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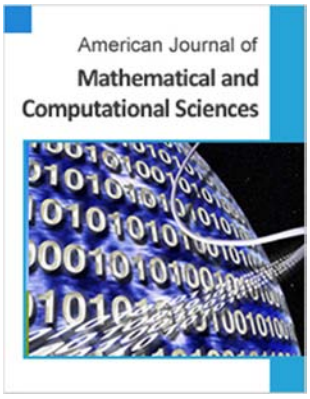

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Lacunary Series Expansions and Weighted Hyperbolic Class $Q^{*}_{p,\omega}$

Alaa Kamal Mohamed, Taha Ibrahim Yassen*

Department of Mathematics, Port Said University, Port Said, Egypt

Email address
alaa_mohamed1@yahoo.com (A. Kamal), taha_hmour@yahoo.com (T. I. Yassen)
*Corresponding author

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Abstract
In this article, we obtained results which characterized $Q^{*}_{p,\omega}$ *and* $Q^{*}_{p,\omega,0}$ spaces of hyperbolic function using integral representation of Hadamard gaps are given. Moreover, we obtain a sufficient and necessary condition for the hyperbolic function f^{*} with Hadamard gaps to belong to $Q^{*}_{p,\omega}$ *and* $Q^{*}_{p,\omega,0}$ on the unite disc $\mathbb{D}$.

1. Introduction

1.1. Analytic Function Spaces

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ be the open unit disc in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denote the classes of analytic functions in the unit disc $\mathbb{D}$. A function $f\in H(\mathbb{D})$ belongs to α-Bloch space $\mathcal{B}^{\alpha}, 0<\alpha<\infty$ if

$$\|f\|_{\mathcal{B}_{\alpha}}=\sup_{z\in\mathbb{D}}(1-|z|)^{\alpha}|f'(z)|<\infty.$$

The little α-Bloch space $\mathcal{B}_{\alpha,0}$ consisting of all $f\in\mathcal{B}_{\alpha}$ such that

$$\lim_{|z|\to 1^{-}}(1-|z|^{2})|f'(z)|=0.$$

Definition 1 *(see [3]) Let a right-continuous and nondecreasing function* $\omega:(0,1]\to(0,\infty)$, *the weighted Bloch space* $\mathcal{B}_{\omega}$ *is defined as the set of all analytic functions* f *on* $\mathbb{D}$ *satisfying*

$$(1-|z|)|f'(z)|\leqslant C\omega(1-|z|),\quad z\in\mathbb{D},$$

for some fixed $C=C_{f}>0$. In the special case where $\omega\equiv 1, \mathcal{B}_{\omega}$ reduces to the classical Bloch space $\mathcal{B}$.

Definition 2 *(see [13]) Let* f *be an analytic function in* $\mathbb{D}$ *and let* $1<p<\infty$. *If*

$$\|f\|^p_{B^p} = \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \,(1-|z|^2)^{p-2}\, dA(z) < \infty,$$

then f belongs to the Besov space B^p. Moreover, if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p \,(1-|z|^2)^{p-2}\, dA(z) = 0,$$

then $f \in B^p_0$.

Let the Green's function of $\mathbb{D}$ with logarithmic singularity at $a\in\mathbb{D}$ be

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right| = \log\frac{1}{|\varphi_a(z)|},$$

where,

$\varphi_a(z)$ denote the Möbius transformations $\varphi_a : \mathbb{D}\to\mathbb{D}$ be defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad for\ z\in\mathbb{D}.$$

Note that $\varphi_a(\varphi_a(z)) = z$, thus $\varphi_a^{-1}(z) = \varphi_a(z)$, and it has the following useful property:

$$1-|\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2},$$
$$1-|\varphi_a(z)|^2 < 2g(z,a).$$

Definition 3 *(see [1]) For $0\leqslant p<\infty$, the spaces Q_p and $Q_{p,0}$ are defined by*

$$Q_p = \{f\in H(\mathbb{D}) : \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2\, g^p(z,a)\, dA(z) < \infty\},$$

$$Q_{p,0} = \{f\in H(\mathbb{D}) : \lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^2\, g^p(z,a)\, dA(z) = 0\}.$$

Proposition 1.1 *[see [1]] Let $0<p\leqslant 1$, and f be analytic in $\mathbb{D}$. Then, the following are equivalent:*

(I) $f\in Q_p \Leftrightarrow \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2\,(1-|\varphi_a(z)|)^p\, dA(z) < \infty$,

(II) $f\in Q_{p,0} \Leftrightarrow \lim_{|a|\to^{-1}} \int_{\mathbb{D}} |f'(z)|^2\,(1-|\varphi_a(z)|)^p\, dA(z) = 0.$

Definition 4 *(see [11]) Let $K:[0,\infty)\to[0,\infty)$, $\omega:(0,1]\to(0,\infty)$, are right-continuous and nondecreasing functions. If $0<p<\infty, -2<q<\infty$, then an analytic function f in $\mathbb{D}$ is said to belong to the space $Q_{K,\omega}(p,q)$ if*

$$\|f\|_{Q_{K,\omega(p,q)}} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p \,\frac{(1-|z|)^q}{\omega^p(1-|z|)}\, K(g(z,a))\, dA(z) < \infty.$$

Definition 5 *[see [17]] Let f be an analytic function in $\mathbb{D}$ and let $0<p<\infty$, $-2<q<\infty$ and $0<s<\infty$. If*

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p\,(1-|z|^2)^q\, g^s(z,a)\, dA(z) < \infty,$$

then $f\in F(p,q,s)$ Moreover, if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p\,(1-|z|^2)^q\, g^s(z,a)\, dA(z) = 0,$$

then $f\in F_0(p,q,s)$.

Now, we introduce the following definition of $Q_{p,\omega}$.

Definition 6 *Let $0\leqslant p<\infty$ and a right-continuous and nondecreasing function $\omega:(0,1]\to(0,\infty)$, the spaces $Q_{p,\omega}$ and $Q_{p,\omega,0}$ are defined by*

$$Q_{p,\omega}=\{f\in H(\mathbb{D}):\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^2\frac{g^p(z,a)}{\omega^2(1-|z|)}dA(z)<\infty\},$$

$$Q_{p,\omega,0}=\{f\in H(\mathbb{D}):\lim_{|a|\to 1^{-1}}\int_{\mathbb{D}}|f'(z)|^2\frac{g^p(z,a)}{\omega^2(1-|z|)}dA(z)=0\}.$$

Then, we state the following proposition without proof.

Proposition 1.2 *Let* $0<p\leqslant 1$, *and* f *be analytic in* $\mathbb{D}$. *Then, the following are equivalent:*

(I) $f\in Q_{p,\omega}\Leftrightarrow \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|\varphi_a(z)|)^p}{\omega^2(1-|z|)}dA(z)<\infty$,

(II) $f\in Q_{p,\omega,0}\Leftrightarrow \lim_{|a|\to^{-1}}\int_{\mathbb{D}}|f'(z)|^2\frac{(1-|\varphi_a(z)|)^p}{\omega^2(1-|z|)}dA(z)=0.$

Two quantities A_f and B_f, both depending on an analytic function f on $\mathbb{D}$, are said to be equivalent, written as $A_f\approx B_f$, if there exists a finite positive constant λ not depending on f such that for every analytic function f on $\mathbb{D}$ we have:

$$\frac{1}{\lambda}B_f\leqslant A_f\leqslant\lambda B_f.$$

If the quantities A_f and B_f, are equivalent, then in particular we have $A_f<\infty$ if and only if $B_f<\infty$.

1.2. Hyperbolic Function Spaces

Let $B(\mathbb{D})$ be a subset of $H(\mathbb{D})$ denote the classes of all the hyperbolic function classes in $\mathbb{D}$, such that $|f(z)|<1$. The hyperbolic function classes are usually defined by using either the hyperbolic derivative $f^*(z)=\frac{|f'(z)|}{1-|f(z)|^2}$ of $f\in B(\mathbb{D})$, or the hyperbolic distance $\rho(f(z),0):=\frac{1}{2}\log(\frac{1+|f(z)|}{1-|f(z)|})$ between $f(z)$ and zero.

Now, we give some definitions of different classes of the hyperbolic functions which recently have been studied intensively in the theory of complex function spaces.

Definition 7 *[see [10]] For* $0<\alpha<\infty$, *a function* $f\in B(\mathbb{D})$ *is said to belong to the hyperbolic* α *-Bloch class* $\mathcal{B}^*_\alpha$ *if*

$$\|f\|_{\mathcal{B}^*_\alpha}=\sup_{z\in\mathbb{D}}f^*(z)(1-|z|^2)^\alpha<\infty.$$

The little hyperbolic Bloch-type class $\mathcal{B}^*_{\alpha,0}$ consists of all $f\in\mathcal{B}^*_\alpha$ such that

$$\lim_{|z|\to 1}f^*(z)(1-|z|^2)^\alpha=0.$$

Note that when $\alpha=1,\mathcal{B}^*$ is the hyperbolic Bloch class and $\mathcal{B}^*_0$ is the little hyperbolic Bloch class [see [12]]. The Schwarz-Pick lemma implies $\mathcal{B}^*_\alpha=B(\mathbb{D})$ for all $\alpha\geqslant 1$ with $\|f\|_{\mathcal{B}^*_\alpha}\leqslant 1$.

Definition 8 *(see [10]) Let* $0\leqslant p<\infty$, *the hyperbolic class* Q^*_p *consists of those functions* $f\in B(\mathbb{D})$ *for which*

$$\|f\|^2_{Q^*_p}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(f^*(z))^2g^p(z,a)dA(z)<\infty.$$

Moreover, we say that $f\in Q^*_p$ belongs to the class $Q^*_{p,0}$ if

$$\lim_{|a|\to 1}\int_{\mathbb{D}}(f^*(z))^2g^p(z,a)dA(z)=0,$$

Also, we introduce the following definition.

Definition 9 *Let* $0<p\leq 1$ *and a right-continuous and nondecreasing function* $\omega:(0,1]\to(0,\infty)$, *the spaces* $Q^*_{p,\omega}$ *and* $Q^*_{p,\omega,0}$ *are defined by*

$$Q^*_{p,\omega}=\{f\in H(\mathbb{D}):\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^2\frac{g^p(z,a)}{\omega^2(1-|z|)}dA(z)<\infty\},$$

$$Q^*_{p,\omega,0}=\{f\in H(\mathbb{D}):\lim_{|a|\to 1^{-1}}\int_{\mathbb{D}}|f'(z)|^2\frac{g^p(z,a)}{\omega^2(1-|z|)}dA(z)=0\}.$$

1.3. Lacunary Series Expansions

Hadamard gaps are known to study some classes and spaces of analytic and hyperholomorphic functions. A wide variety of characterization not only in the type of function spaces, where functions are analytic and hyperholomorphic, but also in the coefficients which extend over Taylor or Fourier series expansions. It is one of the important tasks in the study of function spaces to seek for characterizations of functions by the help of their Taylor or Fourier series expansions.

The analytic function

$$f(z)=\sum_{k=1}^{\infty} a_k z^{n_k} \ \ (with \ \ n_k \in \mathbb{N}\,;\ for\ all \ \ k\in \mathbb{N}),$$

is said to belong to the Hadamard gap class (also known as lacunary series) if there exists a constant $\lambda > 1$ such that $\frac{n_{k+1}}{n_k} \geqslant \lambda$ for all $k \in \mathbb{N}$ [see [8, 18]].

In the past few decades, both Taylor and Fourier series expansions were studied by the help of Hadamard gap class, also called Hadamard's lacunarity condition [see [2, 4, 5, 6, 7, 8] and others].

Recently, we obtain a sufficient condition for functions to be in $Q_{K,\omega}(p,q)$ classes in terms of Taylor coefficients using the Hadamard gap class (see [6]) and gave the following result.

Theorem 1.3 *Let* $0<p<\infty,\ -1<q<\infty.$ *If*

$$f(z)=\sum_{k=0}^{\infty} a_k\, z^{n_k} \in H(\mathbb{D}),$$

has Hadamard gaps. Then the following statements are equivalent:

$(i)\quad f \in Q_{K,\omega}(p,q);$

$(ii)\quad f \in Q_{K,\omega,0}(p,q);$

$$(iii)\quad \sum_{k=0}^{\infty} n_k^{p-q-1}\,|\,a_k\,|^p \,\frac{K(\frac{1}{n_k})}{\omega^p(\frac{1}{n_k})} < \infty.$$

Miao (see [8]) studied a gap series with Hadamard condition as in the following theorems.

Theorem 1.4 *Let* $0<p<\infty.$ *If* $f(z)=\sum_{k=1}^{\infty} a_k z^{n_k}$ *is analytic on* $\mathbb{D}$ *and has Hadamard gaps, that is, if*

$$\frac{n_{k+1}}{n_k} \geqslant \lambda > 1, \quad (k=1,2,...),$$

then the following statements are equivalent:

(I) $\ f\in B^p;\quad (II)\ f\in B_0^p;\quad (III)\sum_{k=1}^{\infty} |\,a_k\,|^p<\infty.$

Theorem 1.5 *(see [14], theorem 1) For* $0<\alpha.$ *If* $f(z)=\sum_{k=1}^{\infty} a_k z^{n_k}$ *is analytic on* $\mathbb{D}$ *and has Hadamard gaps, then the following two propositions hold.*

(I) $f\in \mathcal{B}^{\alpha}$ if and only if

$$\limsup_{k\to\infty} |\,a_k\,|\, n_k^{1-\alpha} < \infty. \tag{1}$$

(II) $f\in \mathcal{B}^{\alpha}$ if and only if

$$\limsup_{k\to\infty} |\,a_k\,|\, n_k^{1-\alpha} = 0. \tag{2}$$

Ruhan Zhao in [see [17]] studied a gap series with Hadamard condition in the following theorem.

Theorem 1.6 *Let* $0<p<\infty,\ -2<q<\infty,\ 0\leqslant s\leqslant 1,$ *and* $q+s>-1.$ *Suppose that* $f(z)=\sum_{k=0}^{\infty} a_k z^{n_k}$ *is analytic in* $\mathbb{D}$ *and has Hadamard gaps, that is, if*

$$\frac{n_{k+1}}{n_k} \geqslant \lambda > 1, \quad (k=1,2,\ldots).$$

Then, the following statements are equivalent:

(I) $f\in F(p,q,s);\qquad (II)\ f\in F_0(p,q,s);\qquad (III)\ \sum_{k=0}^{\infty} n_k^{p-q-s-1}\,|\,a_k\,|^p<\infty.$

2. Preliminaries

We will need the following lemmas in the sequel:

Lemma 2.1 *Let* $0<p<\infty.$ *If* $\{n_k\}$ *is an increasing sequence of positive integers satisfying* $\frac{n_{k+1}}{n_k} \geqslant \lambda > 1$ *for all* $k,$ *then there is a constant* A *depending only on* p *and* λ *such that*

$$A^{-1}(\sum_{k=1}^{\infty}|a_k|^2)^{\frac{1}{2}}\leqslant(\frac{1}{2\pi}\int_0^{2\pi}|\sum_{k=1}^{\infty}a_k e^{in_k\theta}|^p\, d\theta)^{\frac{1}{p}}\leqslant A(\sum_{k=1}^{\infty}|a_k|^2)^{\frac{1}{2}}$$

for any number $a_k\,(k=1,2,...)$.

The above lemma was due to Zygmund [see [18]].

Theorem 2.2 *Let*

$$f(r)=\sum_{n=1}^{\infty}a_n r^n,$$

with $a_n\geqslant 0$. If $\alpha>0,\ p>0$ and $\omega:(0,1]\to(0,\infty)$, then

$$\int_0^1(1-r)^{\alpha-1}(f(r))^p\frac{1}{\omega^p(\log\frac{1}{r})}dr\approx\sum_{n=0}^{\infty}2^{-n\alpha}\,t_n^p\,\frac{1}{\omega^p(\frac{1}{2^n})}\ , \tag{3}$$

where $t_n=\sum_{k\in I_n}a_k,\ n\in\mathbb{N},\ \ I_n=\{k:2^n\leqslant k<2^{n+1};\ k\in\mathbb{N}\}$.

The proof of this theorem can be obtained easily from Theorem 2.1 in [6] with the same steps, so it will be omitted.

We can see that Theorem 2.2 is still satisfied for the function $f(r)=\sum_{n=1}^{\infty}a_n r^{n-1}$.

Lemma 2.3 *[see [1]] Let* $0<p\leqslant 1$, $a\in\mathbb{D}$ *and* $z=re^{i\theta}\in\mathbb{D}$,

$$\int_0^{2\pi}\frac{d\theta}{|1-\bar{a}re^{i\theta}|^{2p}}\leqslant\frac{C}{(1-|a|r)^p},$$

where $C>0$ is a constant.

For our purpose we will use the following inequalities, which follow immediately from Holder's inequality. Let $a_n\geqslant 0$ and let N be a positive integer.

Then for $0<p\leqslant 1$,

$$\frac{1}{N^{1-p}}(\sum_{n=1}^{N}a_n^p)\leqslant(\sum_{n=1}^{N}a_n)^p\leqslant(\sum_{n=1}^{N}a_n^p); \tag{4}$$

for $1\leqslant p<\infty$,

$$(\sum_{n=1}^{N}a_n^p)\leqslant(\sum_{n=1}^{N}a_n)^p\leqslant N^{p-1}(\sum_{n=1}^{N}a_n^p). \tag{5}$$

3. Main Results

In this section, we obtain characterizations of the hyperbolic general family of function spaces $Q^*_{p,\omega}$ and $Q^*_{p,\omega}$ by the coefficients of certain lacunary series expansions in the unit disc. Now, we give the following theorem:

Theorem 3.1 *Let* $0<p\leqslant 1$, *and* $I_n=\{k:2^n\leqslant k<2^{n+1},k\in\mathbf{N}\}$. *Let* $f(z)=\sum_{n=0}^{\infty}a_n z^n$ *be analytic function on* $\mathbb{D}$.

Let

$$(\sum_{n=0}^{\infty}|a_n|z^n)^n\leqslant C\sum_{k=0}^{\infty}|a_n|z^n, \tag{6}$$

If

$$\sum_{n=0}^{\infty}2^{-n(p+1)}(\frac{\sum_{k\in I_n}|d_k|}{\omega(\frac{1}{2^n})})^2<\infty,$$

where

$$d_k=\sum_{k=0}^{n}k|a_k||b_{n-k}|,\quad b_n=\sum_{j=0}^{n}|a_j||a_{n-j}|.$$

Then $f\in Q^*_{p,\omega}$, and $f\in Q^*_{p,\omega,0}$.

Proof:

Let

$$I=\int_{\mathbb{D}}(f^*(z))^2\frac{(1-|\varphi_a(z)|^2)^p}{\omega^2(1-|z|)}dA(z).$$

From [see [14, 16]], we have

$$\sum_{n=0}^{\infty}|z|^n=\frac{1}{(1-|z|)},\qquad\sum_{n=0}^{\infty}(n+1)^{\alpha}|z|^n\approx\frac{C}{(1-|z|)^{\alpha+1}},$$

where C positive constant. Since $f^*(z)=\dfrac{|f'(z)|}{1-|f(z)|^2}$, we have

$$f^*(z)\leqslant\sum_{n=1}^{\infty}n|a_n||z|^{n-1}(\sum_{n=0}^{\infty}|a_n||z|^n)^{2n}. \tag{7}$$

Using Cushy product, we have

$$(\sum_{n=0}^{\infty}|a_n||z|^n)^2=\sum_{n=0}^{\infty}|a_n||z|^n(\sum_{n=0}^{\infty}|a_n||z|^n)$$

$$=\sum_{n=0}^{\infty}(\sum_{j=0}^{n}|a_j||a_{n-j}|)|z|^n$$

$$=\sum_{n=0}^{\infty}|b_n||z|^n. \tag{8}$$

Then, from above and using (6) and (8), we deduce that

$$f^*(z)\leqslant C\sum_{n=1}^{\infty}n|a_n||z|^{n-1}(|z|\sum_{n=0}^{\infty}|b_n||z|^{n-1})$$

$$\leqslant C_1\sum_{n=1}^{\infty}(\sum_{k=0}^{n}k|a_k||b_{n-k}|)|z|^{n-1}$$

$$\leqslant C_1\sum_{n=1}^{\infty}|d_n||z|^{n-1}. \tag{9}$$

From Lemma 2.3, we have

$$I\leqslant C_1\int_{\mathbb{D}}(\sum_{n=1}^{\infty}|d_n||z|^{n-1})^2\frac{(1-|z|^2)^p(1-|a|^2)^p}{|1-\bar{a}z|^{2p}}\frac{1}{\omega^2(1-|z|)}dA(z)$$

$$\leqslant C_1\int_{\mathbb{D}}(\sum_{n=1}^{\infty}|d_n||z|^{n-1})^2(1-|z|^2)^p\frac{(1-|a|^2)^p}{\omega^2(1-|z|)}\frac{1}{|1-\bar{a}z|^{2p}}dA(z)$$

$$\leqslant C_1\int_0^1(\sum_{n=1}^{\infty}|d_n|r^{n-1})^2\frac{(1-|a|^2)^p}{\omega^2(1-r)}(1-r^2)^p(\int_0^{2\pi}\frac{d\theta}{|1-\bar{a}re^{i\theta}|^{2p}})rdr$$

$$\leqslant MC_1\int_0^1(\sum_{n=1}^{\infty}|d_n|r^{n-1})^2(1-r^2)^p\frac{(1-|a|^2)^p}{\omega^2(1-r)}(1-|a|r)^p rdr$$

$$\leqslant 2^{2p}MC_1\int_0^1(\sum_{n=1}^{\infty}|d_n|r^{n-1})^2\frac{(1-r)^p}{\omega^2(1-r)}\frac{(1-|a|)^p}{(1-|a|r)^p}rdr. \tag{10}$$

Where,

$$1-r\leqslant 1-|a|r\leqslant 1+r,\quad 1-|a|\leqslant 1-|a|r\leqslant 1+|a|\leqslant 2. \tag{11}$$

Then, we get

$$I \leqslant 2^{2p} MC_2 \int_0^1 (\sum_{n=1}^{\infty} |d_n| r^{n-1})^2 \frac{(1-r)^p}{\omega^2(1-r)} dr. \quad (12)$$

Using Theorem 2.2, we obtain

$$\int_{\mathbb{D}} (f^*(z))^2 \frac{(1-|\varphi_a(z)|^2)^p}{\omega^2(1-|z|)} dA(z)$$

$$\leqslant 2^{2p} MC_2 \sum_{n=0}^{\infty} 2^{-n(p+1)} (\frac{t_n}{\omega(\frac{1}{2^n})})^2,$$

where

$$t_n = \sum_{k \in I_n} |d_k|.$$

Then we get

$$\|f\|^2_{Q^*_{p,\omega}} = \int_{\mathbb{D}} (f^*(z))^2 \frac{(1-|\varphi_a(z)|^2)^p}{\omega^2(1-|z|)} dA(z)$$

$$\leqslant 2^p MC_2 \sum_{n=0}^{\infty} 2^{-n(p+1)} (\frac{\sum_{k \in I_n} |d_k|}{\omega(\frac{1}{2^n})})^2 < \infty,$$

that is, $f \in Q^*_{p,\omega}$.

To prove that $f \in Q^*_{p,\omega,0} \subset Q^*_{p,\omega}$. If $p = 0$, then $Q^*_{p,\omega,0} = Q^*_{p,\omega}$.

If $0 < p \leqslant 1$, we note that the integral

$$\int_{\delta}^1 (\sum_{n=1}^{\infty} |d_n| r^{n-1})^2 \frac{(1-r)^p}{\omega^2(1-r)} dr \quad (13)$$

is convergent, for

$$\sum_{n=0}^{\infty} 2^{-n(p+1)} (\frac{\sum_{k \in I_n} |d_k|}{\omega(\frac{1}{2^n})})^2 < \infty.$$

Then, for any $\varepsilon > 0$, there is a $\delta \in (0,1)$ such that

$$\int_{\delta}^1 (\sum_{n=1}^{\infty} |d_n| r^{n-1})^2 \frac{(1-r)^p}{\omega^2(1-r)} dr < \varepsilon.$$

Then,

$$I \leqslant C_1 \int_{\mathbb{D}} (\sum_{n=1}^{\infty} |d_n| |z|^{n-1})^2 \frac{(1-|z|^2)^p (1-|a|^2)^p}{|1-\bar{a}z|^{2p}} \frac{1}{\omega^2(1-|z|)} dA(z)$$

$$\leqslant 2^{2p} MC_1 \int_0^1 (\sum_{n=1}^{\infty} |d_n| r^{n-1})^2 \frac{(1-r)^p}{\omega^2(1-r)} \frac{(1-|a|)^p}{(1-|a|r)^p} rdr$$

$$\leqslant 2^{2p} MC_2 \int_0^{\delta} (\sum_{n=1}^{\infty} |d_n| r^{n-1})^2 \frac{(1-r)^p}{\omega^2(1-r)} \frac{(1-|a|)^p}{(1-|a|r)^p} dr + 2^{2p} MC_2 \varepsilon$$

$$\leqslant 2^{2p} MC_2 \frac{(1-|a|)^p}{(1-\delta)^p} \int_0^1 (\sum_{n=1}^{\infty} |d_n| r^{n-1})^2 (\sum_{n=1}^{\infty} |d_n| r^{n-1})^2 \frac{(1-r)^p}{\omega^2(1-r)} dr + 2^{2p} MC_2 \varepsilon$$

$$\leqslant 2^{2p+1} MC_2 \varepsilon.$$

If $1-|a|$ may be sufficiently small, hence

$$\lim_{|a| \to 1^-} \int_{\mathbb{D}} (f^*(z))^2 \frac{(1-|\varphi_a(z)|^2)^p}{\omega^2(1-|z|)} dA(z) = 0.$$

According to the definition of the $Q^*_{p,\omega}$ space, we deduce that $f \in Q^*_{p,\omega,0}$. This completes the proof.

Corollary 3.2 *For* $0 < p \leqslant 1$, *and let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *be analytic function on* $\mathbb{D}$ *with Hadamard gaps. If*

$$(\sum_{n=0}^{\infty} |a_n| z^n)^n \leqslant C \sum_{n=0}^{\infty} |a_n| z^n,$$

If

$$\sum_{n=0}^{\infty} 2^{-n(p+1)} (\sum_{k \in I_n} |d_k|)^2 < \infty,$$

where

$$d_n = \sum_{k=0}^{n} k |a_k| |b_{n-k}|, \quad b_n = \sum_{j=0}^{n} |a_j| |a_{n-j}|.$$

Then $f\in Q^*_{p,0}$.

Theorem 3.3 *Let* $0<p\leqslant 1$. *Suppose that* $f(z)=\sum_{k=1}^{\infty}a_k z^{n_k}$ *is analytic in* $\mathbb{D}$ *and has Hadamard gaps, that is, if*

$$\frac{n_{k+1}}{n_k}\geqslant\lambda>1,\quad (k=1,2,\ldots).$$

Then, the following statements are equivalent:

(I) $f\in Q^*_{p,\omega}$; (II) $f\in Q^*_{p,\omega,0}$; (III) $\sum_{k=0}^{\infty}2^{-k(p+1)}\left(\frac{\sum_{j\in I_k}|d_j|^2}{\omega^2(\frac{1}{2^k})}\right)<\infty.$

Where

$$Q^*_{p,\omega,0}=\lim_{|a|\to 1^-}\int_{\mathbb{D}}(f^*(z))^2\frac{g^p(z,a)}{\omega^2(1-|z|)}dA(z)=0.$$

Since $Q^*_{p,\omega,0}\subset Q^*_{p,\omega,0}$. It is clear that (II) implies (I). We first prove that (III) follows from (I). Applying Lemma 2.1, Theorem 2.2 and using (9), we have

$$\|f\|^2_{Q^*_{p,\omega}}\geqslant\int_{\mathbb{D}}(f^*(z))^2\frac{(1-|z|^2)^p}{\omega^2(1-|z|)}dA(z)$$

$$=\int_{\mathbb{D}}\Big(\sum_{k=1}^{\infty}|d_k||z|^{n_k-1}\Big)^2\frac{(1-|z|^2)^p}{\omega^2(1-|z|)}dA(z)$$

$$\geqslant\frac{2\pi}{A^2}\int_0^1\Big(\sum_{k=1}^{\infty}|d_k|^2 r^{2(n_k-1)}\Big)\frac{(1-r^2)^p}{\omega^2(1-r)}rdr.$$

Then, we obtain

$$\|f\|^2_{Q^*_{p,\omega}}\geqslant\frac{2\pi}{A^2}\int_0^1\Big(\sum_{k=1}^{\infty}|d_k|^2 r^{2(n_k-1)}\Big)\frac{(1-r^2)^p}{\omega^2(1-r)}rdr$$

$$\geqslant\frac{\pi}{A^2}\int_0^1\Big(\sum_{k=1}^{\infty}|d_k|^2 x^{(n_k-1)}\Big)\frac{(1-x)^p}{\omega^2(1-\sqrt{x})}dx.\qquad(14)$$

Then, we deduced that

$$\|f\|^2_{Q^*_{p,\omega}}\geqslant\frac{\pi}{A^2}\sum_{k=1}^{\infty}2^{-k(p+1)}t_k\frac{1}{\omega^2(\frac{1}{2^k})},$$

where

$$t_k=\sum_{j\in I_k}|d_j|^2.$$

Then, we have

$$\|f\|^2_{Q^*_{p,\omega}}\geqslant\frac{\pi}{A^2}\sum_{k=0}^{\infty}2^{-k(p+1)}\left(\frac{\sum_{j\in I_k}|d_j|^2}{\omega^2(\frac{1}{2^k})}\right).$$

Combining the above inequalities yields that (III).

To prove that (II) follows from (III). Assuming that $\sum_{k=0}^{\infty}2^{-k(p+1)}\left(\frac{\sum_{j\in I_k}|d_j|}{\omega(\frac{1}{2^k})}\right)^2<\infty$ and $\frac{n_{k+1}}{n_k}\geqslant\lambda>1$ for all k. Then, by (4) and (5) we have,

$$\sum_{k=0}^{\infty}2^{-k(p+1)}\left(\frac{\sum_{j\in I_k}|d_j|}{\omega(\frac{1}{2^k})}\right)^2\leqslant C_3\sum_{k=0}^{\infty}2^{-k(p+1)}\frac{\sum_{j\in I_k}|d_j|^2}{\omega^2(\frac{1}{2^k})}<\infty.$$

Thus, by Theorem 3.1 $f\in\|f\|^2_{Q^*_{p,\omega,0}}$, and the proof is complete.

Corollary 3.4 *Let* $0<p\leqslant 1$, *and* $f(z)=\sum_{k=1}^{\infty}a_k z^{n_k}$ *is analytic in* $\mathbb{D}$. *Then, the following statements are equivalent:*

(I) $f\in Q^*(p)$; (II) $f\in Q^*_{p,0}$; (III) $\sum_{k=0}^{\infty}2^{-k(p+1)}(\sum_{j\in I_k}|d_j|^2)<\infty.$

4. Conclusion

Characterization for the hyperbolic general family of function spaces $Q^*_{p,\omega}$ using integral representation of lacunary series expansions are given. Moreover, we give a sufficient and necessary condition for the hyperbolic function f^* with Hadamard gaps to belong to $Q^*_{p,\omega}, Q^*_{p,\omega}$.

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