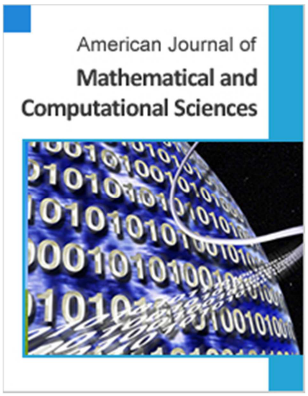
American Journal of Mathematical and Computational Sciences
2017; 2(6): 49-54
http://www.aascit.org/journal/ajmcs

American Association for Science and Technology



Keywords
Exp-function Method,
Fornberg-Whitham Equation,
Soliton Solutions

Received: March 28, 2017
Accepted: September 9, 2017
Published: October 17, 2017

On an Efficient Technique for Solving Nonlinear Fornberg-Whitham Equation

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Citation
Muhammad Hashim, Madiha Afzal, Qazi Mahmood Ul-Hassan, Kamran Ayub, Faiza Yasmeen. On an Efficient Technique for Solving Nonlinear Fornberg-Whitham Equation. *American Journal of Mathematical and Computational Sciences.* Vol. 2, No. 6, 2017, pp. 49-54.

Abstract
In this paper, the exp-function method has been used to find the soliton solutions of Fornberg-Whitham equation. The solution procedure of this method, with the help of symbolic computation of maple software, is of utter simplicity. The exp-function method is a powerful and straightforward mathematical tool to solve the nonlinear equations in mathematical physics.

1. Introduction

The rapid development of nonlinear sciences, leads to new dimensions in the recent past. The detailed study of literature [1-33] reveals that most of the physical problems are mathematically modeled by initial and boundary value problems and hence finding their appropriate solutions are of utmost importance. John Scott Russell was the pioneer who observed the solitary waves in 1834. He observed a large sticking out of water steadily travelling on the Edinburgh-Glasgow canal without any variation of its shape. He observed the thrust out of water and called it ''Great Wave of Translation" was travelling along the channel of water for a long period of time while still retaining its shape. The single humped wave of bulge of water is now called solitary wave or soliton. In 1895, Diederik Korteweg and Gustav de Vries modeled the Korteweg de Vries equation (KdV). They also gave its solitary wave and periodic wave solutions. In 1965, Norman Zabusky and Martin Kruskal investigated numerically the nonlinear interaction of a large solitary waves, and the recurrence of initial states. They discovered that solitary waves undergo nonlinear interaction with KdV equation. The remarkable discovery of Russell that solitary waves possess their identities and their character resemble particle like behavior, motivated Zabusky and Kruskal to call these solitary waves to solitons. A substantial amount of work has been invested for solving the governing equations of these physical models. Several techniques including Hyperbolic function method, Jacobi elliptic method, tanh-coth method, sine-cosine method and homogeneous balance method have been used for the solution of such problems; see [1-6, 16-19] and the references therein. It is always more convenient to tackle ordinary differential equations as opposed to partial differential equations. Ma [1-6] introduced a very efficient transformation, which converts the given partial differential equation to the corresponding ordinary differential equation that can easily be solved by any appropriate technique. It is worth mentioning that Ma et. al presented the Wronskian technique for solving involved non-homogeneous partial differential equations and obtained solution formulae helpful in constructing the existing solution coupled with a number of other new solutions including rational solutions, solitons, negatons and breathers. Recently, Ma, Wu and He [1-6] presented a much more general idea to yield exact solutions to nonlinear wave equations by searching for the so-called frobenius transformations. The physical properties of numerous nonlinear travelling wave solutions have also been determined by constructing and evaluating their graphical results. Most of these techniques encounter the inbuilt deficiencies and involve huge computational work.

Nonlinear evolution equations (NLEEs) has turned out to be one of the most exciting and particularly active areas of research. The appearance of solitary wave solutions in nature is quite common; Bell-shaped sech-solutions, kink-shaped tanh-solutions, wave phenomena in elastic media, plasmas, solid state physics, condensed matter physics, electrical circuits, optical fibers, chemical kinematics, fluids and bio-genetics etc. The basic motivation of this paper is to develop a modified version of exp-function method to construct generalized solitary wave solutions of nonlinear Fornberg-Whitham equation. This modification is based on introducing the homogenous balancing principle phenomena in exp-function method [1-6, 16-19, 21-33]. It is observed that the proposed modification is highly compatible to find solitary wave solutions [7-15, 20-24] of nonlinear problems of diversified physical nature and the same can be extended to other problems even with very strong nonlinearity. To estimate the values of p, q, c and d the following theorems will be used.

Theorem 1. Suppose that $u^{(r)}$ and u^{s} are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and s are both positive integers. Then the balancing procedure using the exp-function ansatz $u(\eta)=\frac{\sum_{n=-c}^{d} a_n \exp[n\eta]}{\sum_{m=-p}^{q} b_m \exp[m\eta]}$, leads to $p=c$ and $d=q, \ \forall\ r,s\geq 2$.

Theorem 2. suppose that $u^{(r)}$ and $u^{(s)}u^{k}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r,s and k are all positive integers. Then the balancing procedure using the exp-function ansatz leads to $p=c$ and $d=q$, $\forall\ r,s,k\geq 1$.

Theorem 3. Suppose that $u^{(r)}$ and $u^{(s)\Omega}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r,s and Ω are all positive integers. Then the balancing procedure using the exp-function ansatz leads to $p=c$ and $d=q$, $\forall\ r,s\geq 1, \forall \Omega \geq 2$.

2. Exp-function Method

Consider the general nonlinear partial differential equation of the type

$$P(u,u_t,u_x,u_{tt},u_{xx},u_{xxx},\ldots)=0. \quad (1)$$

Using a transformation

$$\eta = kx + \omega t, \quad (2)$$

Where k and ω are constants, we can rewrite equation (1) in the following nonlinear ODE,

$$Q\left(u,u',u'',u''',u^{(iv)},\cdots\right)=0. \quad (3)$$

Where the prime denotes derivative with respect to η.

According to the exp-function method, which was developed by He and Wu [11], it is assumed that the wave solutions can be expressed in the following form

$$u(\eta)=\frac{\sum_{n=-c}^{d} a_n \exp[n\eta]}{\sum_{m=-p}^{q} b_m \exp[m\eta]}, \quad (4)$$

Where p,q,c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. Equation (4) can be written in the following equivalent form

$$u(\eta)=\frac{a_c \exp(c\eta)+\ldots+a_{-d}\exp(-d\eta)}{b_p \exp(p\eta)+\ldots+b_{-q}\exp(-q\eta)}. \quad (5)$$

To determine the value of c and p, we balance the linear term of highest order of equation (3) with the highest order nonlinear term. Similarly, to determine the value of d and q, we balance the linear term of lowest order of equation (3) with lowest order nonlinear term.

In this research, exp-function method has been used to obtain new solitary wave solutions for the Fornberg-Whitham equation.

3. Solution Procedure

Consider the nonlinear Fornberg-Whitham equation

$$u_t+u_x-u_{xxt}-uu_{xxx}+uu_x-3u_xu_{xx}=0 \quad t>0, \quad (6)$$

subject to initial condition $u(x,0)=e^{\frac{x}{2}}$.

Introducing a transformation as $\eta = kx+\omega t$, equation (6) can be converted into ordinary differential equation

$$\omega u'+ku'-k^2\omega u'''-k^3uu'''+kuu'-3k^3u'u''=0. \quad (7)$$

Where the prime denotes derivative with respect to η.

The trial solution of the equation (7) can be expressed as follows,

$$u(\eta)=\frac{a_c \exp(c\eta)+\ldots+a_{-d}\exp(-d\eta)}{b_p \exp(p\eta)+\ldots+b_{-q}\exp(-q\eta)}.$$

To determine the value of c and p, we balance the linear term of highest order of equation (7) with the highest order nonlinear term. Proceeding as before, we obtain

$p = c$ and $d = q$.

Case I. we can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $q = d = 1$ equation (5) reduces to

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + b_0 + b_{-1} \exp[-\eta]}. \quad (8)$$

Substituting equation (8) into equation (7), we have

$$\frac{1}{A}\begin{bmatrix} c_5 \exp(5\eta) + c_4 \exp(4\eta) + c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) \\ + c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta) + c_{-5} \exp(-5\eta) \end{bmatrix} = 0$$

$$A = 5\left(b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)\right)^6 \text{ where } c_i\,(i = -5, -4, \ldots\ldots, 4, 5) \quad (9)$$

are constants obtained by Maple 16.

Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

$$\left(c_{-5} = 0, c_{-4} = 0, c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0, c_5 = 0\right) \quad (10)$$

We have following solution sets satisfy the given equation.

1st Solution set:

$$\left\{\omega = \omega, a_{-1} = \frac{a_1 b_{-1}}{b_1}, a_1 = a_1, a_0 = 0, b_{-1} = b_{-1}, b_1 = b_1, b_0 = 0\right\}$$

We therefore, obtained the following generalized solitary solution $u(x,t)$ of equation (6)

$$u(x,t) = \frac{\frac{a_1 b_{-1}}{b_1} e^{-(xk+\omega t)} + a_1 e^{xk+\omega t}}{b_{-1} e^{-(xk+\omega t)} + b_1 e^{xk+\omega t}}$$

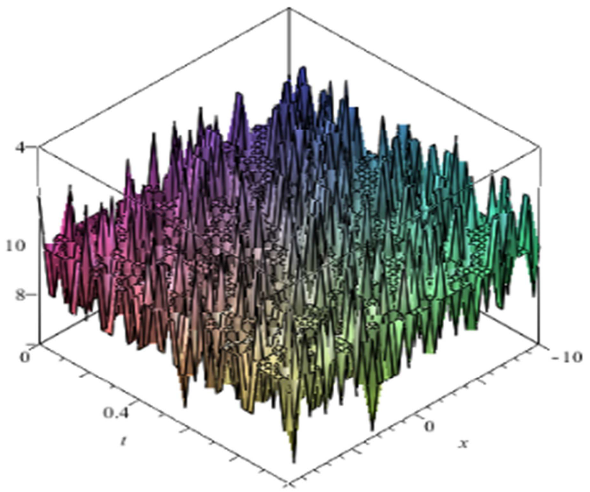

Figure 1. *Solitary wave solution for different values of parameters.*

2nd Solution set:

$$\left\{\omega = \omega, a_{-1} = \frac{a_1 b_{-1}}{b_1}, a_0 = a_0, a_1 = 0, b_{-1} = b_{-1}, b_0 = 0, b_1 = 0\right\}$$

We therefore, obtained the following generalized solitary solution $u(x,t)$ of equation (6)

$$u(x,t) = \frac{\frac{a_1 b_{-1}}{b_1} e^{-(xk+\omega t)} + a_0}{b_{-1} e^{-(xk+\omega t)} + b_0}$$

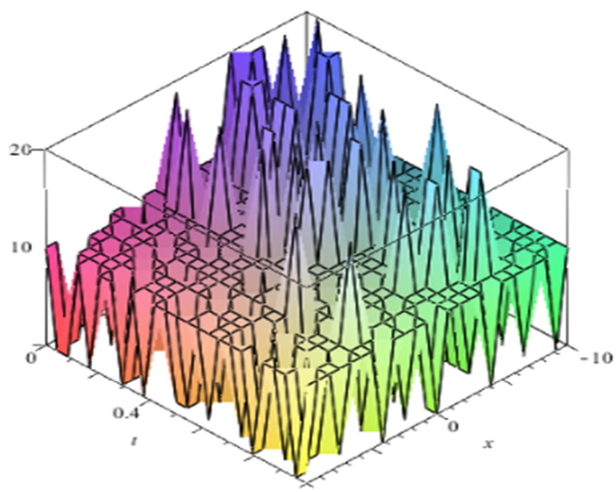

Figure 2. *Solitary wave solution for different values of parameters.*

3rd Solution set:

$$\left\{\omega=\omega, a_{-1}=a_{-1}, a_0=\frac{a_{-1}b_0}{b_{-1}}, a_1=\frac{a_{-1}b_1}{b_{-1}}, b_{-1}=b_{-1}, b_0=b_0, b_1=b_1\right\}$$

We therefore, obtained the following generalized solitary solution $u(x,t)$ of equation (6)

$$u(x,t)=\frac{a_{-1}e^{-(xk+\omega t)}+\frac{a_{-1}b_0}{b_{-1}}+\frac{a_{-1}b_1}{b_{-1}}e^{xk+\omega t}}{b_{-1}e^{-(xk+\omega t)}+b_0+b_1e^{-(xk+\omega t)}}$$

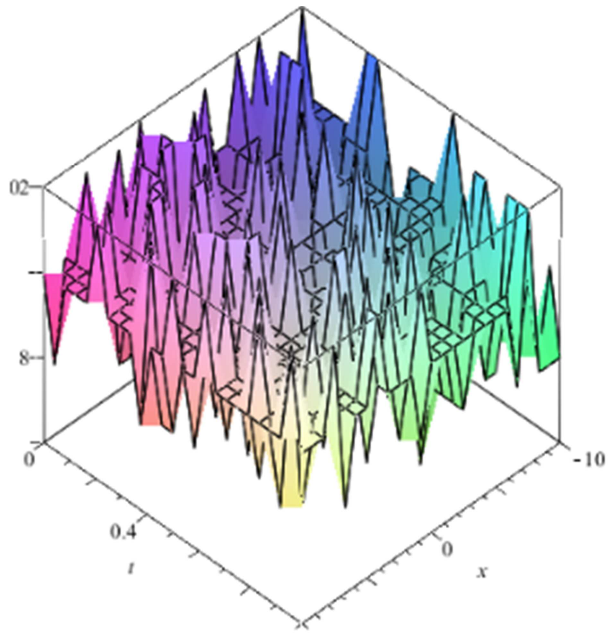

Figure 3. *Solitary wave solution for different values of parameters.*

4th Solution set:

$$\left\{\omega=\omega, a_{-1}=0, a_0=\frac{a_1b_0}{b_1}, a_1=a_1, b_{-1}=0, b_0=b_0, b_1=b_1\right\}$$

We therefore, obtained the following generalized solitary solution $u(x,t)$ of equation (6)

$$u(x,t)=\frac{a_1e^{xk+\omega t}+\frac{a_1b_0}{b_1}}{b_1e^{xk+\omega t}+b_0}$$

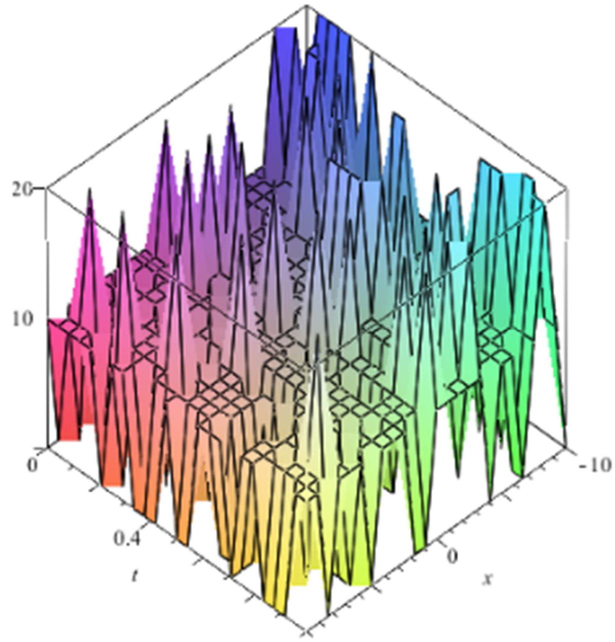

Figure 4. *Solitary wave solution for different values of parameters.*

Case II. If $p=c=2$, and $q=d=1$ then equation (5) reduces to

$$u(\eta)=\frac{a_2\exp[2\eta]+a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_2\exp[2\eta]+b_1\exp[\eta]+b_0+b_{-1}\exp[-\eta]}. \quad (11)$$

Proceeding as before, we obtain
5th Solution set:

$$\left\{\omega=\omega, a_{-1}=0, a_0=a_0, a_1=\frac{a_0b_1}{b_0}, a_2=\frac{a_0b_2}{b_0}, b_{-1}=0, b_0=b_0, b_1=b_1, b_2=b_2\right\}$$

Hence we get the generalized solitary wave solution of equation (6) as follows

$$u(x,t)=\frac{\frac{a_0b_2}{b_0}e^{2(xk+\omega t)}+\frac{a_0b_1}{b_0}e^{xk+\omega t}+a_0}{b_2e^{2(xk+\omega t)}+b_1e^{xk+\omega t}+b_0}$$

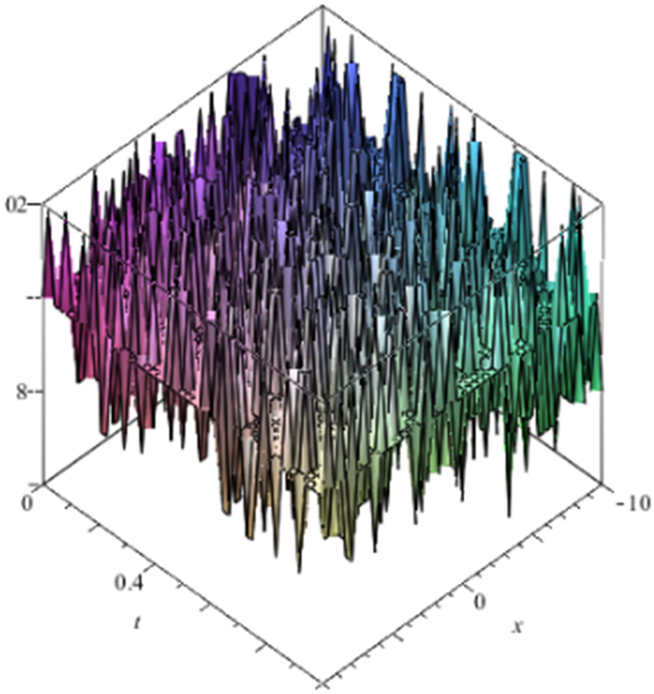

Figure 5. *Solitary wave solution for different values of parameters.*

6th Solution set:

$$\left\{\omega=\omega, a_{-1}=a_{-1}, a_0=\frac{a_{-1}b_0}{b_{-1}}, a_1=\frac{a_{-1}b_1}{b_{-1}}, a_2=\frac{a_{-1}b_2}{b_{-1}}, b_{-1}=b_{-1}, b_0=b_0, b_1=b_1, b_2=b_2\right\}$$

Hence we get the generalized solitary wave solution of equation (6) as follows

$$u(x,t)=\frac{\frac{a_{-1}b_2}{b_{-1}}e^{2(xk+\omega t)}+\frac{a_{-1}b_1}{b_{-1}}e^{xk+\omega t}+\frac{a_{-1}b_0}{b_{-1}}+a_{-1}e^{-(xk+\omega t)}}{b_2e^{2(xk+\omega t)}+b_1e^{xk+\omega t}+b_0++b_{-1}e^{-(xk+\omega t)}}$$

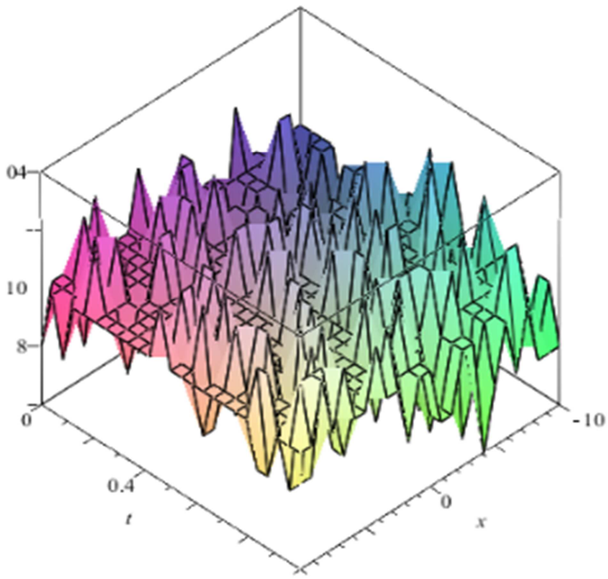

Figure 6. *Solitary wave solution for different values of parameters.*

In both cases, for different choices of c, p, q and d, we get the same soliton solutions which clearly illustrate that final solution does not strongly depends upon these parameters.

4. Conclusion

The generalized solitary solutions to the nonlinear Fornberg-Whitham equation has readily be acquired by using the exp–function method. Mathematical results coupled with the graphical representations reveal the complete compatibility of proposed algorithm for such problems. The application of exp–function method can also be widened to other non-linear evolution equations. The exp-function method is a promising and powerful new method for NLEEs arising in mathematical physics. Its applications are worth further studying in mathematical sciences.

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