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Polynomials of Sampson as Special Case of Polynomials of Gegenbauer

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Abstract: Following the example of Legendre polynomials and allied polynomials, R. Sampson [1] introduced the concept of neoclassical polynomials taking zero value at the ends of the interval [-1, 1]. The analogue of the Legendre polynomials deserves attention. The proposed method is extended to the Chebyshev polynomials of the first kind.

Keywords: Classical Polynomials, Orthogonality, Interval, Differential Equation, Formula, Recurrence Relations

1. Introduction

We call nonclassical orthogonal polynomials polynomials, for which the generating function $\varphi(x,z) \equiv f^{-1}(x,z)$, is $f(x,z)$ the generating function for classical orthogonal polynomials. We will demonstrate this using Legendre polynomials $P_n(x), x \in [-1,1]$. Such polynomials are polynomials $I_n(x)$ first introduced by R. Sampson in 1891 in hydrodynamics for describing axisymmetric slow (Stokes) flows [1], and then undeservedly forgotten. According to the authors of [2], they were also independently obtained by P. Savic [3] and W. L. Haberman and R. M. Sayre [4].

2. Theoretical Backcround

2.1. The Sampson Polynomial Theory

Polynomials $I_n(x)$ are a particular solution of the differential equation

$$(1-x^2)y'' + n(n-1)y = 0, \tag{1}$$

The general solution of which, obtained by R. Sampson, has the form:

$$y(x) = C_1 I_n(x) + C_2 H_n(x),$$

where $I_n(x) = \dfrac{P_{n-2}(x) - P_n(x)}{2n-1}$, $H_n(x) = \dfrac{Q_{n-2}(x) - Q_n(x)}{2n-1}$, $(n \geq 2)$

$P_n(x)$ - the Legendre polynomials, $Q_n(x)$ - the Legendre functions of the second kind, and $I_0(x) = -1$, $I_1(x) = x$, $H_0(x) = x$, $H_1(x) = 1$.

Generating function for polynomials $I_n(x)$ has the form:

$$(1-2xz+z^2)^{1/2} = \begin{cases} -\sum_{n=0}^{\infty} I_n(x) z^n, & |z|<1 \\ z\sum_{n=0}^{\infty} I_n(x) z^{-n}, & |z|>1 \end{cases}$$

where $R = \sqrt{1-2xz+z^2}$ defines the branch of this root, which at the point $z=0$ takes value $R(0) = -1$.

he orthogonality conditions with the weight function $\rho(x) = (1-x^2)^{-1}$ require specification, namely:

$$\int_{-1}^{1} \frac{I_m(x) I_n(x)}{1-x^2} dx = \begin{cases} 0, (m \neq n), (n \geq 2), \\ \dfrac{2}{n(n-1)(2n-1)}, (m = n \geq 2). \end{cases}$$

Rodrigue's formula for polynomials $I_n(x)$ is presented in R. Sampson [1] and has the form:

$$I_n(x) = \frac{1}{n-1}\left(\frac{d}{dx}\right)^{n-2}\left(\frac{x^2-1}{2}\right)^{n-1}, \quad (n \geq 2).$$

2.2. Polynomials Associated with Legendre Polynomials

When $R(0) = 1$, polynomials $\mu_n(x) \equiv -I_n(x)$ were independently introduced in the electrodynamics of periodic structures by Z. Agranovich, V. Marchenko and V.

Shestopalov [5] with the help of the generating function:

$$f(x,t)=\sqrt{(t-\alpha)(t-\bar{\alpha})}=\begin{cases}\sum_{n=0}^{\infty}\mu_n(x)t^n, & |t|<1,\\ -t\sum_{n=0}^{\infty}\mu_n(x)t^{-n}, & |t|>1,\end{cases}\quad (\alpha=e^{i\theta},\ \theta\in[0,\pi]),$$

Below are the expressions for the first six polynomials $\mu_n(x)$:

$$\mu_0(x)=1,\ \mu_1(x)=-x,\ \mu_2(x)=\frac{1}{2}(1-x^2),$$

$$\mu_3(x)=x\mu_2(x),\ \mu_4(x)=\frac{5x^2-1}{4}\mu_2(x),$$

$\mu_5(x)=x\dfrac{7x^2-3}{4}\mu_2(x)$. For comparing, charts of $P_n(x)$ and $\mu_n(x)$: are given below in on Figure 1 and Figure 2.

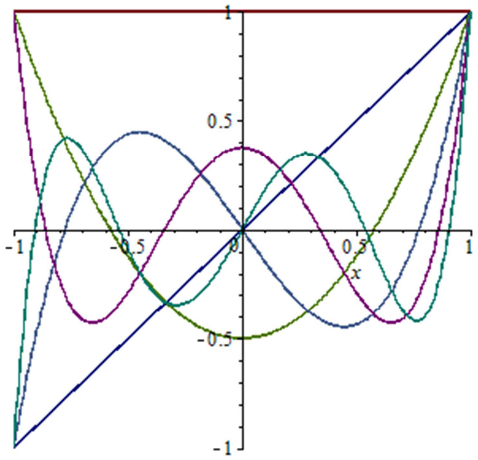

Figure 1. *Legendre Polynomials*

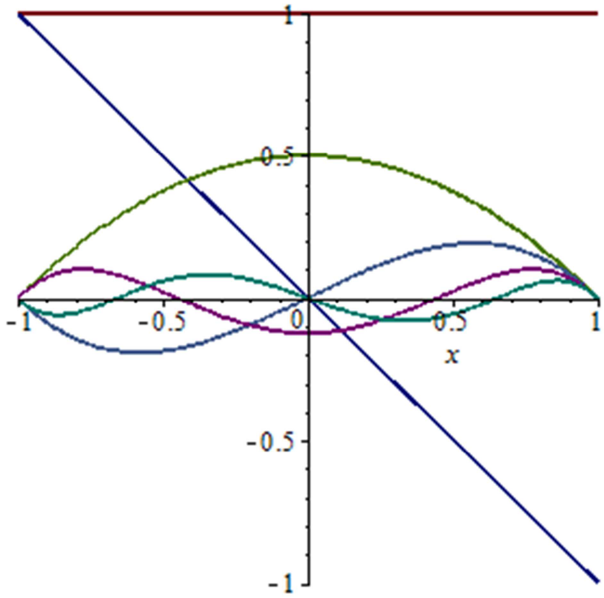

Figure 2. *Polynomials* $\mu_n(x)$.

The order of a polynomial is easily determined from the number of nodes (zeros) of the polynomial. Polynomials $\mu_n(x)$ are expressed in terms of the Legendre polynomials $P_n(x)$:

$$\mu_n(x)=P_n(x)-2xP_{n-1}(x)+P_{n-2}(x),\ (n\geq 2).$$

Considering that at the ends of the interval [-1,1] $P_n(-1)=(-1)^n, P_n(1)=1,$ we obtain:

$$\mu_n(\pm 1)=0,\ (n\geq 2)$$

Table 1. *Legendre polynomials and polynomials* $\mu_n(x)$.

	$P_n(x)$	$\mu_n(x)$
Differential equation	$(1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0$	$(1-x^2)\mu_n''(x)+n(n-1)\mu_n(x)=0;$
The generating function	$(1-2xt+t^2)^{-1/2}$	$(1-2xt+t^2)^{1/2}$
Recurrence relations	$nP_n(x)=(2n-1)xP_{n-1}(x)-(n-1)P_{n-2}(x),$ $P_0(x)=1, P_1(x)=x, (n\geq 2)$ $nP_n(x)=xP_n'(x)-P_{n-1}'(x),$ $(n+1)P_n(x)=P_{n+1}'(x)-xP_n'(x),$	$n\mu_n(x)=(2n-3)x\mu_{n-1}(x)-(n-3)\mu_{n-2}(x),$ $\mu_0(x)=1, \mu_1(x)=-x, (n\geq 2)$ $n\mu_n(x)=x\mu_n'(x)-\mu_{n-1}'(x),$ $(2n-1)\mu_n(x)=\mu_{n+1}'(x)-\mu_{n-1}'(x)$

2.3. Analogue of the Legendre Polynomials

The following representation holds for polynomials $\mu_n(x)$:

$$\mu_n(x)=\tilde{P}_{n-2}(x)\mu_2(x),\ (n\geq 2),$$

where $\tilde{P}_n(x)$– analogue of the Legendre polynomials $P_n(x)$. Below are the expressions for the first six polynomials $\tilde{P}_n(x)$:

$$\tilde{P}_0(x)=1,\ \tilde{P}_1(x)=x,\ \tilde{P}_2(x)=\frac{1}{4}(5x^2-1),\ \tilde{P}_3(x)=\frac{x}{4}(7x^2-3),$$

$$\tilde{P}_4(x)=\frac{1}{8}(21x^4-14x^2+1),\ \tilde{P}_5(x)=\frac{x}{8}(33x^4-30x^2+5),$$

Their graphs are given on Figure 3.

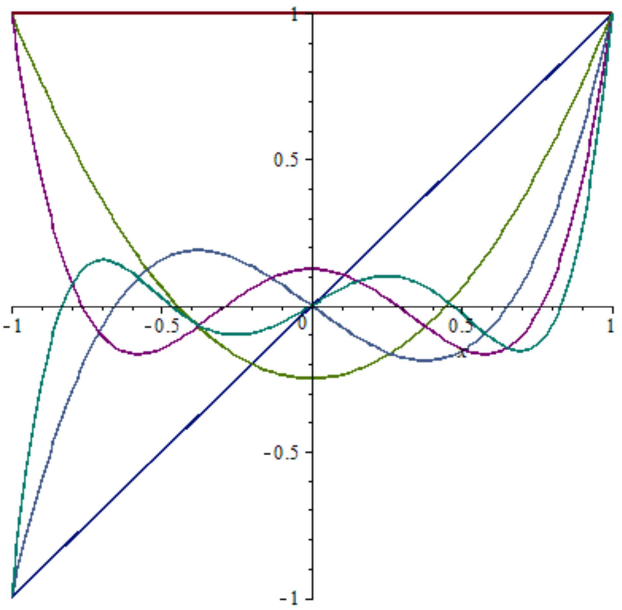

Figure 3. Polynomials $\tilde{P}_n(x)$.

For polynomials $\tilde{P}_n(x)$ the following recurrence formula holds:

$$(n+2)\tilde{P}_n(x) = (2n+1)x\tilde{P}_{n-1}(x) - (n-1)\tilde{P}_{n-2}(x),\ (n \geq 1),\ \tilde{P}_0(x) = 1.$$

2.4. Direct and Inverse Coupling Formulas

Equality is valid:

$$\sum_{n=0}^{\infty} \mu_n(x) z^n \sum_{n=0}^{\infty} P_n(x) z^n = 1, \quad |z| < 1, \tag{2}$$

thus $\sum_{j=0}^{n} \mu_{n-j}(x) P_j(x) = \delta_{0n}$, where δ_{0n} is the Kronecker symbol.

In addition, it follows from (2) that each of the represented series can be regarded as reversible with respect to the other series, which is called the inverse. This leads to direct (3a) and inverse (3b) formulas:

$$\mu_n(x) = -\sum_{j=1}^{n} \mu_{n-j}(x) P_j(x), \quad (n \geq 1) \tag{3a}$$

$$P_n(x) = -\sum_{j=1}^{n} \mu_j(x) P_{n-j}(x), \quad (n \geq 1) \tag{3b}$$

2.5. Polynomials Sampson as a Special Case of Gegenbauer Polynomials

It is known that Legendre polynomials are a special case of Gegenbauer polynomials $P_n^{(\sigma)}(x)$ at σ =1/2 [6-9]. Differential equation (1) for polynomials $\mu_n(x) = y$, obtained by the method of the derived function in [10] coincides with the differential equation for $P_n^{\sigma}(x)$ for the value σ= -1/2 that is forbidden in the theory of classical orthogonal polynomials. Concerning this prohibition, in [8] it is stated on page 176: "Many of the formal relationships remain valid even without this restriction". Below is the solution of differential equation (1) for $\mu_n(x)$, obtained in [11] with the help of mathematical package DEtools and the system of computer mathematics Maple:

2.6. Program for Solving the Differential Equation for Polynomials Sampson

```
restart : with(DEtools) :
df := (1 - x^2)DF^2 + n·(n - 1) :
diffop2de(df, y(x), [DF, x]);
```

$$n(n-1)y(x) + (1-x^2)\left(\frac{d^2}{dx^2}y(x)\right)$$

```
dsolve(%, y(x));
```

$$y(x) = \_C1(1-x^2)hypergeom([1-\frac{n}{2}, \frac{1}{2}+\frac{n}{2}], [\frac{1}{2}], x^2) + \_C2x(1-x^2)hypergeom([1+\frac{n}{2}, \frac{3}{2}-\frac{n}{2}], [\frac{3}{2}], x^2)$$

The polynomials $\mu_n(x)$ were used to simplify the form of the SLAU-2 matrix elements obtaine in [5]. As an example, we give some results:

$$V_0^n(x) = \tfrac{1}{2}\mu_{n+1}(x),\ n > 0;\ V_m^0(x) = \tfrac{1}{2}(m+1)\mu_{m+1}(x),\ m \neq 0.$$

2.7. Polynomials Allied to Chebyshev Polynomials of the First Kind

In a similar way, one can obtain polynomials $t_n(x)$ allied to Chebyshev polynomials of the first kind $T_n(x)$. In particular:

$$t_0(x) = 1,\ t_1(x) = -x,\ t_2(x) = 1 - x^2,\ t_3(x) = xt_2(x),$$
$$t_4(x) = x^2 t_2(x),\ t_5(x) = x^3 t_2(x),$$

and charts of $T_n(x)$ and $t_n(x)$: are given for comparison on Figure 4 and Figure 5.

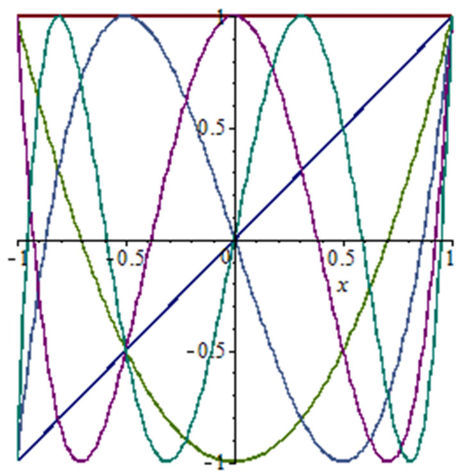

Figure 4. Chebyshev Polynomials of the First Kind.

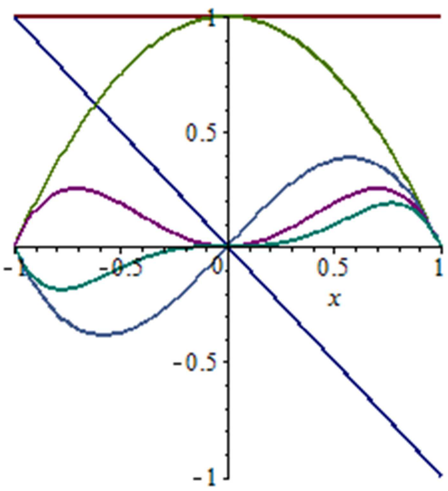

Figure 5. *Polynomials* $t_n(x)$.

For polynomials $t_n(x)$, the following representation holds:

$$t_n(x) = x^{n-2} t_2(x), \quad (n \geq 2).$$

3. Result

Using the theory of reversible and inverse power series and the rules for their multiplication and division with respect to Legendre polynomials and Chebyshev polynomials of the first kind, two classes of orthogonal polynomials are called neoclassical, since they vanish at the endpoints of the interval [-1,1]. The graphs of these polynomials are constructed and their values are calculated for where is the order of the polynomial.

4. Conclusion

It should be noted that the resulting straight lines (3a) and the inverse (3b) formulas for the connection between the Legendre polynomials $P_n(x)$ and polynomials $\mu_n(x)$ are of a general nature. In this way, we can extend the class of orthogonal polynomials that can find application in the approximation of functions, in the calculation of filters, in solving boundary value problems of the mathematical theory of diffraction of electromagnetic waves on periodic structures, in the theory of elasticity, in quantum mechanics and other fields.

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