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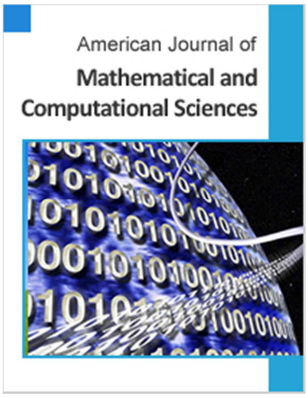

Another Modelling of α-Continuous Multifunctions

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Abstract
In this paper, some new characterization of upper (lower) α-irresolute multifunctions due to Neubrunn and each of Noiri and Nasef will be established. Also, other properties have been presented and some results in are improved. The relationships between upper α-irresolute functions and other related multifunctions are also investigated.

1. Introduction

In 1986, Neubrunn [1] extended the concepts of α-continuous functions (Mashhour et al. [2]) and α-irresolute functions (Maheshwari and Thakur [3]) to multifuntions. Recently, many authors studied some continuity properties of set multifunctions [4, 5]. Noiri and Nasef [6] obtained several new characterizations and properties of upper (lower) α-irresolute multifunctions. Also, Nasef [7] introduced and studied some properties of α-continuity between topological spaces. The purpose of the present paper is devoted to present other new characterizations upper (lower) α-irresolute multifunctions. Moreover, some results in [6] are improved and the relationships between α-irresolute multifunctions and other corresponding ones are discused.

2. Preliminaries

The topological spaces or simply spaces which will be used are (X,τ) and (Y,σ) (or simply X and Y for shortly) without any separation axioms and whenever such properties are needed they will be explicitly assumed. By $f:(X,\tau)\to(Y,\sigma)$ (or simply $f:X\to Y$ will represent a multivalued function. for a multifunction $F:(X,\tau)\to(Y,\sigma)$, the upper and the lower inverse of a subset G of Y are denoted by $F^{+}(G)$ and $F^{-}(G)$ respectively. That is, $F^{+}(G)=\{x\in X: F(x)\subseteq G\}$ and $F^{-}(G)=\{x\in X: F(x)\cap G\neq\varphi\}$. In particular, $F^{-}(y)=\{x\in X: y\in F(x)\}$ for each point $y\in Y$. For each $A\subset X$, $F(A)=\bigcup\limits_{x\in A}F(x)$. Then F is said to be a surjection if $F(X)=Y$, or equivalently, if for each $y\in Y$, there exists $x\in X$ such that $y\in F(x)$. If A is a subset of X, the closure and the interior of A with respect to τ are denoted by $\tau-Cl(A)$ and $\tau-Int(A)$, respectively. A subset A of X is said to be α-open [8] (resp. semiopen [9], preopen [10], β-open [11]) if $A\subset\tau-Int(\tau-Cl(\tau-Int(A)))$ (resp. $A\subseteq\tau-Cl(\tau-Int(A))$, $A\subseteq\tau-Int(\tau-Cl(A))$, $A\subseteq\tau-Cl(\tau-Int(\tau-Cl(A))$). The family of all α-open sets of X containing a point $x\in X$ is denoted by $\alpha(X,x)$.

The family of all α-open (resp. semiopen, preopen) sets in (X,τ) is donoted by $\alpha O(X,\tau)$ or τ^{α} (resp. $SO(X,\tau)$, $PO(X,\tau)$).

The complement of an α-open (resp. semiopen, preopen, β-open) set is said to be α-closed (resp. semiclosed, preclosed, β-closed). Since $\alpha O(X,\tau)$ is a topology for X [[8] Proposition 2], by $\alpha Cl(A)$ (resp. $\alpha Int(A)$), we denote the closure (resp. interior) of A with respect to $\alpha O(X)$. A subset A of X is called an α-neighborhood of a point x of X if there exists $U\in \alpha O(X)$ such that $x\in A\subseteq U$. Following Noiri and Nasef [6], a multifunction $F:(X,\tau)\to (Y,\sigma)$ is said to be upper α- irresloute (resp. lower α-irresolute) at a point $x\in X$ if for each α-open set V such that $F(U)\subseteq V$ (resp. $F(u)\cap V\neq \varphi$ for every $u\in U$). If F is upper α- irresolute (resp. lower α -irresolute) at all points of its domain, then it is called upper α-irresolute (resp. lower α-irresolute).

3. Characterizations

Several characterizations of upper and lower α-irresolute functions have been given in [1, 6, 12] and we show a bit more.

Definition 3.1 A multifunction $F:(X,\tau)\to (Y,\sigma)$ is called α- irresolute at a point $x\in X$ if for each pair $W_i\in \alpha O(Y,\sigma)$, $i=1,2$ such that $F(x)\subseteq W_1$ and $F(x)\cap W_2\neq\varphi$, there exists $H\in \alpha(X,x)$ with $F(H)\subseteq W_1$ such that $F(h)\cap W_2\neq\varphi$ for every $h\in H$.

Therefore, a multifunction $F:(X,\tau)\to (Y,\sigma)$ is called α- irresolute if it has the above property at each point $x\in X$.

Proposition 3.2 Any α -irresolute multifunction $F:(X,\tau)\to (Y,\sigma)$ at any point $x\in X$ is both upper and lower α-irresolute at the same point.

Theorem 3.3 The following are equivalent for a multifunction $F:(X,\tau)\to (Y,\sigma)$:

(1) F is an α-irresolute at a point $x\in X$;

(2) For any $W_1,W_2\in \alpha O(Y,\sigma)$ such that $F(x)\subset W_1$ and $F(x)\cap W_2\neq\varphi$, we have $x\in \tau-Int(\tau-Cl(\tau-Int[F^+(W_1)\cap F^-(W_2)]))$;

(3) For each $W_1,W_2\in \alpha O(Y,\sigma)$ having $F(x)\subset W_1$, $F(x)\cap W_2\neq\varphi$ and for any open set $U\subseteq X$ containing x, there exists a nonempty open set G of X with $G\subseteq U$, $F(G)\subset W_1$ and $F(g)\cap W_2\neq\varphi$ for each $g\in G$.

Proof. $(1)\Rightarrow(2)$: Let $W_i\in \alpha O(Y,\sigma)$, $i=1,2$ with $F(x)\subset W_1$ and $F(x)\cap W_2\neq\varphi$. By hypothesis, there exists $H\in \alpha O(Y,x)$ such that $F(H)\subset W_1$ and $F(h)\cap W_2\neq\varphi$ for each $h\in H$. Thus $x\in H\subseteq F^+(W_1)$ and $x\in H\subseteq F^-(W_2)\neq\varphi$. Hence $x\in H\subseteq F^+(W_1)\cap F^-(W_2)$. Since H is an α -open in X, then $x\in H\subseteq \tau-Int(\tau-Cl(\tau-Int(H)))\subseteq \tau-Int(\tau-Cl(\tau-Int[F^+(W_1)\cap F^-(W_2)]))$.

$(2)\Rightarrow(3)$: Let $W_i\in \alpha O(Y,\sigma)$, $i=1,2$ with $F(x)\subseteq W_1$ and $F(x)\cap W_2\neq\varphi$. But (b) gives $x\in \tau-Int(\tau-Cl(\tau-Int[F^+(W_1)\cap F^-(W_2)]))$. Also, let a nonempty open set U containing x, then $U\cap[\tau-Int[F^+(W_1)\cap F^-(W_2)]]\subseteq U\cap[\tau-IntF^+(W_1)\cap\tau-IntF^-(W_2)]=G$ which is open and $G\subseteq \tau-IntF^+(W_1)$ and also $G\subseteq \tau-IntF^-(W_2)\subseteq F^-(W_2)$. Therefore $F(G)\subseteq W_1$ and $F(g)\cap W_2\neq\varphi$ for each $g\in G$.

$(3)\Rightarrow(1)$: Follows immediately from the observation $\tau(x)\subseteq \alpha(X,x)$.

Lemma 3.4 (Andrijvic′ [13]) For any subset A of a space (X,τ), $\alpha Cl(A)=A\cup\tau-Cl(\tau-Int(\tau-Cl(A)))$.

Theorem 3.5 The following are equivalent for a multifunction $F:(X,\tau)\to (Y,\sigma)$:

(1) F is an α-irresolute;

(2) For any pair $W_1,W_2\in \alpha O(Y,\sigma)$, $F^+(W_1)\cap F^-(W_2)\in \alpha O(X,\tau)$;

(3) For each α -closed sets $K_1,K_2\subseteq Y$, $F^-(K_1)\cup F^+(K_2)$ is an α-closed set in X;

(4) for every sets $B_1,B_2\subseteq Y$, $\tau-Cl(\tau-Int(\tau-Cl[F^-(B_1)\cup B^+(B_2)]))\subseteq F^-(\alpha Cl(B_1)\cup F^+(\alpha Cl(B_2))$;

(5) $\alpha Cl[F^-(B_1)\cup F^+(B_2)]\subseteq F^-(\alpha Cl(B_1)\cup F^+(\alpha Cl(B_2)$ for any subsets $B_1,B_2\subseteq Y$;

(6) $F^-(\alpha Int(B_1)\cap F^+(\alpha Int(B_2)\subseteq \alpha Int[F^-(B_1)\cap F^+(B_2)]$ for any subsets $B_1,B_2\subseteq Y$;

(7) for any point $x\in X$ and for each α- neighborhood N of $F(x)$, then every $W\in \alpha O(Y,\sigma)$ such that $W\cap F(x)\neq\varphi$, $F^+(N)\cap F^-(W)$ is an α -neighborhood of x;

(8) for any point $x\in X$ and for each α- neighborhood N of $F(x)$, then every $W\in \alpha O(Y,\sigma)$ such that $W\cap F(x)\neq\varphi$. There is α- neighborhood U of x such that $F(U)\subseteq N$ and $F(u)\cap W\neq\varphi$ for each $u\in U$.

Proof. $(1)\Rightarrow(2)$: Let $x\in F^+(W_1)\cap F^-(W_2)$ for any $W_1,W_2\in \alpha O(Y,\sigma)$, thus $F(x)\in W_1$ and $F(x)\cap W_2\neq\varphi$. Since F is α -irresolute, then Theorem 3.3, gives $x\in \tau-Int(\tau-Cl(\tau-Int[F^+(W_1)\cap F^-(W_2)]))$.

$(2)\Rightarrow(3)$: Follows immediately from the fact if $V\subseteq Y$, then $F^-(Y-V)=X-F^+(V)$ and $F^+(Y-V)=X-F^-(V)$.

$(3)\Rightarrow(4)$: Let B_1,B_2 be any subsets of Y. Then

$\alpha Cl(B_i)$, $i=1,2$ are α-closed sets in Y. By (c), $F^-(\alpha Cl(B_1))\cup F^+(\alpha Cl(B_2))$ is also α-closed sets in (X,τ), i.e. $\tau-Cl(\tau-Int(\tau-Cl[F^-(\alpha Cl(B_1)\cup F^+(\alpha Cl(B_2)]))\subseteq F^-(\alpha Cl(B_1)\cup F^+(\alpha Cl(B_2))$.

Since $F^+(B_2)\subseteq F^+(\alpha Cl(B_2)$ and $F^-(B_1)\subseteq F^-(\alpha Cl(B_1)$.

Consequently, $\tau-Cl(\tau-Int(\tau-Cl[F^-(B_1)\cup F^+(B_2)]))\subseteq\tau-Cl(\tau-Int(\tau-Cl[F^-(\alpha Cl(B_1)\cup F^+(\alpha Cl(B_2)]))\subseteq F^-(\alpha Cl(B_1)\cup F^+(\alpha Cl(B_2))$

$(4)\Rightarrow(5)$: Follows immediately from Lemma 3.4.

$(5)\Rightarrow(6)$:

$X-\alpha Int[F^-(B_1)\cap F^+(B_2)]\subseteq\alpha Cl[X-(F^-((B_1)\cap F^+(B_2)]$
$=\alpha Cl[(X-F^-(B_1))\cup(X-F^+(B_2))]$
$=\alpha Cl[F^+(Y-B_1)\cup F^-(Y-B_2)\subseteq F^+(\alpha Cl(Y-B_1))\cup F^-(\alpha Cl(Y-B_2))$
$=F^+(Y-\alpha Int(B_1))\cup F^-(Y-\alpha Int(B_2))$
$=(X-F^-(\alpha Int(B_1))\cup(X-F^+(\alpha Int(B_2))$
$=X-[F^-(\alpha Int(B_1))\cap F^+(\alpha Int(B_2))]$

and thus

$\alpha Int[F^-(B_1)\cap F^+(B_2)]\supseteq[F^-(\alpha Int(B_1)\cap F^+(\alpha Int(B_2)]$.

$(6)\Rightarrow(7)$: Let $x\in X$, N be an α-neighborhood of $F(x)$ and W is an α-open set of Y with $F(x)\cap W\neq\varphi$, then there exists two α-open sets U_1 and U_2 such that $U_1\subseteq N$ and $U_2\subseteq W$, $F(x)\subseteq U_1$ and $F(x)\cap U_2\neq\varphi$. Thus $x\in F^+(U_1)\cap F^-(U_2)$. By hypothesis,

$x\in F^+(U_1)\cap F^-(U_2)=F^+(\alpha Int(U_1))\cap F^-(\alpha Int(U_2))$
$\subseteq\alpha Int[F^+(U_1)\cap F^-(U_2)]\subseteq$
$\alpha Int[F^+(N)\cap F^-(W)]\subseteq F^+(N)\cap F^-(W)$.

From the above it follows that $F^+(N)\cap F^-(W)$ is an α-neighborhood of x.

$(8)\Rightarrow(8)$: Let $x\in X$, N be α-neighborhood of $F(x)$ and $W\in\alpha O(Y,\sigma)$ with $F(x)\cap W\neq\varphi$, then $U=F^+(N)\cap F^-(W)$ is an α-neighborhood of x, $F(U)\subseteq N$ and $F(u)\cap W\neq\varphi$ for every $u\in U$.

$(8)\Rightarrow(1)$: It is clear from the hypothesis and this completes the proof.

The following characterizations of upper α-irresolute and lower α-irresolute are due to Noiri and Nasef [6].

Theorem 3.6 The following are equivalent for a multifunction $F:(X,\tau)\to(Y,\sigma)$:

(1) F is upper α-irresolute (resp. lower α-irresolute);
(2) $F^+(W)$ (resp. $F^-(W)$) is an α-open in X for each $W\in\alpha O(Y,\sigma)$;
(3) $F^-(K)$ (resp. $F^+(K)$) is an α-closed in X for each α-closed set K of Y;
(4) $sInt(Cl(F^-(B)))\subseteq F^-(\alpha Cl(B))$ (resp. $sInt(Cl(F^+(B)))\subseteq F^+(\alpha Cl(B))$ for each subset B of Y;
(5) $\alpha Cl(F^-(B))\subseteq F^-(\alpha Cl(B))$ (resp. $\alpha Cl(F^+(B))\subseteq F^+(\alpha Cl(B))$ for each subset B of Y.

Theorem 3.7 The following are equivalent for a multifunction $F:(X,\tau)\to(Y,\sigma)$:

(1) F is lower α-irresolute;
(2) $F(\tau-Cl(\tau-Int(\tau-Cl(H))))\subseteq F(H)$ for each $H\in\alpha O(X,\tau)$;
(3) $F(\alpha Cl(H))\subseteq F(H)$ for each $H\in\alpha O(X,\tau)$.

Proof. $(1)\Leftrightarrow(2)$: Follows by the equivalence of (a) and (e) of Theorem 3.6 and by considering $W=F(H)$.

$(2)\Rightarrow(3)$: Follows by using Lemma 3.4.

$(3)\Rightarrow(1)$: Let $x\in X$ and W be any α-open set of Y having $F(x)\cap W\neq\varphi$. Then $x\in F^-(W)$. By (c), $F(\alpha Cl(F^+(Y-W)))\subseteq F(F^+(Y-W))\subseteq Y-W$. Therefore, $\alpha Cl(F^+(Y-W))\subseteq F^+(Y-W))$. This shows that $F^+(Y-W)$ is α-closed in (X,τ). This implies $F^-(W)$ is α-open. Set $H=F^-(W)$, $H\in\alpha(X,x)$ and $F(h)\cap W\neq\varphi$ for every $h\in H$. Hence F is lower α-irresolute.

Lemma 3.8 *(Popa and Noiri [14]) If $F:X\to Y$ is a multifunction, then for each open set V of Y,* $(\alpha ClF)^-(V)=F^-(V)$.

Proof. Let V be any open set of Y and $x\in(\alpha ClF)^-(V)$. Then $(\alpha ClF)(x)\cap V=\alpha Cl(F(x))\cap V\neq\varphi$ and hence $F(x)\cap V=\neq\varphi$. Since V is open, then we obtain $x\in F^-(V)$ and hence $(\alpha ClF)^-(V)\subseteq F^-(V)$. Conversely, let $x\in F^-(V)$. Then $\varphi\neq F(x)\cap V\subseteq(\alpha ClF)(x)\cap V$ and hence $x\in(\alpha ClF)^-(V)$. Thus we have $F^-(V)\subseteq(\alpha ClF)^-(V)$. Consequently, we obtain $(\alpha ClF)^-(V)=F^-(V)$.

Theorem 3.9 *A multifunction $F:X\to Y$ is lower α-irresolute if and only if $\alpha ClF:X\to Y$ is lower α-irresolute.*

Proof. Necessity, suppose that F is lower α-irresolute. Let $x\in X$ and V be any open set of Y with $(\alpha ClF)(x)\cap V\neq\varphi$. By Lemma 3.8, we have $x\in(\alpha ClF)^-(V)=F^-(V)$ and hence $F(x)\cap V\neq\varphi$. Since F is lower α-irresolute, there exists $U\in\alpha(X,x)$ such that $U\subseteq F^-(V)=(\alpha ClF)^-(V)$. Therefore αClF is lower α-irresolute.

Sufficiency, suppose that αClF is lower α-irresolute, $x\in X$ and let V be any open set of Y with $F(x)\cap V\neq\varphi$. By Lemma 3.8, we have $x\in F^-(V)=(\alpha ClF)^-(V)$. Since

αClF is lower α-irresolute, there exists $U \in \alpha(X,x)$ such that $U \subseteq (\alpha ClF)^{-}(V) = F^{-}(V)$. Therefore F is lower α-irresolute.

4. Some Miscellaneous Results

The following lemma was shown by Mashhour el al. [2] and Rielly and Vamanamurthly [15].

Lemma 4.1 Let A and B be subsets of a topological space X,

(1) If $A \in SO(X) \cup PO(X)$ and $B \in \alpha O(X)$, then $A \cap B \in \alpha O(A)$;

(2) If $A \subseteq B \subseteq X$, $A \in \alpha O(B)$ and $B \in \alpha O(X)$, then $A \in \alpha O(X)$.

Theorem 4.2 If a multifunction $F:(X,\tau) \to (Y,\sigma)$ is upper α-irresolute (resp. lower α-irresolute) and $X_0 \in PO(X,\tau) \cup SO(X,\tau)$, the restriction $F \mid X_0 : (X_0, \tau \mid X_0) \to (Y,\sigma)$ is upper α-irresolute (resp. lower α-irresolute).

Proof. We prove only the assertion for F upper α-irresolute, the proof for F lower α-irresolute being analogous. Let $x \in X_0$ and V be any α-open set of Y such that $(F \mid X_0)(x) \subseteq V$. Since F is upper α-irresolute and $(F \mid X_0)(x) = F(x)$, there exists $U \in \alpha O(X)$ containing x such that $F(U) \subseteq V$. Set $U_0 = U \cap X_0$, then by Lemma 4.1, we have $x \in U_0 \in \alpha O(X_0)$ and $(F \mid X_0)(U_0) \subseteq V$. This shows that $F \mid X_0$ is upper α-irresolute.

Theorem 4.3 A multifunction $F:(X,\tau) \to (Y,\sigma)$ is upper α-irresolute (resp. lower α-irresolute) if for each $x \in X$, there exists $X_0 \in \alpha O(X)$ containing x such that the restriction $F \mid X_0 : (X_0, \tau \mid X_0) \to (Y,\sigma)$ is upper α-irresolute (resp. lower α-irresolute).

Proof. We prove only the assertion for F upper α-irresolute, the proof for F lower α-irresolute being analogous. Let $x \in X$ and V be any α-open set of Y such that $F(x) \subseteq V$. There exists $X_0 \in \alpha(X,x)$ such that $F \mid X_0$ is upper α-irresolute. Therefore there exists $U_0 \in \alpha O(X_0)$ containing x such that $(F \mid X_0)(U_0) \subseteq V$. By Lemma 3.4, $U_0 \in \alpha O(X)$ and $F(u) = (F \mid X_0)(u)$ for every $u \in U_0$. This shows that F is upper α-irresolute.

Corollary 4.4 *Let $\{U_\lambda : \lambda \in \nabla\}$ be an α-open cover of X. A multifunction $F:(X,\tau) \to (Y,\sigma)$ is upper α-irresolute (resp. lower α-irresolute) if and only if the restriction $F \mid U_\lambda : (U_\lambda, \tau \mid U_\lambda) \to (Y,\sigma)$ is upper α-irresolute (resp. lower α-irresolute) for $\lambda \in \nabla$.*

Proof. This is an immediate consequence of Theorems 4.2 and 4.3.

A subset A of a space (X,τ) is called α-compact if every α-open cover of A in (X,τ) has a finite subcover. Hence the concept of an α-compact space in [16] can be restated as: A space (X,τ) is α-compact if and only if (X,τ) is an α-compact subset of itself. From the definition a subset A of (X,τ) is α-compact if and only if A is compact in (X,τ^{α}).

Theorem 4.5 *Let $F:(X,\tau) \to (Y,\sigma)$ be an upper α-irresolute multifunction and $F(x)$ is an α-compact relative to Y for each $x \in X$. If A is an α-compact relative to X, then $F(A)$ is an α-compact relative to Y.*

Proof. Let $\{V_\lambda : \lambda \in \nabla\}$ be any cover of $F(A)$ by α-open sets of Y. For each $x \in A$, there exists a finite subset $\nabla(x)$ of ∇ such that $F(x) \subseteq \bigcup\{V_\lambda : \lambda \in \nabla(x)\}$. Set $V(x) = \bigcup\{V_\lambda : \lambda \in \nabla(x)\}$. Then $F(x) \subseteq V(x) \in \alpha O(Y)$ and there exists $U(x) \in \alpha(X,x)$ such that $F(U(x)) \subseteq V(x)$. Since $\{U(x) : x \in A\}$ is an α-open cover of A, there exists a finite number of points of A, say, $x_1, x_2, \cdots, x_n$ such that $A \subseteq \bigcup\{U(x_i) : i = 1,2,\cdots,n\}$. Therefore we obtain $F(A) \subseteq F(\bigcup_{i=1}^{n} U(x_i)) \subseteq \bigcup_{i=1}^{n} V(x_i) \subseteq \bigcup_{i=1}^{n} \bigcup_{\lambda \in \nabla(x)} V_\lambda$. This shows that $F(A)$ is α-compact relative to Y.

Corollary 4.6 *Let $F:(X,\tau) \to (Y,\sigma)$ be an α-irresolute multifunction and $F(x)$ is an α-compact relative to Y for each $x \in X$. If X is an α-compact, then Y is an α-compact.*

A space X is said to be α-normal if for any pair of disjoint closed subsets A, B of X, there exist disjoint $U, V \in \alpha O(X)$ such that $A \subseteq U$ and $B \subseteq V$.

Theorem 4.7 If Y is an α-normal space and $F_i : X_i \to Y$ is an upper α-irresolute multifunction such that F_i is punctually closed for each $i = 1,2$, then the set $\{(x_1,x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \varphi\}$ is an α-closed set in $X_1 \times X_2$.

Proof. Let $A = \{(x_1,x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \varphi\}$ and $(x_1,x_2) \notin A$. Then $F_1(x_1) \cap F_2(x_2) = \varphi$. Since Y is an α-normal and F_i is punctually closed for $i = 1,2$, there exist disjoint $V_1, V_2 \in \alpha O(X)$ such that $F_i(x_i) \subseteq V_i$ for $i = 1,2$. Since F_i is an upper α-irresolute, $F_i^{+}(V_i) \in \alpha O(X_i, x_i)$ for $i = 1,2$. Set $U = F_1^{+}(V_1) \times F_2^{+}(V_2)$, then $U \in \alpha O(X_1 \times X_2)$ and $(x_1,x_2) \in U \subseteq (X_1 \times X_2) - A$. This shows that $(X_1 \times X_2) - A \in \alpha O(X_1 \times X_2)$ and hence A is an α-closed in $X_1 \times X_2$.

For a multifunction $F : X \to Y$ the graph $G(F)$ of F is defines as follows:

$G(F) = \{(x,y) \in X \times Y : x \in X \text{ and } y \in F(x)\}$.

Theorem 4.8 If Y is a Hausdorff space and $F : X \to Y$ is an upper α-irresolute multifunction such that $F(x)$ is compact for each $x \in X$, then the graph $G(F)$ is α-closed in $X \times Y$.

Proof. Let $(x, y) \in X \times Y - G(F)$. Then $y \in Y - F(x)$. For each $z \in F(x)$, there exist disjoint open sets $V(z), W(z)$ of Y such that $z \in V(z)$ and $y \in W(z)$. The family $\{V(z): z \in F(x)\}$ is an open cover of $F(x)$ and there exist a finite number of points in $F(x)$, say, $z_1, z_2, \cdots, z_n$ such that $F(x) \subseteq \bigcup\{V(z_i): 1 \le i \le n\}$ and $W = \bigcap\{W(z_i): 1 \le i \le n\}$. Since $F(x) \subseteq V$ and F is an upper α-irresolute, there exists $U \in \alpha(X, x)$ such that $F(U) \subseteq V$. Therefore we obtain $F(U) \cap W = \varphi$ and hence $(U \times W) \cap G(F) = \varphi$. Since $U \times W$ is α-open in $X \times Y$ and $(x, y) \in U \times W \subseteq X \times Y - G(F)$, $X \times Y - G(F)$ is α-open and hence $G(F)$ α-closed in $X \times Y$.

Theorem 4.9 If $F: X \to Y$ and $G: Y \to Z$ are lower α-irresolute (resp. upper α-irresolute) multifunctions, then $G \circ F: X \to Z$ is lower α-irresolute (resp. upper α-irresolute) multifunction.

Proof. Let V be an α-open set of Z. Since $(G \circ F)^{-}(V) = F^{-}(G^{-})(V)$ and F, G are lower α-irresolute multifunctions, $(G \circ F)^{-}(V)$ is an α-open set of X. Thus $G \circ F$ is lower α-irresolute. The upper one is satisfies by a similar argument.

5. Matual Relationships

Definition 5.1 A multifunction $F: X \to Y$ is said to be:

(1) upper precontinuous [17] (resp. upper quasi continuous [18], upper α-continuous [19], upper β-continuous [[14], [19]]) if for each $x \in X$ and each open set V of Y containing $F(x)$, there exists $U \in PO(X, x)$ (resp. $U \in SO(X, x)$, $U \in \alpha(X, x)$, $U \in \beta(X, x)$) such that $F(U) \subseteq V$.

(2) lower precontinuous (resp. lower quasi continuous, lower α-continuous, lower β-continuous) if for each $x \in X$ and each open set V of Y such that $F(x) \cap V \ne \varphi$, there exists $U \in PO(X, x)$ (resp. $U \in SO(X, x)$, $U \in \alpha(X, x)$, $U \in \beta(X, x)$) such that $F(ux) \cap V \ne \varphi$ for every $u \in U$.

(3) upper (lower) precontinuous (resp. upper (lower) quasi continuous, upper (lower) α-continuous, upper (lower) β-continuous) if it has this property at each point of X.

Remark 5.2 For a multifunction $F: X \to Y$ the following implication hold:

Upper quasi-cont.

$$\begin{tikzcd}
 & & \text{Upper quasi-cont.} \arrow[dr] & \\
\text{Upper } \alpha\text{-irresolute} \arrow[r] & \text{Upper } \alpha\text{-cont.} \arrow[ur] \arrow[dr] & & \text{Upper } \alpha\text{-irresolute} \\
 & & \text{Upper precont.} \arrow[ur] &
\end{tikzcd}$$

Upper precont.

We now show that none of these implications are reversible.

Example 5.3 Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3\}$. Define a topology $\tau = \{\varphi, X, \{a\}, \{b, c\}, \{a, b, c\}$ on X and a topology $\sigma = \{\varphi, Y, \{1\}\}$ on Y. A multifunction $F: (X, \tau) \to (Y, \sigma)$ is defined as follows:

$$F(x) = \begin{cases} \{1\}, & if \quad x = a; \\ Y, & if \quad x = b \ or \ c; \\ \{1, 2\}, & if \quad x = d. \end{cases}$$

It can be easily observed that F is upper α-continuous. But F is not upper α-irresolute, since $\{1, 2\} \in \sigma^{\alpha}$ while $F^{+}(\{1, 2\}) = \{a, b\}$ is not α-open in (X, τ).

Example 5.4 Let $X = \{a, b, c\}$ and $Y = \{y: y \in \{0, \pm 1, \pm 2\}\}$. Define a topology $\tau = \{\varphi, X, \{b\}, \{c\}, \{b, c\}$ on X and a topology $\sigma = \{\varphi, Y, \{0, 1, -1, -2\}\}$ on Y. Consider the following multifunction $F: (X, \tau) \to (Y, \sigma)$

$$F(x) = \begin{cases} \{0\}, & if \quad x = a; \\ \{1, -1\}, & if \quad x = b; \\ \{2, -2\}, & if \quad x = c. \end{cases}$$

Then F is upper β-continuous, but not upper precontinuous, since $\{0, 1, -1, -2\} \in \sigma$ but $F^{+}(\{0, 1, -1, -2\}) = \{a, b\}$ is not preopen in (X, τ).

Example 5.5 Let X and Y be as in Example 5.4 with two topologies $\tau = \{\varphi, X, \{b, c\}$ on X and $\sigma = \{\varphi, Y, \{1\}, \{-1\}, \{1, -1\}\}$ on Y. Define a multifunction $F: (X, \tau) \to (Y, \sigma)$ as shown in Example 5.4. one can deduce that F is upper precontinuous but not upper α-continuous.

Example 5.6 Let X, Y and τ be as in Example 5.3. Define a topology $\sigma = \{\varphi, Y, \{1, 3\}\}$ on Y. A multifunction

$F:(X,\tau)\to(Y,\sigma)$ is defined as follows: $F(a)=\{1\}$, $F(b)=\{3\}$, $F(c)=\{2,3\}$ and $F(d)=\{1,2\}$. Then F is upper β-continuous but not upper quasi-continuous because $\{1,3\}\in\sigma$ but $F^{+}(\{1,3\})=\{a,b\}$ is not open in (X,τ).

6. Conclusion

The field of mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. Therefore, the theory of multifunctions is one of the most important subject in topology. On the other hand, topology plays a significant rule in quantum physics, hight energy physics and superstring theory [20]. Thus we study the class of upper (lower) α-irresolute multifuctions which may have possible applications in quantum physics and superstring theory.

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