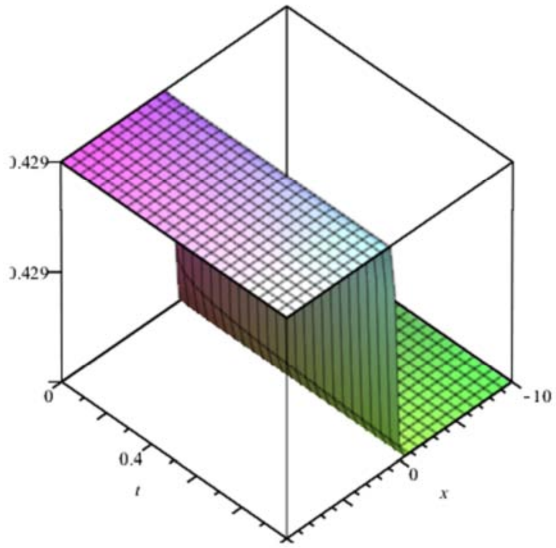

Figure 21. *6th solution set for* $\alpha=.25$.

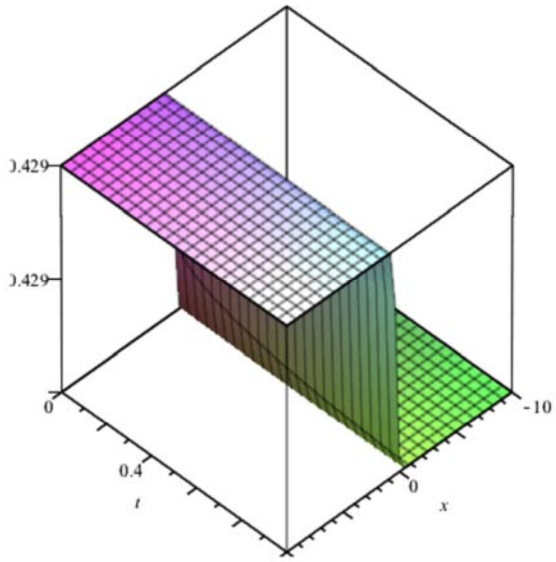

Figure 22. *6th solution set for* $\alpha=.50$.

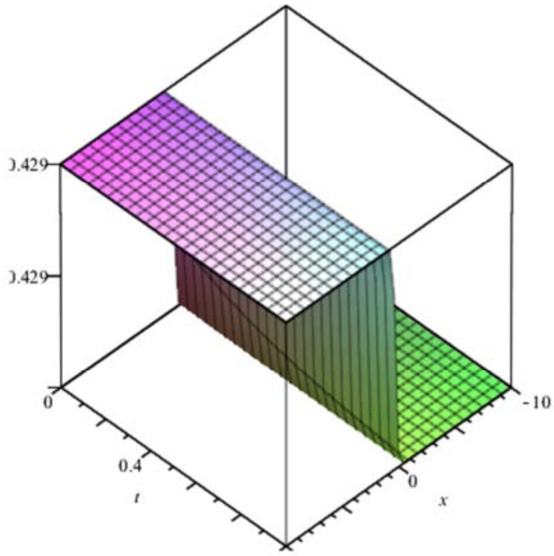

Figure 23. *6th solution set for* $\alpha=.75$.

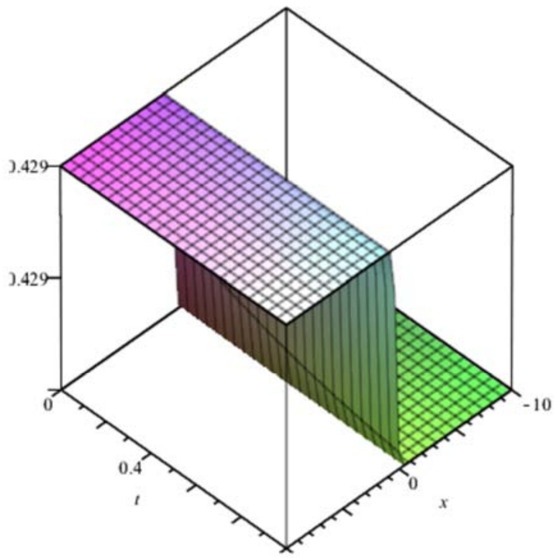

Figure 24. *6th solution set for* $\alpha=1$.

6. Conclusions

Exp-function method is applied to construct solitary solutions of the nonlinear Burgers equation of fractional order. The reliability of proposed algorithm is fully supported by the computational work, the subsequent results and graphical representations. It is observed that Exp-function method is very convenient to apply and is very useful for finding solutions of a wide class of nonlinear problems of fractional orders.

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