New Procedures for Selecting the \( k \)-Best Exponential Populations Better than a Standard

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Citation

Abstract
Suppose several two-parameter exponential populations are given. The scale parameters are assumed unequal and can be either known or unknown. This paper discusses how to select those populations having their location parameters better than a standard under the indifference zone formulation. A striking feature of these procedures is that no statistical tables are needed for their implementation.

1. Introduction
Let \( \pi_1, ..., \pi_K \) denote \( K \) (\( \geq 2 \)) exponential distributions or populations. Suppose \( n_i \) independent observations \( X_{i1}, ..., X_{in_i} \) are taken from \( \pi_i \) having probability density function given by
\[
f(x : \mu_i, \sigma_i) = \sigma_i^{-1} \exp[-(x - \mu_i) / \sigma_i] \quad (x \geq \mu_i > 0, \sigma_i > 0)
\]
where \( \mu_i \) is the location parameter and \( \sigma_i \), the scale parameter. If \( \mu_i = 0 \) and \( \sigma_i = 1 \), the random variable is said to have the standard exponential distribution and we write \( X \sim E(0,1) \). In life testing contexts, the location parameter represents the minimum guarantee time, it is estimated by \( \bar{X}_i = \min(X_{i1}, ..., X_{in_i}) \). whereas the scale parameter is estimated by \( S_i = \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i) / (n_i - 1) \). We denote by \( X_{i(1)} \leq ... \leq X_{i(n_i)} \) the order statistics from \( \pi_i \). If \( \sigma_1 = ... = \sigma_K = \sigma \) is unknown, the maximum likelihood estimate of \( \sigma \) is given by
\[
S := \frac{\sum_{i=1}^{K} \left( \sum_{j=1}^{n_i} (X_{i(j)} - X_{i(1)}) \right)}{\sum_{i=1}^{K} (n_i - 1)}.
\]
The ordered minimum guarantee lifetimes are denoted by \( \mu_{i1} \leq ... \leq \mu_{iK} \) with their corresponding ordered estimates \( \hat{X}_{i1} \leq ... \leq \hat{X}_{iK} \). The population \( \pi_{iK} \) associated with \( \mu_{iK} \) is called the ‘best’ population. Notice that \( \{K\} \) is a fixed unknown index and \( \{K\} \) is a random index. Thus, the index set \( G \) of the \( k \) best populations is given by
\[
G := \{ \{K\}, \{K-1\}, ..., \{K-k+1\} \}.
\]
2. Statistical Ranking and Selection

Ranking procedures are statistical techniques for comparing populations when they can be ordered according to certain feature. Selection procedures are normally designed to choose the best or those populations better than a standard, a control or the average. Two pioneers in the field of ranking and selection are Bechhofer (1954) [1] and Gupta (1965) [2]. Their theory was originally developed for normal populations - several normal populations are ranked and the one with the largest mean is selected. Bechhofer’s ‘indifference zone’ approach selects only one population. Most procedures in this category are either one-stage or two-stage. When the scale parameters $\sigma_i$ are unknown and possibly unequal, a two-stage procedure or even a three-stage procedure is usually needed. Initially, a sample $X_{i1},...,X_{in_i}$ of size $n_i (\geq 2)$ is taken from $\pi_i$. From this, estimates of $\mu_i$ and $\sigma_i$ are calculated from respective formulas:

$$\tilde{X}_{in_i} = \min(X_{i1},...,X_{in_i})$$

and

$$S_i = \sum_{j=1}^{n_i}(X_{ij} - \tilde{X}_{in_i})/(n_i - 1).$$

If necessary, a second sample of size $n_i - n_{i0}$ is taken from $\pi_i$, where

$$n_i = \max\left\{n_{i0}, \left\langle S_i/c\right\rangle + 1\right\}$$

and $\left\langle x\right\rangle$ denotes the largest integer smaller than $x$ and $c$ is a suitably chosen constant. Combining both samples, the estimate of $\mu_i$ is then taken to be

$$\tilde{X}_i = \min\{\tilde{X}_{in_i} : X_{i(n_i + 1)},...,X_{in_i}\}.$$ 

A two-stage procedure stops here, and the final decision of which populations are to be selected is based on these $\tilde{X}_i$. The $n_i$ are so chosen that the population selected is precisely the best with a pre-specified probability. For three-stage procedures, second-stage sampling is still governed by the formula $n_i = \max\left\{n_{i0}, \left\langle S_i/c\right\rangle + 1\right\}$, but the third-stage sample size is determined from

$$n_i = \max\left\{n_i, \left\langle S_i^* / c^*\right\rangle + 1\right\}$$

where $c^*$ is another constant chosen by the experimenter and $S_i^*$ is calculated based on data from all three stages. An estimate of $\mu_i$ is then the minimum of all observed values. In Gupta’s ‘subset selection approach, samples of the same size are taken. Information extracted from these samples is then used to determine a subset which contains the best population with a pre-specified probability.

3. Distributional and Other Relevant Results

Results (R.1) – (R.4) below can be found in many textbooks, such as Lawless (1982) [3].

(R.1) $2n_i(\tilde{X}_i - \mu_i)/\sigma_i \sim \chi^2(2)$ or equivalently $n_i(\tilde{X}_i - \mu_i)/\sigma_i \sim E(0,1)$.

(R.2) $2(n_i - 1)S_i/\sigma_i \sim \chi^2(2(n_i - 1))$.

(R.3) If $\sigma_i = ... = \sigma_K = \sigma$ is unknown then $2nS/\sigma \sim \chi^2(2n)$, where

$$n = \sum_{i=1}^{K}(n_i - 1).$$

(R.4) The random variables in (R.1) and (R.2) are independent, so that

$$n_i(\tilde{X}_i - \mu_i)/S_i \sim F(2,2(n_i - 1)).$$

Also, it is a simple matter to show:

(R.5) Let $F_1, F_2, ..., F_r$ be independent, identically distributed $F(2,2m)$ random variables. Then the cumulative distribution function $\Phi(.)$ of $\max_{1\leq i \leq r} F_i$ is given by

$$\Phi(x) = \left[1 - \left(1 + \frac{x}{m}\right)^{-m}\right]^{-r} (x \geq 0).$$

We need an inequality due to Hochberg (1974) [4]:

(R.6) $|a_1t_1 - a_2t_2| \leq \max(a_1, a_2) \max(|t_1|, |t_2|, |t_1 - t_2|)$, all quantities being real with $a_1, a_2$ positive. Random variables $T := (T_1, ..., T_K)$ are said to be associated if

$$\text{Cov}(f(T), g(T)) \geq 0$$

for all nondecreasing functions $f$ and $g$ for which $E(f(T), Eg(T))$ and $E(f(T)g(T))$ all exist. The following properties of associated random variables (Tong (1980), p. 87- p. 89) [5] will be used:

(R.7a) Independent random variables are associated.

(R.7b) Nondecreasing functions of associated random variables are associated.

(R.7c) If $Y_1, Y_2, ..., Y_n$ are associated random variables, then for real $c_1, ..., c_k$,

$$P(Y_i \leq c_i, ..., Y_k \leq c_k) \geq \prod_{i=1}^{k} P(Y_i \leq c_i).$$

4. Two New Procedures

4.1. Scale Parameters Known

We say there is correct selection (CS) if only those
populations with indices belonging to G are selected. However, if the difference between \( \mu_i \) and \( \mu_j \) is not large, it might not matter whether we choose the population corresponding to \( \mu_i \) or the population corresponding to \( \mu_j \). This consideration motivated the concept of the least favorable configuration (LFC). If we let \( \mu = (\mu_1, ..., \mu_K) \), the LFC for the classical selection problem consists of all arrangements of the components of \( \mu \) for which

\[
\mu_{[K-k+1]} - \mu_{[K-k]} \geq \delta , \ i.e.
\]

\[
P[\text{select } G \mid LFC_{classical}] \geq P_S \text{ where } (K+1)^{-1} < P_S < 1
\]

When \( \sigma_1 = ... = \sigma_K = \sigma \) is known, Barr and Rizvi (1966) [6] and Raghavachari and Starr (1970) [7] considered the goal of selecting the \( k \geq 1 \) best populations. For the case of unknown \( \sigma \) and the goal is to select the best population, i.e. \( k = 1 \), Desu, Narula and Villarreal (1977) [8] have proposed a two-stage procedure.

When a standard having known mean \( \mu_S \) is stipulated, a sensible least favorable configuration would be:

\[
LFC_S = \left\{ \mu \mid \mu_{[K]} \geq \mu_S + \delta_S, ..., \mu_{k-K+1} \geq \mu_S + \delta_S, \mu_{k-K+1} - \mu_{k-K} \geq \delta \right\}
\]

Our goal is,

\[
P[\text{select } G \mid LFC_S] \geq P_S \text{ where } (K+1)^{-1} < P_S < 1,
\]

which means that with probability at least \( P_S \), all populations sufficiently better than both the standard and their closest competitors are selected. On the other hand, we want to leave room for not selecting any population if all of them are substandard. The least favorable configuration for this goal, the second goal, is the set:

\[
LFC_{0} = \left\{ \mu \mid \mu_{[K]} \leq \mu_S + \delta_R \right\}
\]

Mathematically,

\[
P[\text{no selection} \mid LFC_{0}] \geq P_0 \text{ where } (K+1)^{-1} < P_0 < 1,
\]

which means that, with probability at least \( P_0 \), no population is selected when none is sufficiently better than the standard. The quantities \( \mu_S, \delta_S, \delta_R, P_S, P_0 \) are assumed known or chosen in advance.

Lastly, the population associated with \( \tilde{X}_i \) is selected if \( \tilde{X}_i \geq \mu_S + d \) where \( d = d(\mu_S, \delta_S, \delta_R, P_S, P_0) \) is chosen to fulfill those two goals simultaneously. It is obvious that a legitimate \( d \) may not exist for certain combinations of its arguments. Using the indifference zone approach, Bristol & Desu (1985) [9] have proposed procedures for selecting the best, i.e. \( k = 1 \), two-parameter exponential population sufficiently better than the standard and its closest competitor assuming a common known or unknown scale parameter. Apart from their paper, this area is largely untouched. This paper handles the selection of at least one population, i.e. \( k \geq 1 \), when the scale parameters are unequal, known or unknown, using two-stage procedures. An algorithmic version of this indifference zone procedure is presented below.

**Algorithm E**

E1) Choose a value for: \( \delta_S, \delta_R, P_S, P_0, n_0(\geq 2) \) and \( k(\geq 1) \).

For \( 1 \leq i \leq K \), perform E2) to E5):

E2) Take a sample \( \{X_{ij} \mid j = 1, ..., n_0\} \).

E3) Compute \( \tilde{X}_i \).

E4) Let \( n_i = \max_n \left\{ n_0, \left\lfloor c\sigma_i / \delta \right\rfloor + 1 \right\} \) where \( c := -\ln \left(1 - P_0^{1/(2k-1)}\right) \).

E5) If necessary, take a second-stage sample of size \( n_i - n_0 \), and compute \( \tilde{X}_i \) again based on the combined sample.

E6) Calculate \( d := d\left(\mu_S, \delta_S, \delta_R, P_S, P_0\right) \).

E7) If \( \delta_R \leq d \leq \delta_S \), take populations having \( \tilde{X}_i \geq \mu_S + d \) as the \( k \)-best populations.

The proof that our goals are achieved through Algorithm E is as follows.

We have, for the first goal,
\[ P\left[ \text{select } G \mid LFC_S \right] \]
\[ = P\left[ \bar{X}_i > \bar{X}_j, \bar{X}_i \geq \mu_s + d \quad \forall \ i \in G, \ j \notin G \mid LFC_S \right] \]
\[ = P\left[ (\bar{X}_j - \mu_j) - (\bar{X}_i - \mu_i) + (\mu_j - \mu_i + \delta) < \delta, \bar{X}_i \geq \mu_s + d \quad \forall \ i \in G, \ j \notin G \mid LFC_S \right] \]
\[ \geq P\left[ (\bar{X}_j - \mu_j) - (\bar{X}_i - \mu_i) < \delta, \bar{X}_i \geq \mu_s + d \quad \forall \ i \in G, \ j \notin G \right] \]
\[ \quad \left( \because \mu_j - \mu_i + \delta < 0 \text{ under } LFC_S \right) \]
\[ = P\left[ \frac{n_j (\bar{X}_j - \mu_j)}{\sigma_j} - \frac{n_i (\bar{X}_i - \mu_i)}{\sigma_i} < \delta, \bar{X}_i \geq \mu_s + d \quad \forall \ i \in G, \ j \notin G \right] \]
\[ = P\left[ E_j \sigma_j - E_i \sigma_i < \delta, E_i \geq \frac{n_i}{\sigma_i} (\mu_s - \mu_i + d) \quad \forall \ i \in G, \ j \notin G \right] \quad (\text{by (R.1)}) \]
\[ \geq P\left[ E_j \sigma_j - E_i \sigma_i < \delta, E_i \geq \frac{n_i}{\sigma_i} (d - \delta_s) \quad \forall \ i \in G, \ j \notin G \right] \quad (\because \mu_i \geq \mu_s + \delta_s) \]
\[ \geq P\left[ E_j \sigma_j - E_i \sigma_i < \delta \quad \forall \ i \in G, \ j \notin G \right] \quad (\text{by (E7)}) \]
\[ \geq \prod_{i \in G, \ j \notin G} P\left[ E_j \sigma_j - E_i \sigma_i < \delta \right] \quad (\text{being positively correlated & by (R.7)}) \]
\[ \geq \prod_{i \in G, \ j \notin G} P\left[ E_j \sigma_j - E_i \sigma_i < \delta \right] \]
\[ \geq \prod_{i \in G, \ j \notin G} P\left[ \max \left( \frac{\sigma_j}{n_j}, \frac{\sigma_i}{n_i} \right) \max \left( E_j, E_i \right) - \left| E_j - E_i \right| < \delta \right] \quad (\text{by (R.6)}) \]
\[ = \prod_{i \in G, \ j \notin G} P\left[ \max \left( E_j, E_i \right) < \delta \max \left( \frac{\sigma_j}{n_j}, \frac{\sigma_i}{n_i} \right) \right] \quad \left( \because \max \left( E_j, E_i \right) = \max \left( E_j, E_i \right) \right) \]
\[ = \prod_{i \in G, \ j \notin G} P\left[ \max \left( E_j, E_i \right) < \delta \max \left( \frac{\sigma_j}{n_j}, \frac{\sigma_i}{n_i} \right) \right] \quad (\text{by (E4)}) \]
\[ = \prod_{i \in G, \ j \notin G} P\left[ \max \left( E_j, E_i \right) < c \right] \quad (\text{by (E4)}) \]
\[ = \left[ 1 - e^{-c} \right]^{2k(K-k)} \]
\[ = P_S \quad (\text{by (E4)}) \]

And for the second goal,
\[
P\left[ \text{no select} \mid LFC_0 \right] \\
= P\left[ \tilde{X}_i \leq \mu_S + d \quad \forall \ i \in G \mid LFC_0 \right] \\
= P\left[ \tilde{X}_i - \mu_i \leq \mu_S - \mu_i + d \quad \forall \ i \in G \mid LFC_0 \right] \\
= P\left[ E_i \frac{\sigma_i}{n_i} \leq \mu_S - \mu_i + d \quad \forall \ i \in G \mid LFC_0 \right] \\
\geq P\left[ E_i \frac{\sigma_i}{n_i} \leq \mu_S - (\mu_S + \delta_R) + d \quad \forall \ i \in G \right] \quad \left( \because \mu_{K^i} \leq \mu_S + \delta_R \text{ under } LFC_0 \right) \\
= P\left[ E_i \leq \frac{n_i}{\sigma_i} (d - \delta_R) \quad \forall \ i \in G \right] \\
\geq P\left[ E_i \leq \frac{c}{\delta} (d - \delta_R) \quad \forall \ i \in G \right] \quad \left( \because n_i > \frac{c\sigma_i}{\delta} \right) \\
= P\left[ \max_{i \in G} E_i \leq \frac{c}{\delta} (d - \delta_R) \right] \\
= 1 - \exp\left( \frac{-c}{\delta} (d - \delta_R) \right)^{k} \\
= P_0
\]

A more realistic situation would be one having unknown scale parameters which is presented next.

### 4.2. Scale Parameters Unknown

**Algorithm F**

F1) Choose a value for: \( \delta_S, \delta, \delta_R, P_S, P_0, n_0(\geq 2) \) and \( k (\geq 1) \).

For \( 1 \leq i \leq K \), perform F2) to F5):

F2) Take a sample \( \{ X_{ij} \ ; \ j = 1, \ldots, n_0 \} \).

F3) Compute \( \tilde{X}_i \) and \( S_i \).

F4) Let \( n_i = \max\left\{ n_0, \left( cS_i / \delta \right) + 1 \right\} \) where

\[
c := (n_0 - 1) \left( 1 - P_S^{1/(2k(K-k))} \right)^{-1/(n_0-1)} - 1.
\]

F5) If necessary, take a second-stage sample of size \( n_i - n_0 \), and compute \( \tilde{X}_i \) again based on the combined sample.

F6) Calculate \( d := c^{-1} \delta (n_0 - 1) \left( 1 - P_S^{1/k} \right)^{-1/(n_0-1)} - 1 + \delta_R \).

F7) If \( \delta_R \leq d \leq \delta_S \), choose, from among populations having \( \tilde{X}_{(i)} \geq \mu_S + d \) the \( k \)-best populations.

The proof that our goals are achieved through Algorithm F is as follows.

For the first goal,
\[ P[ \text{select } G | LFC_S ] \]
\[ = P[ \bar{X}_i > \bar{X}_j , \bar{X}_i \geq \mu_S + d \ \forall i \in G, j \notin G | LFC_S ] \]
\[ = P[ (\bar{X}_j - \mu_j) - (\bar{X}_i - \mu_i) + (\mu_j - \mu_i + \delta) < \delta, \bar{X}_i \geq \mu_S + d \ \forall i \in G, j \notin G | LFC_S ] \]
\[ \geq P[ (\bar{X}_j - \mu_j) - (\bar{X}_i - \mu_i) < \delta, \bar{X}_i \geq \mu_S + d \ \forall i \in G, j \notin G ] \quad (\because \mu_j - \mu_i + \delta < 0 \ \text{under } LFC_S ) \]
\[ = P \left[ \frac{n_j (\bar{X}_j - \mu_j)}{S_j} - \frac{n_i (\bar{X}_i - \mu_i)}{S_i} < \frac{\delta}{n_j}, \bar{X}_i \geq \mu_S + d \ \forall i \in G, j \notin G \right] \]
\[ = P \left[ F_j \frac{S_j}{n_j} - F_i \frac{S_i}{n_i} < \delta, F_j \geq \frac{n_j}{S_j} (\mu_S - \mu_i + d) \ \forall i \in G, j \notin G \right] \quad (\text{by (R.4)}) \]
\[ \geq P \left[ F_j \frac{S_j}{n_j} - F_i \frac{S_i}{n_i} < \delta, F_j \geq \frac{n_j}{S_j} (d - \delta_S) \ \forall i \in G, j \notin G \right] \quad (\because \mu_i \geq \mu_S + \delta_S ) \]
\[ = P \left[ F_j \frac{S_j}{n_j} - F_i \frac{S_i}{n_i} < \delta \ \forall i \in G, j \notin G \right] \quad (\text{by (F4)}) \]
\[ \geq \prod_{i \in G, j \notin G} P \left[ F_j \frac{S_j}{n_j} - F_i \frac{S_i}{n_i} < \delta \right] \quad (\text{being positively correlated \& (by R.7)}) \]
\[ \geq \prod_{i \in G, j \notin G} P \left[ \max \left( \frac{S_j}{n_j}, \frac{S_i}{n_i} \right) \frac{\max(F_j, F_i)}{\min(F_j, F_i)} - F_i < \delta \right] \quad (\text{by (R.6)}) \]
\[ = \prod_{i \in G, j \notin G} P \left[ \max(F_j, F_i) < \frac{\delta}{\max \left( \frac{S_j}{n_j}, \frac{S_i}{n_i} \right)} \right] \quad (\because \max(F_j, F_i, [F_j - F_i]) = \max(F_j, F_i)) \]
\[ = \prod_{i \in G, j \notin G} P \left[ \max(F_j, F_i) < c \right] \quad (\text{by (F4)}) \]
\[ = \left[ 1 - \left( 1 + \frac{c}{n_0 - 1} \right)^{-\frac{(n_0 - 1)}{2k(K-k)}} \right] \quad (\text{by (R.5)}) \]
\[ = P_S \quad (\text{by (F4)}) \]

For the second goal,
\[ P[\text{no select } \mid \text{LFC}_0] \]

\[ = P[\tilde{X}_i \leq \mu_S - d \quad \forall i \in G \mid \text{LFC}_0] \]

\[ = P[\tilde{X}_i - \mu_i \leq \mu_S - \mu_i + d \quad \forall i \in G \mid \text{LFC}_0] \]

\[ = P[F_i \frac{S_i}{n_i} \leq \mu_S - \mu_i + d \quad \forall i \in G \mid \text{LFC}_0] \]

\[ \geq P[F_i \frac{S_i}{n_i} \leq \mu_S - (\mu_S + \delta_R) + d \quad \forall i \in G] \quad (\because \mu_{[1]} \leq \mu_S + \delta_R \text{ under LFC}_0) \]

\[ = P[F_i \leq \frac{n_i}{S_i} (d - \delta_R) \quad \forall i \in G] \]

\[ \geq P[F_i \leq \frac{c}{\delta} (d - \delta_R) \quad \forall i \in G] \quad (\because n_i > \frac{cS_i}{\delta}) \]

\[ = P[\max_{i \in G} F_i \leq \frac{c}{\delta} (d - \delta_R)] \]

\[ = \left[ 1 - \left( 1 + \frac{c(d - \delta_R)}{\delta(n_0 - 1)} \right)^{-k} \right]^{k} \quad \text{(by (3.5))} \]

\[ = P_0 \quad \text{(by (F4))} \]

5. Examples

Samples of size \(n_0 = 20\) are generated from \(K = 6\) exponential populations having the following location-scale parameter \((\mu, \sigma)\) combinations:

<table>
<thead>
<tr>
<th>(i)</th>
<th>((10.3))</th>
<th>((9.3))</th>
<th>((7.2))</th>
<th>((5.2))</th>
<th>((3.1))</th>
<th>((2.1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>14.6</td>
<td>12.1</td>
<td>7.5</td>
<td>5.7</td>
<td>5.0</td>
<td>3.9</td>
<td></td>
</tr>
<tr>
<td>11.4</td>
<td>12.1</td>
<td>7.6</td>
<td>7.3</td>
<td>3.1</td>
<td>3.0</td>
<td></td>
</tr>
<tr>
<td>12.2</td>
<td>11.9</td>
<td>7.3</td>
<td>9.1</td>
<td>3.1</td>
<td>4.4</td>
<td></td>
</tr>
<tr>
<td>13.7</td>
<td>11.0</td>
<td>7.3</td>
<td>7.3</td>
<td>3.2</td>
<td>2.2</td>
<td></td>
</tr>
<tr>
<td>19.5</td>
<td>10.2</td>
<td>12.2</td>
<td>15.5</td>
<td>3.2</td>
<td>5.1</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Random samples generated from various two-parameter exponential populations.

The stipulated standard is \(\mu_S = 6\). We want the best two populations better than the standard, i.e. \(k = 2\). The location parameter and scale parameter estimates based on stage-one samples are tabulated below.

<table>
<thead>
<tr>
<th>(i)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{X}_{n_0})</td>
<td>10.4</td>
<td>9.0</td>
<td>7.3</td>
<td>5.1</td>
<td>3.0</td>
<td>2.0</td>
</tr>
<tr>
<td>(\hat{S})</td>
<td>2.7</td>
<td>2.4</td>
<td>2.2</td>
<td>2.0</td>
<td>1.1</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Table 2. Summary measures of stage-one samples generated from various two-parameter exponential populations.

Using these summary measures, it was found that \(n_1 = \ldots = n_6 = 20\), so that second-stage sampling is not necessary. Now suppose the following parameters were chosen:
\( \delta_S = 0.8, \quad \delta = 1.25, \quad \delta_R = 0.3, \quad P_1 = 0.95, \quad P_0 = 0.8. \)

Using Algorithm F,
\[
c = (n_0 - 1)(1 - P_S^{1/(K-k)})^{-1/(n_0 - 1)} = 19(1 - 0.95^{1/16})^{-1/19} - 1 = 6.707.
\]

It follows that,
\[
d = c^{-1}\delta(n_0 - 1)(1 - P_{1/k}^{1/(n_0 - 1)} - 1) + \delta_R = 6.707^{-1}(23.75)(0.106^{-1/19} - 1) + 0.3 = 0.745.
\]

For this procedure to work, \( \delta_R \leq d \leq \delta_S \), i.e. \( 0.3 \leq d \leq 0.8 \). Since \( d \) indeed lies within this range, we have the following selection rule:

“Select the population associated with \( \tilde{X}_{(6)} \) if \( \tilde{X}_{(6)} \geq 6 + 0.745 = 6.745 \), and the population associated with \( \tilde{X}_{(5)} \) if \( \tilde{X}_{(5)} \geq 6.745 \).”

Finally, populations associated with (10,3) and (9,3) are selected.

As a further example, Wu & Wu (2005) [10] reported data on the duration of remission of four drugs used in the treatment of leukemia. For each drug, a two-parameter exponential distribution was found to fit the data well (Wu et al. 2010) [11]. As the location parameters correspond to minimum guarantee remission times, the larger the location parameter, the better the population. The four scale parameters were found to be significantly different (Wu et al. 2010) [11] using a chi square test.

<table>
<thead>
<tr>
<th>Drug 1</th>
<th>Drug 2</th>
<th>Drug 3</th>
<th>Drug 4</th>
</tr>
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<td>2.951</td>
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</table>

For these and using Algorithm F, \( c = 4.5151 \) from which \( d = 1.1778 \). For this procedure to work, \( \delta_R \leq d \leq \delta_S \), i.e. \( 0.5 \leq d \leq 1 \). Since \( d \) lies outside this range, no selection is made. However, if one lessens the upper bound by letting \( \delta_S = 1.5 \), say, then this value of \( d \) is legitimate giving the following selection rule:

“Select the population associated with \( \tilde{X}_{(4)} \) if \( \tilde{X}_{(4)} \geq 3 + 1.1778 = 4.1778 \).”

Therefore, drug 4 is chosen. Suppose now \( k = 2 \) while everything else remains the same, i.e. we want the best two populations better than the standard. We now have \( c = 4.8711 \) and \( d = 1.3249 \). This value of \( d \) is legitimate. The selection rule is:

“Select the population associated with \( \tilde{X}_{(4)} \) if \( \tilde{X}_{(4)} \geq 4.3249 \) and the population associated with \( \tilde{X}_{(3)} \) if \( \tilde{X}_{(3)} \geq 4.3249 \).”

The final choice is again drug 4. We can interpret the result in the following way. Although \( \tilde{X}_{(3)} = 3.071 \) is better than the standard \( \mu_S = 3 \), it is not judged to be much better than the standard and is therefore rejected.

### 6. Conclusion

The exponential distribution provides a good approximation to many types of lifetime data, especially those within maintenance and reliability theory. Moreover, it leads to much simpler statistical procedures when compared with the Weibull and gamma distribution.

Experimenters often use a control or standard treatment as a benchmark against which to compare the specific treatments. If we have several treatments, one naturally wants to select the one with the longest mean lifetime or those treatments with mean lifetime above the standard and falling not too much below the best one.
In this article, two-stage indifference zone procedures for selecting the best \( k \) exponential populations better than a standard are presented. These procedures are novel in that they work for systems with unknown and possibly unequal location and scale parameters as well as incorporating the minimum significant indifference as a decision making parameter.

**References**


