On an Efficient Technique for Solving Nonlinear Fornberg-Whitham Equation

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Citation

Abstract
In this paper, the exp-function method has been used to find the soliton solutions of Fornberg-Whitham equation. The solution procedure of this method, with the help of symbolic computation of maple software, is of utter simplicity. The exp-function method is a powerful and straightforward mathematical tool to solve the nonlinear equations in mathematical physics.

1. Introduction

The rapid development of nonlinear sciences, leads to new dimensions in the recent past. The detailed study of literature [1-33] reveals that most of the physical problems are mathematically modeled by initial and boundary value problems and hence finding their appropriate solutions are of utmost importance. John Scott Russell was the pioneer who observed the solitary waves in 1834. He observed a large sticking out of water steadily travelling on the Edinburgh-Glasgow canal without any variation of its shape. He observed the thrust out of water and called it “Great Wave of Translation” was travelling along the channel of water for a long period of time while still retaining its shape. The single humped wave of bulge of water is now called solitary wave or soliton. In 1895, Diederik Korteweg and Gustav de Vries modeled the Korteweg de Vries equation (KdV). They also gave its solitary wave and periodic wave solutions. In 1965, Norman Zabusky and Martin Kruskal investigated numerically the nonlinear interaction of a large solitary waves, and the recurrence of initial states. They discovered that solitary waves undergo nonlinear interaction with KdV equation. The remarkable discovery of Russell that solitary waves possess their identities and their character resemble particle like behavior, motivated Zabusky and Kruskal to call these solitary waves to solitons. A substantial amount of work has been invested for solving the governing equations of these physical models. Several techniques including Hyperbolic function method, Jacobi elliptic method, tanh-coth method, sine-cosine method and homogeneous balance method have been used for the solution of such problems; see [1-6, 16-19] and the references therein. It is always more convenient to tackle ordinary differential equations as opposed to partial differential equations. Ma [1-6] introduced a very efficient transformation, which converts the given partial differential equation to the corresponding ordinary differential equation that can easily be solved by any appropriate
Using a transformation
\[ \eta = kx + \omega t, \]  \tag{2} \]

Where \( k \) and \( \omega \) are constants, we can rewrite equation (1) in the following nonlinear ODE,
\[ Q(u, u', u'', u^{(iv)}, \cdots) = 0. \]  \tag{3} \]

Where the prime denotes derivative with respect to \( \eta \).

According to the exp-function method, which was developed by He and Wu [11], it is assumed that the wave solutions can be expressed in the following form
\[ u(\eta) = \frac{\sum_{\nu=0}^{d} a_{\nu} \exp[n\eta]}{\sum_{\nu=0}^{d} b_{\nu} \exp[m\eta]} \]  \tag{4} \]

Where \( p, q, c, \) and \( d \) are positive integers which are known to be further determined, \( a_{\nu} \) and \( b_{\nu} \) are unknown constants. Equation (4) can be written in the following equivalent form
\[ u(\eta) = \frac{a_{\nu} \exp(c\eta) + \cdots + a_{q} \exp(-d\eta)}{b_{p} \exp(p\eta) + \cdots + b_{q} \exp(-q\eta)}. \]  \tag{5} \]

To determine the value of \( c \) and \( P \), we balance the linear term of highest order of equation (3) with the highest order nonlinear term. Similarly, to determine the value of \( d \) and \( Q \), we balance the linear term of lowest order of equation (3) with lowest order nonlinear term.

In this research, exp-function method has been used to obtain new solitary wave solutions for the Fornberg-Whitham equation.

### 3. Solution Procedure

Consider the nonlinear Fornberg-Whitham equation
\[ u_{t} + u_{x} - u_{xxx} - uu_{x} + 3u_{x}u_{xx} = 0 \quad t > 0, \]  \tag{6} \]

subject to initial condition \( u(x,0) = e^{\frac{x}{2}} \).

Introducing a transformation as \( \eta = kx + \omega t \), equation (6) can be converted into ordinary differential equation
\[ \ddot{u} + k u' - k^{2} \dddot{u} - k^{3} uu' + ku' - 3k^{3} u'' = 0. \]  \tag{7} \]

Where the prime denotes derivative with respect to \( \eta \).

The trial solution of the equation (7) can be expressed as follows,
\[ u(\eta) = \frac{a_{\nu} \exp(c\eta) + \cdots + a_{q} \exp(-d\eta)}{b_{p} \exp(p\eta) + \cdots + b_{q} \exp(-q\eta)}. \]

To determine the value of \( c \) and \( P \), we balance the linear term of highest order of equation (7) with the highest order nonlinear term. Proceeding as before, we obtain

### 2. Exp-function Method

Consider the general nonlinear partial differential equation of the type
\[ P(u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0. \]  \tag{1} \]
Case I. we can freely choose the values of $c$ and $d$, but we will illustrate that the final solution does not strongly depend upon the choice of values of $c$ and $d$. For simplicity, we set $p = c = 1$ and $q = d = 1$ equation (5) reduces to

$$\frac{1}{A} \left[ c_1 \exp(5\eta) + c_4 \exp(4\eta) + c_5 \exp(3\eta) + c_4 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) \right] = 0$$

where

$$\begin{align*}
A &= 5(b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^6 \\
&= 5(c_i \exp(\eta) + b_{-1} \exp(-\eta))^6
\end{align*}$$

are constants obtained by Maple 16.

Substituting equation (8) into equation (7), we have

$$\left( \begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{array} \right) \left( \begin{array}{c}
- \omega_1 \\
- \omega_2 \\
- \omega_3 \\
- \omega_4 \\
- \omega_5 \\
- \omega_6
\end{array} \right) = 0$$

We have following solution sets satisfy the given equation.

1st Solution set:

$$\left\{ \begin{array}{c}
\omega_1 = a_1, a_2 = b_1, a_3 = a_4 = a_5 = a_6 = 0, b_{-1} = b_1, b_0 = 0
\end{array} \right\}$$

We therefore, obtained the following generalized solitary solution $u(x,t)$ of equation (6)

$$u(x,t) = \frac{a_1 b_1}{b_1} e^{-kt + \omega x} + \frac{a_1}{b_1} e^{kt + \omega x}$$

2nd Solution set:

$$\left\{ \begin{array}{c}
\omega_1 = -\omega_2, a_3 = a_4 = a_5 = a_6 = 0, b_{-1} = b_1, b_0 = 0
\end{array} \right\}$$

We therefore, obtained the following generalized solitary solution $u(x,t)$ of equation (6)

$$u(x,t) = \frac{a_1 b_1}{b_1} e^{-kt - \omega x} + \frac{a_1}{b_1} e^{kt - \omega x}$$

3rd Solution set:

$$\left\{ \begin{array}{c}
\omega_1 = -\omega_2, a_3 = a_4 = a_5 = a_6 = 0, b_{-1} = b_1, b_0 = 0
\end{array} \right\}$$

We therefore, obtained the following generalized solitary solution $u(x,t)$ of equation (6)

$$u(x,t) = \frac{a_1 b_1}{b_1} e^{-kt - \omega x} + \frac{a_1}{b_1} e^{kt - \omega x}$$

Figure 1. Solitary wave solution for different values of parameters.

Figure 2. Solitary wave solution for different values of parameters.
\[
\left\{ \omega = \omega, a_{-1} = a_{-1}, a_0 = \frac{a_1b_0}{b_0}, a_1 = \frac{a_2b_1}{b_1}, b_{-1} = b_{-1}, b_0 = b_0, b_1 = b_1 \right\}
\]

We therefore, obtained the following generalized solitary solution \( u(x,t) \) of equation (6)

\[
u(x,t) = \frac{a_1e^{ix+\alpha} + a_1b_0}{b_1e^{ix+\alpha} + b_0}\]

**Figure 3.** Solitary wave solution for different values of parameters.

**Figure 4.** Solitary wave solution for different values of parameters.

4\textsuperscript{th} Solution set:

\[
\left\{ \omega = \omega, a_{-1} = 0, a_0 = \frac{a_1b_0}{b_0}, a_1 = a_1, b_{-1} = b_{-1}, b_0 = b_0, b_1 = b_1 \right\}
\]

We therefore, obtained the following generalized solitary solution \( u(x,t) \) of equation (6)

\[
u(x,t) = \frac{a_1e^{ix+\alpha} + a_1b_0}{b_1e^{ix+\alpha} + b_0}\]

Case II. If \( p = c = 2 \), and \( q = d = 1 \) then equation (5) reduces to

\[
u(\eta) = \frac{a_2 \exp[2\eta] + a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_2 \exp[2\eta] + b_1 \exp[\eta] + b_0 + b_{-1} \exp[-\eta]}
\]

(11)

Proceeding as before, we obtain

5\textsuperscript{th} Solution set:

\[
\left\{ \omega = \omega, a_{-1} = 0, a_0 = a_0, a_1 = \frac{a_2b_0}{b_0}, a_2 = \frac{a_0b_1}{b_1}, b_{-1} = b_{-1}, b_0 = b_0, b_1 = b_1, b_2 = b_2 \right\}
\]

Hence we get the generalized solitary wave solution of equation (6) as follows

\[
u(x,t) = \frac{a_2b_2 e^{2ix+\alpha} + a_0b_1 e^{ix+\alpha} + a_0}{b_2 e^{2ix+\alpha} + b_1 e^{ix+\alpha} + b_0}
\]
Figure 5. Solitary wave solution for different values of parameters.

6th Solution set:

\[
\begin{aligned}
\omega &= \omega, a_{-1} = a, a_0 = \frac{a_0 b_0}{b_{-1}}, a_1 = \frac{a_1 b_1}{b_{-1}}, a_2 = \frac{a_2 b_2}{b_{-1}}, b_{-1} = b, b_0 = b, b_1 = b_1, b_2 = b_2 \\
\end{aligned}
\]

Hence we get the generalized solitary wave solution of equation (6) as follows

\[
\begin{aligned}
&u(x, t) = \frac{a_{-1} b_{-1} e^{2(ik+\alpha t)} + a_{-1} b_{-1} e^{ik+\alpha t} + a_{-1} b_{-1} e^{-(ik+\alpha t)}}{b_{-1} e^{2(ik+\alpha t)} + b_{-1} e^{ik+\alpha t} + b_{-1} e^{-(ik+\alpha t)}}
\end{aligned}
\]

In both cases, for different choices of \( c, \rho, q \) and \( d \), we get the same soliton solutions which clearly illustrate that final solution does not strongly depend on these parameters.

4. Conclusion

The generalized solitary solutions to the nonlinear Fornberg-Whitham equation has readily be acquired by using the exp-function method. Mathematical results coupled with the graphical representations reveal the complete compatibility of proposed algorithm for such problems. The application of exp-function method can also be widened to other non-linear evolution equations. The exp-function method is a promising and powerful new method for NLEEs arising in mathematical physics. Its applications are worth further studying in mathematical sciences.

References


