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Extension of the Method of Basic Trajectories by G. I. Marchuk for Modeling of Road Pavements via Laser Scanned Data

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Abstract: For polynomial splines of *n*-th degree with non-uniform knots the new type wavelet, semi-orthogonal according to scalar product with derivatives, is offered. With use of splitting on even and odd knots the algorithm of wavelet-decomposition via solution of band system of linear algebraic equations is received. The task of definition of factors of differential equations of nonlinear dynamic system is decided. For group of homogeneous objects of exponential type the equation of system is used in linearized form. The problems of modeling of surfaces of highways with use of data of laser scanning are described. The examples of imposing of the designed road pavement on the previously processed laser measurements are shown.

Keywords: Non-Uniform Laser Measurements, Splines of Odd Degree, Multi-Scale Analysis, Wavelets, Differential Equations, Data Processing, Highways, Modeling

1. Introduction

The majority of complex processes and phenomena of nature, engineering and public life have dynamic character, that is the meanings of variable values are connected not only with the current, but also with the earlier on time meanings of external and internal characteristics. The attempts of describing such processes and phenomena by linear methods of identification based on concepts of polynomial trend and the sum of harmonic oscillations, do not guarantee the adequate analysis and authentic prediction even on limited time intervals. It is caused by the local instability of linear dynamic systems. Really, any such system has the limited spectrum of own frequencies and under influence of compelling force of the appropriate frequency comes into condition of resonance. On the other hand, we notice frequently surprising independence of complex natural and social-economic systems on external influences.

At the end of the last century the works have appeared, in which it is shown, that the steady acyclic decisions can arise for rather simple systems of nonlinear differential equations of the third order. It is accepted to name such decisions "strange attractors", and their external manifestation is viewed by that the observable meanings of variable values make chaotic spread-spectrum sequences – "determined chaos" [1]. Usually the presence of such irregularities is treated as result of influence of stochastic random noise and is withdrawn from analytical description of model during identification.

In the paper the more modest task of development of method for restoration of nonlinear differential equations, simulating complex dynamic system on observable realizations of its variables, is put. Thus the laws of change of parameters are represented as vector splines, and the spline-coefficients are defined iteratively with the help of vector system of equations constructed on the basis of property of accuracy of computational schemes on polynomials. The special attention is given to the case, when the dynamic system represents group of homogeneous objects which are taking place in the same conditions. In this connection the concept of group of exponential type (for which equilibrium average trajectories are monotonous on all the interval of observation [2, page 162]) is entered.

The results of numerical modeling of road pavements by use of laser scanned data and wavelet-decomposition are given.

2. The Bases of the Theory of Spline-Wavelets

Wavelets are by now a widely accepted tool in signal and image processing as well as in numerical simulation. Application of algorithms based on construction of wavelet decompositions is today one of the most popular ways for filtration and compression of numerical information such as temporal series and spatial fields, thanks to their high efficiency [3, 4]. In the field of numerical analysis, methods based on wavelets are successfully used especially for preconditioning of large systems arising from discretization of elliptic partial differential equations, sparse representations of some types of operators and adaptive solving of functional equations [5].

In the theory of multi-scale analysis wavelets name the basis of the set, filling the difference between approximating spaces on rich and coarse grids (see [6, page 41]). In the classical case of approximation on the uniform grid, infinitely continued at both ends, such basis is generated by compressions and shifts of the unique wave function which is looking like short or quickly fading splash, which refers to as wavelet. Because of compression wavelets reveal with a different degree of details the distinction in the characteristics of the measured signal, and by shifts are capable to analyze properties of the signal in different points on the entire investigated interval. At the analysis of non-stationary signals the local property of wavelets provides the essential advantage before transformation of Fourier, which gives only global items of information about the properties of the researched signal, as basic functions, used at it (the sine and cosine), have the infinite support. At the decision of tasks of numerical analysis, as wavelets transform system of basic functions with the distributed (allocated) parameters to system with concentrated parameters, such basis appears much more effective from the point of view of conditionality and convergence.

As the basis for construction of wavelets is the presence of so-called scaling relations such, that each basic function on coarse grid can be expressed as linear combination of basic functions on a rich grid. In particular, splines – smooth functions which have been stuck together from pieces of polynomials of degree n on the enclosed sequence of grids have such relations. In case of uniform grids on the entire numerical axes these relations are well known [7], as well as some cases of approximation on the finite interval. About the practically important case of measurements given on the non-uniform grid, it is known much less. In [8] scaling relations for cubic splines were formally written out for any knots. However explicit expressions of scaling factors were not specified. Therefore in the given work we offer the elementary way of decision of this problem based on application of the idea of local approximation, exact on splines, and well known lemma of De Boor - Fix [9].

The second, that underlies any wavelet-transformation, are scaling relations for wavelets themselves. Such relations are known for the cases of orthogonal and biorthogonal wavelets, that allow by infinite iteration procedure to receive their graphic representation, but do not deliver analytical expressions to use as trial functions, for example, in a method like type of Galerkin. As against it, semi-orthogonal (see [10, page 112]) and non-orthogonal [11] wavelets are defined explicitly as linear combinations of basic splines on a rich grid. Characteristic property of semi-orthogonal wavelets, which sometimes is the basis of appropriate numerical method [12] for construction of wavelet-transformation, is that wavelet-decomposition provides construction of least square approximation of splines on a rich grid by means of splines on a coarse grid. It gives serious advantages at the decision of task of compression of discrete numerical information. However given advantages degrade at differentiation of received spline-wavelet decomposition. On our point of view, the optimum decision is spline-wavelets, ensuring least square approximation of derivative of splines on a rich grid by means of derivative of splines on a coarse grid. For the first time such wavelets were investigated in the case of cubic splines [13, 14]. Thus it appears that these wavelets have very simple construction. In addition, they have deserved the common recognition at the decision of differential equations [15] and numerically were realized as standard program [16] in MatLab. The attempt of generalization of given construction to the case of non-uniform grid was undertaken in [8]. But the resulting expressions have turned out to be very confusing.

At last, the third under the account, but not on importance task of the theory of wavelets, is the problem of evaluation of coefficients of wavelet-decomposition for given function. In the case of orthogonal and biorthogonal wavelets the decision is reduced to application of local averaging filters. We consider it with lack, as information for accounting each factor on coarse grid is used not completely. As against it, the factors of semi-orthogonal (see [9, page 115]) and non-orthogonal [10] wavelets are evaluated from systems of linear algebraic equations, but the good conditionality for them is not guaranteed. In case of measurements given on non-uniform grid, these problems are aggravated by the problem of instability concerning an arrangement of knots of a grid [17]. In [8] at calculations it was offered to take advantage of the unique property "Point Value Vanishing" of constructed wavelets to receive algorithm of discrete wavelet-transformation requiring the calculation of factors of some interpolation spline. Also it was announced the generalization of given algorithm to spline-wavelets of any degree. To the similar result leads developed by the first author earlier [4] universal reception of even-odd splitting on the basis of finite implicit relations connecting basic functions of spline space on a rich grid, basic functions on a coarse grid and wavelets.

2.1. Polynomial Splines of one Variable

Let on the interval of observation $[0, T]$ the grid of time markings Δ: $0 = t_0 < t_1 <\ldots < t_N = T$ is entered. Ordinary polynomial spline of the degree n and defect 1 refers to as the function $S_n(t)$, which coincides with polynomial of the degree not above n on every time piece $[t_{i-1}, t_i]$, $i = 1, 2,.., N$, and has continuous derivatives up to the order n–1. The set of all the splines $S_n(t)$ forms the finite-dimensional space $S_n(\Delta)$ with dimension $N + n$. To receive basic functions of the space $S_n(\Delta)$, we add to the grid Δ the fictitious time markings $t_{-n} <... < t_{-1} < t_0$, $t_{N+1} < …< t_{N+2} < … < t_{N+n}$ and construct for the function $\varphi_n(v,t) = \left[\max(0, v-t)\right]^n$ the divided differences of (n+1)-st order on values of argument $v = t_i, ..., t_{i+n+1}$. Then the functions ([17, page 86]) $B_n^i(t) = (t_{i+n+1} - t_i)\varphi_n\left[t_i,...,t_{i+n+1};t\right]$ are normalized B-splines. They are spline-functions of degree n of defect 1. They are different from zero on intervals (supports) (t_i, t_{i+n+1}) and are identically equal to zero outside of them. Or else, at the points t_i, t_{i+n+1} the derivatives $B_n^i(t)$ up to the order n–1 are equal to zero. It imposes $2n$ of conditions on parameters of received spline, and one else free parameter in this case is determined so that normality condition $\sum_{=-n}^{N-1} B_n^i(t) = 1, 0 \le t \le T,$ takes place. Every spline $S_n(t) \in S_n(\Delta)$ can be uniquely presented as linear combination of B-splines $S_n(t) = \sum_{i=-n}^{N-1} d_i B_n^i(t),$ where d_i – constant factors.

We don't establish any conditions of continuity in the boundary points of considered interval. It corresponds that the basis of B-splines provides correct representation of elements of the space $S_n(\Delta)$ only on the interval [0, T]. Thus to include into consideration the case of defect > 1, it is necessary to attribute to the knots of the grid Δ multiplicity, equal to the defect of spline in these points, and to renumber the knots taking into account their multiplicity. Through B-splines the power functions (monomials) of t up to the degree n can be expressed, and it is the initial moment for construction of approximation formulas.

Let $d_i = \xi_i^\mu = (-1)^\mu \mu! \psi_i^{(n-\mu)}(0)/n!,$ where $0 \le \mu \le n$, $\psi_i(v) = (v - t_{i+1})...(v - t_{i+n})$. Then the equality $S_n(t) \equiv t^\mu, 0 \le t \le T,$ is fair. Moreover, if

$$\lambda_{\tau_i,i}^n(x) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k \psi_i^{(n-k)}(\tau_i)\, x^{(k)}(\tau_i)$$

with any values of argument τ_i on the open interval (t_i, t_{i+n+1}), then $\lambda_{\tau_i,i}^n\left(B_n^j\right) = \delta_i^j$ (Kronecker delta) at all j. Thus the formula of local approximation of the form

$$S_n(f;t) = \sum_{i=-n}^{N-1} \lambda_{\tau_i,i}^n(f) B_n^i(t)$$

– is exact on splines with knots t_i. In it there is the essence of the lemma of De Boor - Fix [9].

2.2. The Construction of Polynomial Spline-Wavelets

Let n initial and n final knots are chose so, that for the case of odd n: $0 = t_{(1-n)/2} = … = t_0, t_N = … = t_{N+(n-1)/2} = T$. Then at $d_{-n} = … = d_{-(n+1)/2} = d_{N-(n+1)/2} = … = d_{N-1} = 0$ the entered functions satisfy to homogeneous boundary conditions

$$S(0) = S'(0) = … = S^{(n-1)/2}(0) = S(T) = S'(T) = … = S^{(n-1)/2}(T) = 0.$$

Henceforth we shall believe, that $N = 2^L$, and designate the received grid as Δ^L, and the appropriate set of splines of the degree n – as V_L. Into basic functions we shall also enter the value L, designating $B_n^i(t) = N_{i+(n+1)/2}^L(t)$ and entering thus, as it is accepted for splines of odd degree, numbering by central knot in the support. For any grid Δ^L spline $S^L(t) \in V_L$ can be presented on the interval [0, T] of parameter t as

$$S^L(t) = \sum_{i=1}^{2^L-1} C_i^L N_i^L(t), 0 \le t \le T. \quad (1)$$

As the grid Δ^{L-1} is received from Δ^L by means of removal of every second knot (not counting fictitious knots), the appropriate space V_{L-1} with the basic functions $N_i^{L-1}(t)$, not equal to zero on double supports (t_{2i-n-1}, t_{2i+n+1}), is enclosed in V_L. The two-scale relation between basic functions in V_L and V_{L-1}, not touching the ends of the piece of approximation, was received in [19] for the case of non-uniform grid, in a special way infinitely continued, in very complex, in our opinion, form. We shall use instead for calculations very simple expressions following from the lemma of De Boor - Fix. In particular, for the case n = 3 it is possible to receive that

$$N_i^{L-1}(t) = \sum_{k=0}^{4} p_{i,k} N_{2i-2+k}^L(t), i = 2,3,…2^{L-1} - 2. \quad (2)$$

Here, taking into account, that (see [18, page 96])

$$\psi_j(t_j) = 0, \psi_j'(t_j) = -h_j \cdot h_{j-1}, \psi_j''(t_j) = 2(h_j - h_{j-1}), \psi_j'''(t_j) = 6, h_j = t_{j+1} - t_j,$$

$$p_{i,1} = \frac{t_{2i-1} - t_{2i-4}}{t_{2i+2} - t_{2i-4}}, p_{i,0} = p_{i,1}\frac{t_{2i-3} - t_{2i-4}}{t_{2i} - t_{2i-4}},$$
$$p_{i,2} = p_{i,1}\frac{t_{2i+2} - t_{2i+1}}{t_{2i+2} - t_{2i-2}} + \frac{(t_{2i+1} - t_{2i-2})(t_{2i-1} - t_{2i+4})}{(t_{2i-2} - t_{2i+4})(t_{2i+2} - t_{2i-2})}, \quad (3)$$
$$p_{i,3} = \frac{t_{2i+1} - t_{2i+4}}{t_{2i-2} - t_{2i+4}}, p_{i,4} = p_{i,3}\frac{t_{2i+3} - t_{2i+4}}{t_{2i} - t_{2i+4}}.$$

The scaling relations between basic functions in V_L and V_{L-1} for the case of two-multiple knot on the left edge of the interval look like (2) with i=1, $p_{1,0}$=0 and

$$p_{i,1} = \frac{t_1 - t_0}{t_4 - t_0},$$
$$p_{i,2} = p_{i,1}\frac{t_6 - t_3}{t_6 - t_0} + \frac{(t_3 - t_0)(t_4 - t_1)}{(t_4 - t_0)^2}, \quad (4)$$
$$p_{i,3} = \frac{t_3 - t_6}{t_0 - t_6}, p_{i,4} = p_{i,3}\frac{t_5 - t_6}{t_2 - t_6},$$

whereas for the case of two-multiple knot on the right edge of the interval look like (2) with $i=2^{L-1}-1$, $p_{i,4}$=0 and

$$p_{i,3} = \frac{t_{2^L} - t_{2^L-1}}{t_{2^L} - t_{2^L-4}},$$
$$p_{i,2} = p_{i,3}\frac{t_{2^L-6} - t_{2^L-3}}{t_{2^L-6} - t_{2^L}} + \frac{(t_{2^L-1} - t_{2^L-4})(t_{2^L} - t_{2^L-3})}{(t_{2^L} - t_{2^L-4})^2}, \quad (5)$$
$$p_{i,1} = \frac{t_{2^L-3} - t_{2^L-6}}{t_{2^L} - t_{2^L-6}}, p_{i,0} = p_{i,1}\frac{t_{2^L-5} - t_{2^L-6}}{t_{2^L-2} - t_{2^L-6}}.$$

The wavelet space W_{L-1} is defined as orthogonal complement of V_{L-1} up to V_L in Hilbert space (see [18, pages 63, 60 and 175]

$$H_0^{(n+1)/2}[0,T] = \{f \in C[0,T], f^{(n+1)/2} \in L^2[0,T],$$
$$f(0) = f'(0) = \ldots = f^{(n-1)/2}(0) = f(T) = f'(T) = \ldots = f^{(n-1)/2}(T) = 0\},$$

in which the scalar product is defined as

$$< f, g >= \int_0^T f^{(n+1)/2} g^{(n+1)/2}.$$

The characteristic property of functions $g \in W_{L-1}$ is that $g(t_{2i})=0$, $i=1,2,\ldots, 2^{L-1}$, whence, in particular, for the case n = 3 it immediately follows, that the set of basic wavelets satisfies the following two-scale relations [8]:

$$\psi_k^{L-1}(t) = q_{k,0}N_{2k-2}^L(t) + N_{2k-1}^L(t) + q_{k,2}N_{2k}^L(t), \quad (6)$$

where

$$q_{k,0} = -\frac{N_{2k-1}^L(t_{2k-2})}{N_{2k-2}^L(t_{2k-2})}, q_{k,2} = -\frac{N_{2k-1}^L(t_{2k})}{N_{2k}^L(t_{2k})}, k = 1,2,\ldots 2^{L-1},$$

while $q_{1,0} = q_{2^{L-1},2} = 0$.

It is obvious, that the support of given wavelets is rather small (t_{2k-4}, t_{2k+2}), that is less than the support of cubic B-spline. Nevertheless, the authors did not manage to find the references to use constructed wavelets in practical calculations, probably because of the complexity of offered in [8] computing algorithm.

2.3. The Construction of the Block of Filters

Any function in V_L can be written down as the sum of some function in V_{L-1} and some function in W_{L-1}. It is convenient to write down factors of spline and basic functions, as $C^L = \left[C_1^L, C_2^L, ..., C_{2^L-1}^L\right]^T$, $\varphi^L = \left[N_1^L, N_1^L, ..., N_{2^L-1}^L\right]$. Then the equation (1) corresponds as $S^L(t) = \varphi^L(t)\, C^L$. Similarly, at the level of coarsening L we shall write down basic wavelet-functions as matrix-line $\psi^L = \left[\psi_1^L, \psi_2^L, ..., \psi_{2^L}^L\right]$. The appropriate wavelet-factors we shall collect in the vector $D^L = \left[D_1^L, D_2^L, ..., D_{2^L}^L\right]^T$. Then with use of designation for block matrixes the process of obtaining C^L from C^{L-1} and D^{L-1} can be written down as (see [9, page 114]):

$$C^L = \left[P^L \mid Q^L\right]\left[\frac{C^{L-1}}{D^{L-1}}\right]. \quad (7)$$

The blocks of matrix P^L are made of factors of appropriate scaling relations. For the case n = 3 they are the relations (2)-(5), as each wide basic function inside the piece of approximation consists of five, and at the edges of the interval – of four narrow basic functions. The elements of columns of matrix Q^L are made of factors of wavelet, and for the case n = 3 they are factors of the relation (6). The example of matrix $[P^L \mid Q^L]$, corresponding to $L = n = 3$, is presented below:

$$\left[P^3 \mid Q^3\right] = \left[\begin{array}{ccc|cccc} p_{1,1} & 0 & 0 & 1 & 0 & 0 & 0 \\ p_{1,2} & p_{2,0} & 0 & q_{1,2} & q_{2,0} & 0 & 0 \\ p_{1,3} & p_{2,1} & 0 & 0 & 1 & 0 & 0 \\ p_{1,4} & p_{2,2} & p_{3,0} & 0 & q_{2,2} & q_{3,0} & 0 \\ 0 & p_{2,3} & p_{3,1} & 0 & 0 & 1 & 0 \\ 0 & p_{2,4} & p_{3,2} & 0 & 0 & q_{3,2} & q_{4,0} \\ 0 & 0 & p_{3,3} & 0 & 0 & 0 & 1 \end{array}\right].$$

2.4. Algorithm with Application of Splitting

The inverse process of decomposition of factors C^L on the rougher version C^{L-1} and the detailing factors D^{L-1} consists in the decision of system of linear equations (7). Thus the system (7) is expedient for splitting on even and odd knots. Has a place, for example, the following

Theorem 1. Let for the case $n = 3$ the values C_i^{L-1} of spline coefficients on the coarse grid Δ^{L-1} are calculated from the decision of three-diagonal system of linear equations of the form

$$\begin{bmatrix} N_1^{L-1}(t_2) & N_2^{L-1}(t_2) & 0 & \cdots & 0 & 0 \\ N_1^{L-1}(t_4) & N_2^{L-1}(t_4) & N_3^{L-1}(t_4) & \ddots & \vdots & \vdots \\ 0 & N_2^{L-1}(t_6) & N_3^{L-1}(t_6) & \ddots & 0 & 0 \\ 0 & 0 & N_3^{L-1}(t_8) & \ddots & N_{2^{L-1}-1}^{L-1}(t_{2^L-6}) & 0 \\ \vdots & \vdots & \vdots & \ddots & N_{2^{L-1}-1}^{L-1}(t_{2^L-4}) & N_{2^{L-1}}^{L-1}(t_{2^L-4}) \\ 0 & 0 & 0 & \cdots & N_{2^{L-1}-1}^{L-1}(t_{2^L-2}) & N_{2^{L-1}}^{L-1}(t_{2^L-2}) \end{bmatrix} \cdot \begin{bmatrix} C_1^{L-1} \\ C_2^{L-1} \\ \vdots \\ C_{2^{L-1}-1}^{L-1} \end{bmatrix} := \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{2^{L-1}-1} \end{bmatrix}, \tag{8}$$

where $f_i = S^L(t_{2i}),\ i = 1,2,\ldots,2^{L-1}-1$. Then the values of wavelet-coefficients are equal

$$D_1^{L-1} = C_1^L - p_{1,1}C_1^{L-1},$$
$$D_i^{L-1} = C_{2i-1}^L - p_{i-1,3}C_{i-1}^{L-1} - p_{i,1}C_i^{L-1},\ i = 2,3,...,2^{L-1}-1,$$
$$D_{2^{L-1}}^{L-1} = C_{2^L-1}^L - p_{2^{L-1}-1,3}C_{2^{L-1}-1}^{L-1}.$$

Resolvability of the system (8) follows from uniqueness of appropriate interpolating spline (see [17, page 141]).

3. Modeling by Splines on the Results of Observation

Let for dynamic object the mathematical description of interaction of state variables **x** is presented as the system of nonlinear differential equations concerning time t, written down in the vector form [20]

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}, \mathbf{u}, \mathbf{\Pi}, t), \tag{9}$$

where $\mathbf{x} = \{x_1, x_2, \ldots, x_m\}$ – investigated parameters (state variables of the object); $\mathbf{u} = \{u_1, u_2, \ldots\}$ – input variables (managing influences); $\mathbf{\Pi} = \{\Pi_1(t), \Pi_2(t),\ldots\}$ – parameters of the system; $f(\mathbf{x}, \mathbf{u}, \mathbf{\Pi}, t) = \{f_1(\mathbf{x}, \mathbf{u}, \mathbf{\Pi}, t),\ldots, f_m(\mathbf{x}, \mathbf{u}, \mathbf{\Pi}, t)\}$ – vector function, generally nonlinear, describing dynamic connection of parameters.

The state-space approach to identification of dynamic system is, that unknown parameters **Π** (t) and state variables **x**(t) are estimated on the measured values of observable parameters (observation variables)

$$\mathbf{y}(t) = g(\mathbf{x}, \mathbf{z}, \mathbf{\Pi}, t), \tag{10}$$

where g – the known vector or scalar, generally nonlinear, function, $\mathbf{z}(t) = \{z_1(t), z_2(t),\ldots\}$ – parameters of external environment (random errors).

Let's assume, that on the interval of observation $[0, T]$ the state variables **x**(t) can be adequately presented as the vector spline $\mathbf{S_x}(t) = \sum_{i=-n}^{N-1} \mathbf{d}_i B_n^i(t)$. As it is well known, for this purpose it is necessary, that the functions **x**(t) belong to the vector space **C**$[0, T]$ – continuous on $[0, T]$ functions, and for high degree of adequacy (small error of approximation) it is better, if **x**(t) belong to the vector space $\mathbf{W}_\infty^{n+1}[0,T]$ – absolutely continuous up to derivative of the order n functions with bounded almost everywhere n + 1-st derivative.

Thus, for numerical evaluation of spline $\mathbf{S_x}(t)$ on the vector of observation **y**(t) it is necessary to solve approximately nonlinear functional equation $g(\mathbf{S_x}, \mathbf{z}, \mathbf{\Pi}, t) = \mathbf{y}(t)$.

For example, nonlinear variant of the method of least squares generates non-stationary iterative process, in which at each step k = 1, 2 … for finding of parameters $\mathbf{d}_i^{(k)}$ of vector spline $\mathbf{S}_\mathbf{x}^{(k)}(t) = \sum_{=-n}^{N-1} \mathbf{d}_i^{(k)} B_n^i(t)$ it is required to solve the following extreme problem

$$\min \int_0^T \left\| \Delta\mathbf{y}_{k-1}(t) - \frac{\partial g}{\partial \mathbf{x}}\left(\mathbf{S}_\mathbf{x}^{(k-1)}, \mathbf{z}, \mathbf{\Pi}, t\right) \sum_{i=-n}^{N-1} \left(\mathbf{d}_i^{(k)} - \mathbf{d}_i^{(k-1)}\right) B_n^i(t) \right\|^2 dt.$$

Here ‖ · ‖ – usual Euclidean norm in the vector space of functions **y**(t). Also the designation $\Delta\mathbf{y}_{k-1}(t) = \mathbf{y}(t) - g\left(\mathbf{S}_\mathbf{x}^{(k-1)}, \mathbf{z}, \mathbf{\Pi}, t\right)$ is entered and $C_{k-1}(t) = \frac{\partial g}{\partial \mathbf{x}}\left(\mathbf{S}_\mathbf{x}^{(k-1)}, \mathbf{z}, \mathbf{\Pi}, t\right)$ – the appropriate Jacobian matrix (see [21, page 112]).

Then it is easy to receive, that parameters $\mathbf{d}_i^{(k)}$ satisfy to the vector equalities

$$\sum_{j=-n}^{N-1} \left(\mathbf{d}_j^{(k)} - \mathbf{d}_j^{(k-1)}\right) \int_0^T B_n^j(t) B_n^i(t)\,dt = \int_0^T B_n^j(t) M_{k-1}(t) \Delta\mathbf{y}_{k-1}(t)\,dt \tag{11}$$

at $i = -n, \ldots, N-1$.

The designation here is entered: $M_{k-1}(t) = \left(C_{k-1}^T(t)\, C_{k-1}(t)\right)^{-1} C_{k-1}(t)$.

The more simple approach consists in the following [21, 22]. Let [23]

$$\Lambda_i(\mathbf{y}) = \sum_{j=-\dot{r}}^{\dot{p}} \alpha^i_j(k)\,\lambda^{l-\dot{p}-\dot{r}}_{\tau_i,i+j}\left(L_i^{l-\dot{p}-\dot{r}} M_{k-1}(t)\mathbf{y}\right),\ \alpha_0^i(k) = 1,$$

$$i = -n, ..., N-1,$$

the set of continuous linear vector functionals in linear vector space of real-valued functions, into which the mapping g translates the space $\mathbf{C}[0, T]$, and vector parameters $\mathbf{d}_i^{(k)}$ are determined iteratively with the help of vector system of equations

$$\sum_{j=-\dot{r}}^{\dot{p}} \alpha_j^i(k)\left(\mathbf{d}_{i+j}^{(k)} - \mathbf{d}_{i+j}^{(k-1)}\right) = \Lambda_i(\Delta\mathbf{y}_k),\ i = -n,...,N-1, \quad (12)$$

where $\dot{r} = r\phi_0(i, r-n-1),\ \dot{p} = p\phi_0(N-p, i)$.

Here $L_i^p\mathbf{x}$ – operators of approximation, exact on polynomials of the degree not higher p, $0 \le p \le n$, for example, least squares polynomials or interpolation Newton polynomials, or on splines with knots t_i. The form of received approximation depends on choice of factors $\alpha_j^i(k)$. In particular, with use of this approach it is possible to reduce the fullness of matrix of system of equations in comparison with the method of least squares and even to realize recurrent scheme for calculation of spline-factors, choosing triangular form of matrix and keeping thus approximation properties of received solution.

Theorem 2 [23]. Let the set $\{\Delta\mathbf{x}(t)\}$ contains the set $\mathbf{P}_l$ of all vector polynomials of any degree up to l, $p + r \le l \le n$, and the determinant of system of equations (12) is not equal to zero. Then for each iteration spline $\Delta\mathbf{S}(t) = \sum_{=-n}^{N-1}\left(\mathbf{d}_i^{(k)} - \mathbf{d}_i^{(k-1)}\right)B_n^i(t)$ to coincide with the decision $\Delta\mathbf{x}(t)$ of equation

$$\frac{\partial g}{\partial \mathbf{x}}\left(\mathbf{S}_\mathbf{x}^{(k-1)}, \mathbf{z}, \mathbf{\Pi}, t\right)\Delta\mathbf{x}(t) = \Delta\mathbf{y}(t)$$

with the right part $\Delta\mathbf{y}(t)$, calculated at anyone $\Delta\mathbf{x} \in \mathbf{P}_l$, it is necessary and enough the equalities

$$\sum_{j=-\dot{r}}^{\dot{p}} \alpha^i_j(k)\left(\lambda^{l-\dot{p}-\dot{r}}_{\tau_i,i+j}\left(L_i^{l-\dot{p}-\dot{r}} M_{k-1}(t)\,C_{k-1}(t)\,\mathbf{T}_\mu\right) - \Xi^{\mu}_{i+j}\right) =$$

$$= \sum_{j=-\dot{r}}^{\dot{p}} \alpha^i_j(k)\left(\lambda^{l-\dot{p}-\dot{r}}_{\tau_i,i+j}(\mathbf{T}_\mu) - \Xi^{\mu}_{i+j}\right) = 0,\ \mu = l-\dot{p}-\dot{r}+1,...,l,$$

been executed on all vectors $\mathbf{T}_\mu = \overrightarrow{\left[t^\mu\right]},\ \Xi^\mu_{i+j} = \overrightarrow{\xi^\mu_{i+j}},\ \mu = 0,1,...,l,$ in space of state variables $\{\mathbf{x}(t)\}$ with components equal t^μ , ξ^μ_{i+j} , accordingly, for all $1 \le \rho \le m$.

In particular, if $L_i^{l-\dot{p}-\dot{r}}$ – identical operator, then

$$\Lambda_i(\mathbf{y}) = \sum_{j=-\dot{r}}^{\dot{p}} \alpha^i_j(k)\,\lambda^{l-\dot{p}-\dot{r}}_{\tau_i,i+j}\left(M_{k-1}(t)\mathbf{y}\right),\ \alpha^i_j(k) = 1.$$

In view of Taylor's formula for polynomial $\psi_i(v)$ of degree n there is fair

$$\lambda^{l-\dot{p}-\dot{r}}_{\tau_i,i+j}(\mathbf{T}_\mu) - \Xi^{\mu}_{i+j} = ... = \overrightarrow{\left[-\frac{\mu!}{n!}\sum_{k=l-\dot{p}-\dot{r}+1}^{\mu} \psi_i^{(n-k)}(\tau_i)\frac{\tau_i^{(\mu-k)}}{(\mu-n)!}\right]}$$

and consecutive substitution into equations (10) gives

$$\sum_{j=-\dot{r},\, j\neq 0}^{\dot{p}} \alpha^i_j(k)\psi^{(n-\mu)}_{i+j}(\tau_i) = -\psi^{(n-\mu)}_{i+j}(\tau_i),\ \mu = l-\dot{p}-\dot{r}+1,..l,$$

for each vector component.

The received systems of equations are solvable uniquely concerning factors $\alpha^i_j(k)$, as functions $\psi_{i+j}(v)$ are linearly independent solutions of differential equation $\psi^{(n-1)}(v) = 0$ and, hence, the matrix of the system is the minor of nondegenerate Wronskian matrix (see [21, page 272]).

Nevertheless, the matrices in the equations (11), (12) are badly conditioned for the basis of B-splines [10]. This bad conditionality can be explained by the fact that each function of basis of B-splines represents only small part of solution. Therefore in case of any extensive change of observation vector the sharp transition from one basis to another is required. And in this case the iterative method of decision which is used for big rarefied systems differs in slow convergence to the exact solution. It is clear, that use of basis of B-splines in interactive mode can lead to long pauses during which the system will recalculate the new solution for the observation vector (every time when the user adds or changes restrictions for the solution).

To solve the problem of weak conditionality and slow convergence not B-splines but wavelets received from them are useful as finite elements. It is possible to consider that in case of the choice of basis of wavelets the change in observations extend from one point to another much quicker because, making changes in this point, it is possible to address onto the top of hierarchy, to basic functions with wider supports from which it is possible to go down to basic functions with narrower supports. Such basis is much more effective for interactive mode of identification of objects (curves and surfaces) by variation methods.

In [25] the method of acceleration of calculations at which all geometrical restrictions are given to one level of wavelet-hierarchy is presented and only curvature of received spline-curve is minimized. This reception allows keeping details of initial curve which are usually lost in usual method of variation modeling. Often, on the contrary, wavelet-transformation is used directly for decomposition of initial curve on various frequency components. High-frequency noise can be filtered during reconstruction. At the same time it is required that the algorithm of filtration is limited to elimination of undesirable defects and preserving isogeometrical performance of the curve (see [26]).

4. Identification of Nonlinear Dynamic Systems

After wavelet-analysis of values $x_{10}, ..., x_{mN}$ at moments of time $t_j,\ j = 0,1,..., N,$ and definition as splines (interpolating or smoothing type) of all the parameters $\mathbf{x} = \{x_1, ..., x_m\}$ it is possible to determine the right part $f(\mathbf{x}, \mathbf{u}, \mathbf{\Pi}, t)$ of differential equation of dynamic system, approximating derivatives of appropriate components of vector approximation spline [27]. Thus, first, the incorrectness of the problem of calculation of numerical values of coefficients at nonlinearities as result of irregular knots of interpolation is possible; secondly, the choice of used non-linearity is substantially subjective owing to basic impossibility to unambiguously restore the equations operating the dynamics of the process [28].

4.1. Linearization of Equations of System

If dynamic system represents the group of homogeneous objects distinguished by initial conditions and, probably, weak perturbations in the right part of operator (9), the alternative approach to the task of identification of system is possible.

Group modeling is understood as modeling on data, measured on group from M of objects, with purpose of revealing basic laws of investigated processes [23]. In this case dynamic properties of entire system are investigated on experimental data averaged on group of homogeneous, concerning parameters, individual objects making the dynamic system. The application of the stated above technique for values averaged on group of parameters of the system gives some equilibrium trajectory $\mathbf{x}_{spl}(t), \mathbf{u}_0(t)$ and for forecasting of processes occurring in complex dynamic system, its equation is used in linearized form. Thus the components of state vector $\mathbf{x}_{ind} = \{x_{ind_1}, ..., x_{ind_m}\}$ of the individual object which is included in homogeneous group, are expressed through deviations from the basic law presented by splines, as

$$x_{ind_i}(t) = x_{spl_i}(t) + \Delta x_i(t),\ i = 1,...,m.$$

Definition. Let processes in complex dynamic system take place in the stationary steady mode that is the equilibrium trajectory is monotonous on the entire interval of observation (in this case $\dot{x}_{spl_i}(t) \neq 0\ \forall i, t$). Then it is possible to name such group as group of exponential type (see [2, page 162].

Theorem 3 [23]. For nonlinear dynamic system of exponential type the representation is fair $\Delta\dot{x}(t) = E(t)\Delta x(t),$ where the elements of matrix $E(t)$ are factors of the form

$$e_{ij}(t) = \frac{\ddot{x}_{spl_i}(t)}{\dot{x}_{spl_j}(t)},\ i, j = 1,...,m, \text{ at the moment of time } t.$$

The received expression is basic for modeling of individual trajectories on measurements on homogeneous group. For this purpose in the parameters of model (initial conditions and factors) are entered small casual perturbations. Thus the condition of stability of transitive matrix $E(t)$ at any moment of the interval of modeling has the essential importance (which is, that all eigenvalues of matrix should be on module < 1 [29]), otherwise small changes in parameters of model answered the large uncontrollable indignation in trajectories and the group "collapses".

4.2. Development of the Method of Basic Trajectories with Reference to the New Technology of Laser Scanning in Road Reconstruction

The experience of use of cubic splines as adequate tool of mathematical representation of the picket method of tracing of reconstructing highways has won recognition of road builders [30]. The purpose of given work is to construct the novel method of approximation of linear-extended spatial objects and to prove its application for decision of tasks of processing of results of laser scanning in road reconstruction. Laser scanning is rather new method in 3D-measurements of high accuracy. Overall objectives of laser scanning of highways – restoration of mathematical model of surface of the road pavement both detection of cracks and damages at places demanding the repair [31]. From the mathematical point of view the presence of scheduled axial line of the road allows to transform bends of the axial line to some rectangular area [32], to which it is possible to apply effectively the stated above method of G. I. Marchuk to axial trajectories on the basis of parametrical identification of nonlinear differential equations and two-dimension interpolation splines on rectangular grid. Advantage of such approach is the opportunity of preservation of structural lines of the road (for example edges), as against popular method of restoration of surface by triangulation of chaotic points. It is very important that thus the construction and application of wavelet-processing of scanned information is considerably facilitated, that guarantees high accuracy of restoration of thin structure of road pavement in places demanding repair, and essential compression in places of the road which are not requiring repair. For this purpose in subsection 2.2 the polynomial adapted differentiating wavelet-filter was constructed. However the problem is complicated by that for today the highways in Russia and Tomsk area have a pitiable view, and at some all plans of their construction irrevocably are lost. Thus, there is an objective need of creation such intellectual software, which operatively and precisely can receive and (or) restore the necessary data for control of the condition of roads, including preliminary data processing with purpose of removal of reflections from roadside both environmental landscape and filling of the misses created by passing automobiles, and definition of the axis of the road

[33]. By lack of the received in the theorem 3 representations is that it is fair to within errors only of the second order. To increase the accuracy of the offered above approach to parametrical identification of nonlinear differential equations it is possible to include in decomposition of individual trajectories the second partial derivatives (see [21, page 334]) at the appropriate complication of the used mathematical tool as for vector-valued functions the second partial derivatives form not the matrix (the Hessian), but the tensor of the rank 3 (see [21, page 515]).

In given work it is offered to consider the extension of the method of G. I. Marchuk to the case of several basic trajectories, as which in relation to highway it is admissible to use well distinguishable on the scanned image brows, borders and other, linearly extended elements of highway. Creation of the new method of approximation of linearly extended spatial objects and justification of its application for solution of problems of processing of results of laser scanning in road construction will be the result of work. The urgency of the work is caused by insufficient accounting of features of existing road strip at designing repairs of highways in conditions of continuously growing intensity of transport flows on roads of Russia and Tomsk area.

5. The Examples

5.1. The Example of Differentiating Spline-Wavelet of the Third Degree

Let's consider as test the function of Harten [34], given on the piece [0, 1]:

$$f(t)=\begin{cases}\frac{1}{2}\sin(3\pi t), & t\le\frac{1}{3},\\ |\sin(4\pi t)|, & \frac{1}{3}<t\le\frac{2}{3},\\ -\frac{1}{2}\sin(3\pi t), & t>\frac{1}{3}.\end{cases}$$

It is piecewise-smooth function equal to zero at points t = 0 and 1. It has breaks of the first kind at points t = 1/3 and 2/3 and break of the first derivative in the point t = 1/2. Its first and second derivatives – piecewise-smooth functions also. The task consists in attempt of calculation of second derivative of the function of Harten, using investigated in the section 2.2 differentiating spline-wavelet of the third degree.

Let's assume

$$\psi^L{}_i(t) := 0.083\cdot 8^{\frac{-L}{2}}\,\psi^L{}_i(2^L t),\ i=1,2^L-1,$$

$$\psi^L{}_i(t) := 0.097\cdot 8^{\frac{-L}{2}}\,\psi^L{}_i(2^L t),\ i=2,3,\dots,2^L-2.$$

Then $\psi_i^L(t)$ are normalized so, that $\left\|\frac{d^2}{dt^2}\psi_i^L(t)\right\|_{L_2(0,1)}=1$ for $i=1,2,\dots,2^L-1$.

As first derivative of the function of Harten in the points t = 0 and 1 is not equal to zero, we shall subtract the values of cubic interpolating polynomial $f'(0)t(1-t)^2-f'(1)(1-t)t^2$ with subsequent addition of given polynomial to the results of wavelet-synthesis. Besides we shall use instead of initial factors of decomposition on basis of B-splines the values, poorly distinguished from them, of function. In the world literature this reception is rather popular and refers to as «Wavelet Crime» [35, 36].

In Figure 1 the result of reconstruction of the second derivative of the spline of 3-rd degree under condition of zeroing wavelet-coefficients on module smaller 185 is presented. It is shown clearly alternating behavior on intervals of smoothness of the broken line of least square approximation of the second derivative. The finite differences approximation of the second derivative which isn't shown in drawing differs from the line a little, demonstrating emissions up to $\pm 10^3$ in points of break of function and its first derivative.

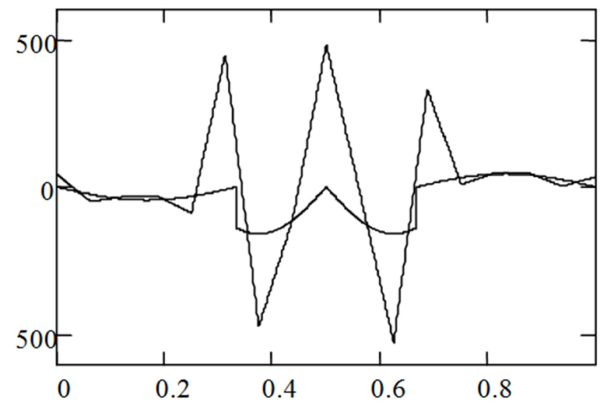

Figure 1. The diagram of wavelet-reconstruction of the second derivative of the spline of 3-rd degree.

5.2. The Example of Wavelet-Compression of One Track of Laser Scanning

The total length of the track is more than 4 km, in Figure 2 the piece of the profile of height to length 450 m long is shown.

There is achieved the compression with factor 15.9. The greatest error does not exceed 3.5 sm.

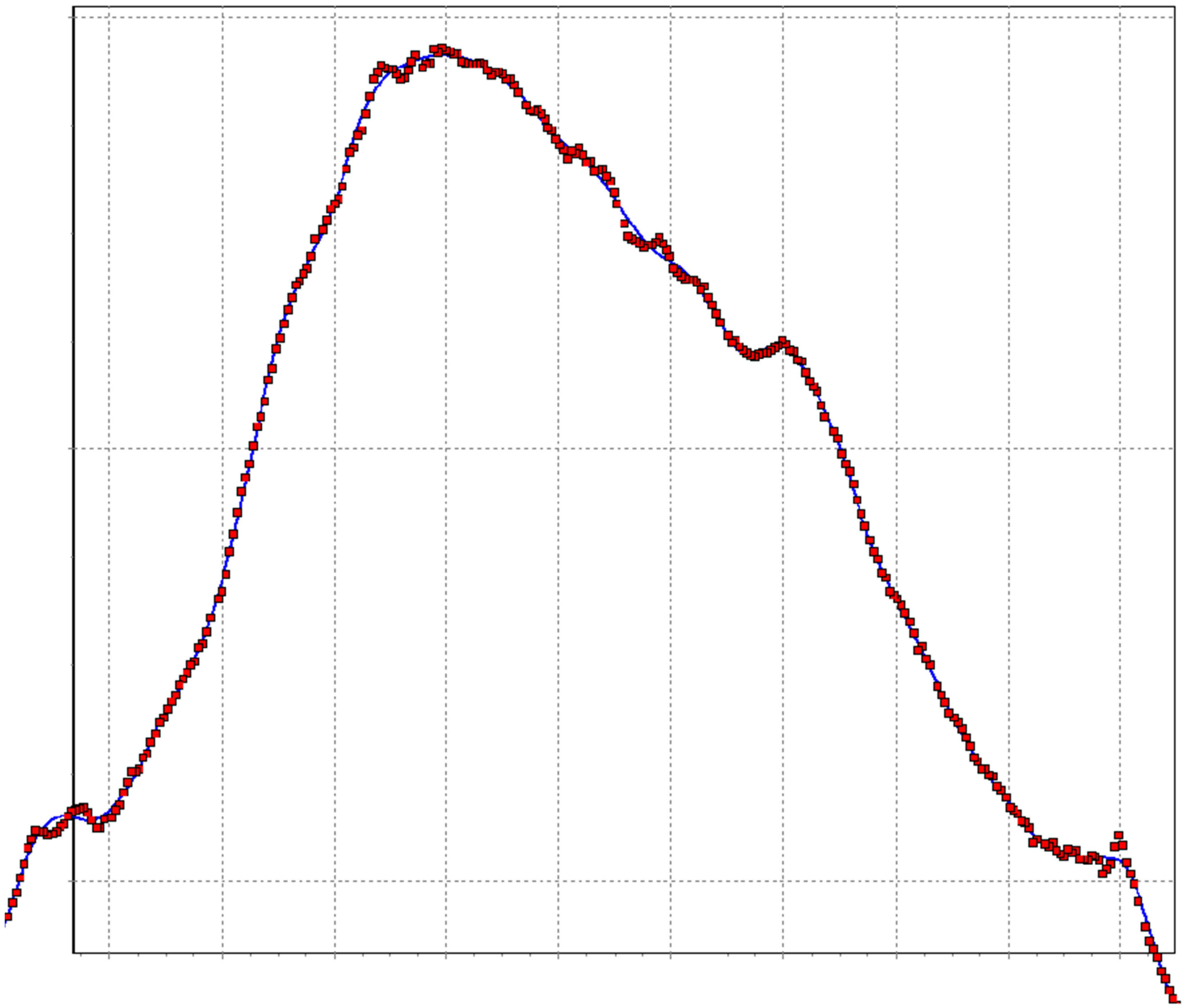

Figure 2. *The result of compression of information for one track.*

5.3. The Example of Modeling of Surfaces

We chose for application the tensor decomposition technique [37] as the array of cross scans of highway has prevailing length in one of two directions. Important auxiliary task at interpolation of nonclosed surfaces via bicubic splines with zero boundary conditions is the creation of surface of Coons [38]. The equation of bicubic surface of Coons, in which the values on the boundaries of the rectangular grid coincide with the values of approximated surface, is necessary to subtract from initial coordinates. Then, the corrected values of coordinates are at the edges nullified, and it is required to add the subtracted earlier equation to bicubic spline received after wavelet processing. The above stated algorithms were base of the software package [32] for processing materials of laser scanning. The results of visualization of the processed data are given in Figure 3.

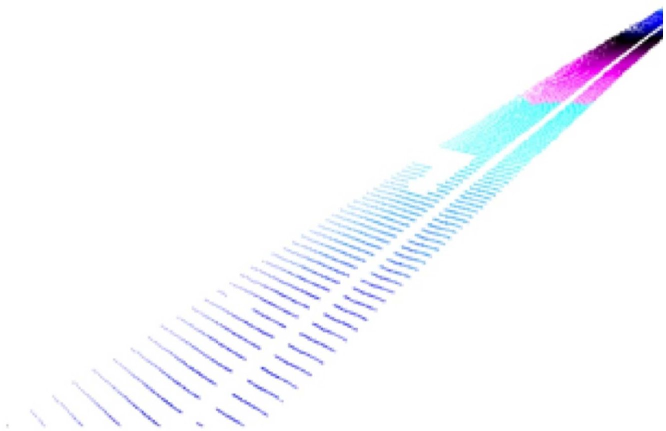

Figure 3. *3D-view of laser points after preliminary filtration.*

The results of visualization of the surface of highway projected on the restored spline-surface are given in Figure 4.

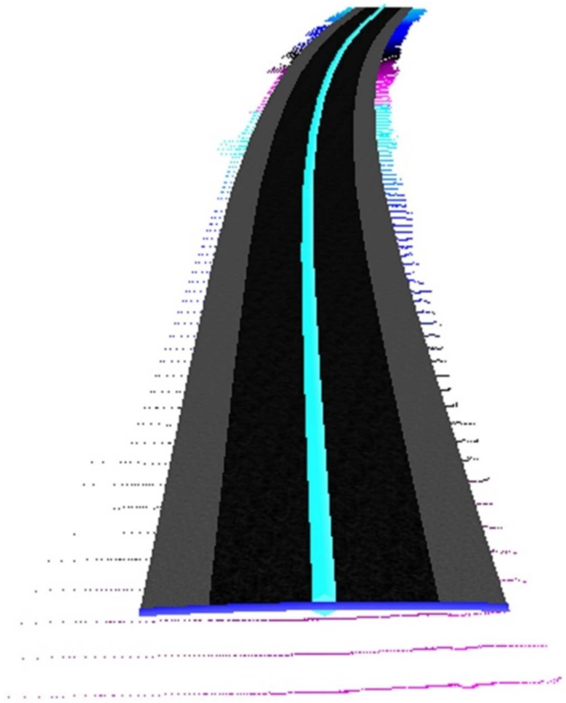

Figure 4. Imposing of the projected road pavement after filling the misses of given data.

6. Conclusion

In work the scheme of construction of spline-wavelets, semi-orthogonal according to their scalar product with derivatives is submitted. The implicit relations of decomposition with splitting on even and odd knots giving new opportunities for creation of effective computing algorithms for construction and use of spline-wavelets on non-uniform grids are received. The construction and the inverting of the block of filters in problems of processing of regular signals and two-dimensional fields are considered. The results of numerical experiments on application of splines and wavelets for compression of data and results of imposing of the designed road pavement on the processed points of laser measurements are presented.

The numbers and schedules following from experiments show that wavelet-transformation – the powerful tool of analysis and planning of repairs of highways with use of information technologies of processing of large volumes of data of laser measurements. We believe that multi-scale wavelet-decomposition has the potential which can be used when processing images and video records. And we expect that together with algorithm of detection of cracks and damages of road pavement it will be possible further to develop intellectual system of restoration of highways.

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