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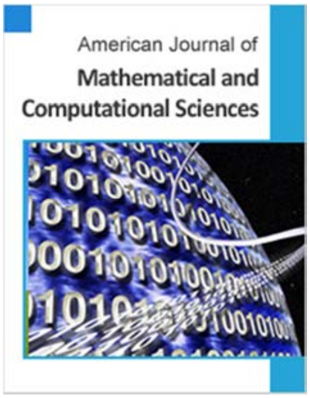

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Tolerance Solution and Control Solution of Max-plus Interval Inequalities

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Abstract
In recent years, with the introduction of max-plus algebra architecture to the field of interval mathematics, some equivalent propositions of various solutions in max-plus algebra have been paid close attention to by many scholars in the world. In the paper, first of all, in the structure of max-plus algebras, we define tolerance solution and control solution of interval inequalities, and then we established equivalent conditions of tolerance solution and control solution of max-plus algebra. Finally, we give some corresponding examples to illustrate.

1. Introduction

As an uncertain system of equations and inequalities, interval linear equations and inequalities have been studied by many scholars recently. For example, in [1-2], the authors consider the linear interval equations and interval equations of various solutions; in [3-4], Researching AE solution is the beginning in interval uncertain linear equations and inequalities; in [5-8], the authors discuss the problem which is interval optimization the solution of uncertain interval local solutions of equations and inequalities. In recent years, with max-plus algebra has appeared in interval mathematics, The system of max-plus algebra is also concerned. In [9, 13-14], the weak solution strong solutions, tolerance solutions and control solutions of the interval equations under max-plus are obtained. The necessary and sufficient conditions and relevant inferences are obtained. After the definitions of the left local solution and right local solution are introduced into the max-plus algebra, see e.g. [10-12], and the definition of the left and right local solutions under the max-plus algebra and related conclusions are given. Since then, in [15-19], many authors have redefined max-plus algebras in the variable $x \in [\underline{x}, \overline{x}]$, equivalent conditions of various solutions of the $\boldsymbol{A} \otimes x = \boldsymbol{b}$ and $\boldsymbol{A} \otimes x \le \boldsymbol{b}$ are obtained. Some authors applied the properties of max-plus algebra to analyze some important characteristics of max-plus linear discrete event which the event consists of production system, queuing system and array processor, see e.g. [20-23]. In this paper, firstly, we define tolerance solution and control solution of interval inequalities in the structure of max-plus algebras, and then we established equivalent conditions of tolerance solution and control solution of max-plus algebra. Finally, we give corresponding examples to illustrate.

2. Preliminaries

We remark $\mathbb{R}$ as real sets. Defined interval as $\boldsymbol{a} = [\underline{a}, \overline{a}]$, where $\overline{a}$, $\underline{a} \in \mathbb{R}$, and

$\underline{a} \le \overline{a}$, then $\boldsymbol{a} = [\underline{a}, \overline{a}] = \{x \in \mathbb{R} \mid \underline{a} \le x \le \overline{a}\}$. All sets of interval sign as $\mathbb{IR}$. When $a = \underline{a} = \overline{a}$, it is called degenerate interval, namely real. We define that $\underline{A}, \overline{A} \in \mathbb{R}$. $\underline{A} \le \overline{A}$ said each element of the matrix $\underline{A}$ of the position corresponding are less than or equal to $\overline{A}$ matrix elements, interval matrix $\boldsymbol{A}$ is a set of real matrices, denoted as $\boldsymbol{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \le A \le \overline{A}\}$. Denoted all interval matrices under $m\times n$ -orders as $\mathbb{IR}^{m\times n}$,and all interval matrices under n -orders as $\mathbb{IR}^{n}$. During the last 50 years special algebras structures appeared, where defined $(\oplus, \otimes)$ which is a semi-group operation. We denoted that $(\oplus, \otimes) = (max, +)$. The operation corresponding real sets on $\boldsymbol{A}^2 \otimes \boldsymbol{x} \le \boldsymbol{b}^2$. then, we have $\mathbb{R}_{max} = \mathbb{R}_{max}(max, +, -\infty)$. in $\mathbb{R}_{max}$,assume $\alpha \in \mathbb{R}_{max}$,then

$$\alpha \oplus -\infty = max\{\alpha, -\infty\} = \alpha \, , \alpha \otimes (-\infty) = \alpha + (-\infty) = -\infty \, .$$

For matrix where all elements belong to $\mathbb{R}_{max}$, it labeled $\mathbb{R}_{max}^{m\times n}$. If $x \in \mathbb{R}_{max}^{n}$ is a column vector made by $x_1, x_2, ..., x_n$. Then $x^T = (x_1, x_2, ..., x_n)$, when arbitrary $x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n$, arbitrary $y = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n$, denoted as

$$x^T \otimes y = (x_1 \otimes y_1) \oplus \cdots \oplus (x_n \otimes y_n) = \bigoplus_{j=1}^{n} (x_j \otimes y_j)$$

when $x, y \in \mathbb{R}_{max}^{n}$,

$$x^T \otimes y = \bigoplus_{j=1}^{n} (x_j \otimes y_j) = \max_{1\le j\le n}\{x_j + y_j\}$$

For a $m\times n$ matrix $A \in \mathbb{R}_{max}^{m\times n}$, and $i = 1,...,m, j = 1,...,n,$ then, $(A \otimes x) \in \mathbb{R}_{max}^{m}$ have operations, for $i = 1,...,m$,

$$(A \otimes x)_i = \bigoplus_{j=1}^{n} (a_{ij} + x_j)$$

$$(A \otimes x) = ((A \otimes x)_1, \cdots, (A \otimes x)_m)^T$$

and $a_{ij} \in \mathbb{R}_{max}, x_j \in \mathbb{R}_{max}$

$$(A \otimes x)_i = \bigoplus_{j=1}^{n} \left(a_{ij} + x_j\right) = \max_{1\le j\le n}\left\{a_{ij} + x_j\right\}$$

if $A_{i\cdot}$ sign as the ith rows of matrix A . Obviously, for $i = 1, \cdots, m$.

$$(A \otimes x)_i = A_{i\cdot} \otimes x$$

when $(\oplus, \otimes)$ in $\mathbb{R}_{max}$, $-\infty, 0$ is zero element. then $\alpha \oplus -\infty = \max\{\alpha, -\infty\} = \alpha$, $\alpha \otimes 0 = \alpha + 0 = \alpha$.

Labeled matrix $A = \left(a_{ij}\right) \in \mathbb{R}^{m\times n}$, matrix $C = \left(c_{ij}\right) \in \mathbb{R}^{m\times n}$, vector $x = \left(x_j\right) \in \mathbb{R}^n$, vector $y = \left(y_j\right) \in \mathbb{R}^n$, where $i \in \{1, 2, \cdots, m\}, j \in \{1, 2, \cdots, n\}$. if $\forall i \in \{1, 2, \cdots, m\}, \forall j \in \{1, 2, \cdots, n\}$ $a_{ij} \le c_{ij}$, then $A^{m\times n} \le C^{m\times n}$. If $j \in \{1, 2, \cdots, n\}$, $x_j \le y_j$, then $x \le y$.

Lemma [2.1] (isotone continuous function)

Matrix $A = (a_{ij}) \in \mathbb{R}_{max}^{m\times n}$, matrix $C = \max(c_{ij}) \in \mathbb{R}_{max}^{m\times n}$, if $A^{m\times n} \le C^{m\times n}$, then $A \otimes x \le C \otimes x$.

Proof. According to the definition of max-plus algebras in $\mathbb{R}_{max}^{m\times n}$,

$$(A \otimes x)_i = \bigoplus_{j=1}^{n} (a_{ij} + x_j) = \max_{1\le j\le n}\{a_{ij} + x_j\}$$

$$(C \otimes x)_i = \bigoplus_{j=1}^{n} (c_{ij} + x_j) = \max_{1\le j\le n}\{c_{ij} + x_j\}$$

We know that, if $\forall i \in \{1, 2, \cdots, m\}$, $\forall j \in \{1, 2, \cdots, n\}$, $A^{m\times n} \le C^{m\times n}$ we have $a_{ij} \le c_{ij}$. Then $(a_{ij} + x_j) \le (c_{ij} + x_j)$, therefore $A \otimes x \le C \otimes x$.

3. Tolerance Solution and Control Solution of Interval Inequalities

Consider interval inequalities

$$\boldsymbol{A}^1 \otimes x \le \boldsymbol{b}^1 \tag{1}$$

$$\boldsymbol{A}^2 \otimes x \le \boldsymbol{b}^2 \tag{2}$$

Where $\boldsymbol{A}^1 \in \mathbb{IR}_{max}^{m\times n}$, $\boldsymbol{A}^2 \in \mathbb{IR}_{max}^{m\times n}$, $\boldsymbol{b}^1 \in \mathbb{IR}_{max}^{m}$, and $\boldsymbol{b}^2 \in \mathbb{IR}_{max}^{m}$.

3.1. Tolerance Solution of Inequalities

Consider interval inequalities of eq. (1) and eq. (2)

Where $\boldsymbol{A}^1 \in \mathbb{IR}_{max}^{m\times n}$, $\boldsymbol{A}^2 \in \mathbb{IR}_{max}^{m\times n}$, $\boldsymbol{b}^1 \in \mathbb{IR}_{max}^{m}$, and $\boldsymbol{b}^2 \in \mathbb{IR}_{max}^{m}$.

Definition 3.1.1 A tolerance solution of eq. (1) and eq. (2), For each $A^1 \in \boldsymbol{A}^1$ and each $A^2 \in \boldsymbol{A}^2$, if there exist $b^1 \in b^1$, exist $b^2 \in b^2$, such that $A^1 \otimes x \le b^1$ and $A^2 \otimes x \ge b^2$.

Theorem 3.1.1 A vector $x \in \mathbb{R}_{max}^{n}$ is a tolerance solution of eq. (1) and eq. (2). If and only if

$$\overline{A^1} \otimes x \le \overline{b^1} \tag{3}$$

$$\underline{A^2} \otimes x \geq \underline{b^2} \tag{4}$$

Proof. If vector $x \in \mathbb{R}^n_{max}$ is a tolerance solution of eq. (1) and eq. (2), then we know by definition 3.1.1 that, for each $A^1 \in \boldsymbol{A}^1$, each $A^2 \in A^2$, exist $b^1 = b^1_0$, and exist $b^2 = b^2_0$, we have

$$A^1 \otimes x \leq b^1_0 \leq \overline{b^1} \tag{5}$$

$$A^2 \otimes x \geq b^2_0 \geq \underline{b^2} \tag{6}$$

Because A^1, A^2 are arbitrary, therefore, if $A^1 = \overline{A^1}$, we obtain eq. (3) and if $A^2 = \underline{A^2}$, we obtain eq. (4).

For the opposite implication, let us suppose that $x \in \mathbb{R}^n_{max}$ fulfills eq. (3) and eq. (4), but it is not a tolerance solution of (1) and (2). For each $b^1 \in b^1$ and each $b^2 \in b^2$, there exist $\tilde{A}^1 \in \boldsymbol{A}^1$ and $\tilde{A}^2 \in \boldsymbol{A}^2$, such that either

$$\tilde{A}^1 \otimes x \not\leq b^1 \tag{7}$$

or

$$\tilde{A}^2 \otimes x \not\geq b^2 \tag{8}$$

By isotone continuous function of lemma [2.1], for each $b^1 \in b^1$, there exist $\tilde{A}^1 \in \boldsymbol{A}^1$ and exist $i_0 \in \{1, \cdots, n\}$ such that either

$$\bigoplus_{j=1}^{n} (\overline{a}^1_{i_0 j} \otimes x_j) \geq \bigoplus_{j=1}^{n} (\tilde{a}^1_{i_0 j} \otimes x_j) > b^1_{i_0 j} \tag{9}$$

or for each $b^2 \in b^2$, exist $\tilde{A}^2 \in \boldsymbol{A}^2$ and exist $i_1 \in \{1, \cdots, n\}$

$$\bigoplus_{j=1}^{n} (\underline{a^2_{i_1 j}} \otimes x_j) \leq \bigoplus_{j=1}^{n} (\tilde{a}^2_{i_1 j} \otimes x_j) < b^2_{i_1 j} \tag{10}$$

In the former case, because b^1 is arbitrary, therefore, if $b^1_{i_0 j} = \overline{b^1_{i_0 j}}$, then $\bigoplus_{j=1}^{n} (\overline{a}^1_{i_0 j} \otimes x_j) > \overline{b^1_{i_0 j}}$ holds, it contradicts eq. (3) and in the latter case, because b^2 is arbitrary, therefore, if $b^2_{i1j} = \underline{b^2_{i_1 j}}$, we have $\bigoplus_{j=1}^{n} (\underline{a^2_{i_1 j}} \otimes x_j) < \underline{b^2_{i_1 j}}$, but eq. (4) is not fulfilled. Therefore, we proof that vector $x \in \mathbb{R}^n_{max}$ is a tolerance solution of interval inequalities of eq. (1) and eq. (2). This completes the proof.

Corollary 1. From the proof of the above theorem, we can see that when the row number of $\boldsymbol{A}^1$ and $\boldsymbol{A}^2$, and $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ is inconsistent, this conclusion still holds.

3.2. Control Solution of Inequalities

Consider interval inequalities of eq. (1) and eq. (2)

Where $\boldsymbol{A}^1 \in \mathrm{I}\mathbb{R}^{m \times n}_{max}$, $\boldsymbol{A}^2 \in \mathrm{I}\mathbb{R}^{m \times n}_{max}$, $\boldsymbol{b}^1 \in \mathrm{I}\mathbb{R}^{m}_{max}$, and $\boldsymbol{b}^2 \in \mathrm{I}\mathbb{R}^{m}_{max}$.

Definition 3.2.1 A control solution of eq. (1) and eq. (2), For each $b^1 \in b^1$ and each $b^2 \in b^2$, if there exist $A^1 \in \boldsymbol{A}^1$, exist $A^2 \in \boldsymbol{A}^2$, such that $A^1 \otimes x \leq b^1$ and $A^2 \otimes x \geq b^2$.

Theorem 3.2.1 A vector $x \in \mathbb{R}^n_{max}$ is a control solution of eq. (1) and eq. (2). If and only if

$$\underline{A^1} \otimes x \leq \underline{b^1} \tag{11}$$

$$\overline{A^2} \otimes x \geq \overline{b^2} \tag{12}$$

Proof. If vector $x \in \mathbb{R}^n_{max}$ is a control solution of eq. (1) and eq. (2), then we know by definition 3.2.1 and isotone continuous function of lemma [2.1] that, for each $b^1 \in b^1$, and each $b^2 \in b^2$, there exist $A^1 = A^1_0$, and exist $A^2 = A^2_0$, such that

$$\underline{A^1} \otimes x \leq A^1_0 \otimes x \leq b^1 \tag{13}$$

$$\overline{A^2} \otimes x \geq A^2_0 \otimes x \geq b^2 \tag{14}$$

Because b^1, b^2 are arbitrary, therefore, if $b^1 = \underline{b^1}$, we obtain (11) and if $b^2 = \overline{b^2}$, we obtain eq. (12).

For the opposite implication, let us suppose that $x \in \mathbb{R}^n_{max}$ fulfills (11) and eq. (12), but it is not a control solution of eq. (1) and eq. (2). Then we know by definition 3.2.1 that, for each $A^1 \in \boldsymbol{A}^1$ and each $A^2 \in \boldsymbol{A}^2$, there exist $\tilde{b}^1 \in b^1$ and $\tilde{b}^2 \in b^2$, such that either

$$A^1 \otimes x \not\leq \tilde{b}^1 \tag{15}$$

or

$$A^2 \otimes x \not\geq \tilde{b}^2 \tag{16}$$

then, $\exists i_0 \in [1, \cdots, m]$, we have

$$\bigoplus_{j=1}^{n} (a^1_{i_0 j} \otimes x_j) > \tilde{b}^1_{i_0 j} \geq \underline{b^1_{i_0 j}} \tag{17}$$

or $\exists i_1 \in [1, \cdots, m]$, we have

$$\bigoplus_{j=1}^{n} (a^2_{i_1 j} \otimes x_j) < \tilde{b}^2_{i_1 j} \leq \overline{b^2_{i_1 j}} \tag{18}$$

then, in the case of eq. (17), As a result of $a^1_{i_0 j}$ is arbitrary. Then, if we set $a^1_{i_0 j} = \underline{a^1_{i_0 j}}$, we have

$$\bigoplus_{j=1}^{n}(\underline{a^1_{i_0 j}} \otimes x_j) > \underline{b^1_{i_0 j}} \tag{19}$$

However, eq. (19) and eq. (11) contradict each other.

Therefore, we proof that vector $\boldsymbol{x} \in \mathbb{R}^n_{max}$ is a control solution of eq. (1) and eq. (2).

In the case of eq. (18), As a result of $a^2_{i_1 j}$ is arbitrary, then, when $\overline{a^2_{i_1 j}}$, we have

$$\bigoplus_{j=1}^{n}(\overline{a^2_{i_1 j}} \otimes x_j) < \overline{b^2_{i_1 j}} \tag{20}$$

However, eq. (20) and eq. (12) contradict each other. Therefore, we proof that vector $\boldsymbol{x} \in \mathbb{R}^n_{max}$ is a control solution of interval inequalities of eq. (1) and eq. (2). This completes the proof.

Corollary 2. From the proof of the above theorem, we can see that when the row number of $\boldsymbol{A}^1$ and $\boldsymbol{A}^2$, $\boldsymbol{b}^1$ and $\boldsymbol{b}^2$ is inconsistent, this conclusion still holds.

4. Examples of Tolerance Solution of Inequalities

Example 4.1

Consider interval inequalities

$$[1,3] \otimes x \le [8,12] \tag{21}$$

$$[2,4] \otimes x \ge [3,6] \tag{22}$$

From the analysis, we know $x = 3$ is a tolerance solution of eq. (21) and eq. (22).

Proof. First of all, by the necessary and sufficient condition of the tolerance solution of interval inequalities in the max-plus algebra, we see that

$$\overline{A^1} \otimes x = 3+3 = 6 < 12 = \overline{b^1} \tag{23}$$

$$\underline{A^2} \otimes x = 2+3 = 5 > 3 = \underline{b^2} \tag{24}$$

Then, we solve the above problem according to the definition of the tolerance solution of interval inequalities in the max-plus algebra.

By isotone continuous function of lemma [2.1], eq. (23) and eq. (24). For each $A^1 \in \boldsymbol{A}^1$ and each $A^2 \in \boldsymbol{A}^2$, we have

$$A^1 \otimes x \le \overline{A^1} \otimes x = 3+3 = 6 < 12 = \overline{b^1} \tag{25}$$

$$A^1 \otimes x \ge \underline{A^2} \otimes x = 2+3 = 5 > 3 = \underline{b^2} \tag{26}$$

Therefore, $x = 3$ is a tolerance solution of eq. (21) and eq. (22).

Example 4.2

Consider interval inequalities

$$\begin{bmatrix} [4,6] & [2,7] \\ [5,8] & [7,10] \end{bmatrix} \otimes x \le \begin{bmatrix} [7,11] \\ [10,17] \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} [-1,4] & [2,5] \\ [-2,7] & [4,11] \\ [3,7] & [-4,1] \end{bmatrix} \otimes x \ge \begin{bmatrix} [4,8] \\ [-1,10] \\ [3,9] \end{bmatrix} \tag{28}$$

From the analysis, we know $x = [2,3]^{\mathrm{T}}$ is a tolerance solution of eq. (27) and eq. (28).

Proof. First of all, by the necessary and sufficient condition of the tolerance solution of interval inequalities in the max-plus algebra, we see that

$$\overline{A^1} \otimes x = \begin{bmatrix} 6 & 7 \\ 8 & 10 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \end{bmatrix} \le \begin{bmatrix} 11 \\ 17 \end{bmatrix} = \overline{b^1} \tag{29}$$

$$\underline{A^2} \otimes x = \begin{bmatrix} -1 & 2 \\ -2 & 4 \\ 3 & -4 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 5 \end{bmatrix} \ge \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \underline{b^2} \tag{30}$$

Then, we solve the above problem according to the definition of the tolerance solution of interval inequalities in the max-plus algebra.

By isotone continuous function of lemma [2.1], eq. (29) and eq. (30). For each $A^1 \in \boldsymbol{A}^1$ and each $A^2 \in \boldsymbol{A}^2$, we have

$$\underline{A^1} \otimes x = \begin{bmatrix} 4 & 2 \\ 5 & 7 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix} \le \begin{bmatrix} 7 \\ 10 \end{bmatrix} = \underline{b^1} \le b^1 \quad 31)$$

$$\overline{A^2} \otimes x = \begin{bmatrix} 4 & 5 \\ 7 & 11 \\ 7 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \\ 9 \end{bmatrix} \ge \begin{bmatrix} 8 \\ 10 \\ 9 \end{bmatrix} = \overline{b^2} \ge b^2 \tag{32}$$

Therefore, $x = [2,3]^{\mathrm{T}}$ is a tolerance solution of inequalities of eq. (27) and eq. (28).

Example 4.3

Consider interval inequalities

$$\begin{bmatrix} [2,3] & [-2,1] \\ [-2,7] & [5,8] \end{bmatrix} \otimes x \le \begin{bmatrix} [4,7] \\ [10,20] \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} [4,7] & [-2,7] \end{bmatrix} \otimes x \ge [6,10] \tag{34}$$

From the analysis, we know $x = [2,3]^{\mathrm{T}}$ is a control solution of eq. (33) and eq. (34).

Proof. First of all, by the necessary and sufficient condition of the control solution of interval inequalities in the max-plus algebra, we see that

$$\underline{A^1} \otimes x = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \le \begin{bmatrix} 4 \\ 10 \end{bmatrix} = \underline{b^1} \tag{35}$$

$$\overline{A^2} \otimes x = \begin{bmatrix} 7 & 7 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 10 \geq 10 = \overline{b^2} \quad (36)$$

Then, we solve the above problem according to the definition of the control solution of interval inequalities in the max-plus algebra.

By isotone continuous function of lemma [2.1], eq. (35) and eq. (36). For each $b^1 \in b^1$ and each $b^2 \in b^2$, we have

$$\underline{A^1} \otimes x = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 10 \end{bmatrix} = \underline{b^1} \leq b^1 \quad (37)$$

$$\overline{A^2} \otimes x = \begin{bmatrix} 7 & 7 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 10 \geq 10 = \overline{b^2} \geq b^2 \quad (38)$$

Therefore, $x = [2,3]^{\mathrm{T}}$ is a control solution of inequalities of eq. (33) and eq. (34).

5. Conclusion

First of all, by defining the tolerance solution and the control solution of interval inequalities of the max-plus algebra, equivalence condition of the tolerance solution and the control solution of the interval inequalities are given. And then, by means of the definitions of the tolerance solution and the control solution of interval inequalities of the max-plus algebra and the necessary and sufficient conditions obtained in this paper, we give corresponding examples to illustrate the tolerance solution and the control solution of interval inequalities of the max-plus algebra. It is a complete complement to various solutions of interval inequalities under max-plus algebra, and its related properties are worth further study. At the same time, its understanding is also the beginning of the study of multi-variables interval equations.

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