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Derivation of Some Formulations of Average from One Technique of Construction of Mean

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Abstract: Average is a vital measure which is used not only in research but almost in every day to day work. A number of definitions/formulations are available to be used in different situations. On the other hand, it is possible to construct more and more definitions/formulations of average by a single technique and one such technique has been identified for construction of definition/formulation of average. A number of definitions/formulations of average have been constructed by the technique identified. This paper describes the definitions/formulations of average constructed in the study.

Keywords: Average, Definitions, Formulations, Construction, One Technique

1. Introduction

The term "average" [5, 10, 15] as a commonly speaking term, is used to infer a characteristic of an aggregate of individuals but not of any individual. Average has become a vital concept behind and averaging has become a vital tool of most of the measures used in analysis of data. It is Pythagoras [16] who constructed three definitions/formulations of averages namely arithmetic mean, geometric mean and harmonic mean. Several definitions/formulations of average have already been developed independently, which can also be formulated from the concept of Pythagorean mean. Different formulations of average are suitable for handling data in different situations [1, 5, 15]. The existing formulations of average are not sufficient to handle data of different types. There exist many types of data for which formulations of average are not available to handle with them. Thus, there is necessity of searching for / constructing of more and more definitions / formulations of average that can be suitable for handling data of various characteristics. In this connection, it can be thought that if some technique can be found for defining average in different situations then lots of definitions / formulations can be constructed for average. Accordingly, an attempt has here been made on searching for such technique and consequently one technique has been identified for constructing of definition/formulation of average. By the technique identified a number of definitions/formulations have been constructed for average. This paper describes the definitions/formulations of average constructed in the study.

2. Some Existing Averages

Here

$$x_1,\ x_2, \ldots\ldots\ldots\ldots\ldots,\ x_n$$

denote the *n* numbers of a list and/or the values assumed by a variable *x*.

2.1. Pythagorean Means

The three most common averages, due to Pythagoras, are

Arithmetic Mean, Geometric Mean & Harmonic Mean [2, 3, 4, 12]. In statistics, these three means are used as measures of central tendency of numerical data [14, 20].

Arithmetic Mean

The arithmetic mean of the *n* numbers, as defined by Pythagoras, is

$$\frac{1}{n}(x_1 + x_2 + \ldots\ldots\ldots\ldots\ldots + x_n) \quad (1)$$

Geometric Mean

The geometric mean of the *n* numbers is defined by

$$(x_1 x_2 x_3\ \ldots\ldots\ldots\ldots\ldots x_n)^{1/n} \quad (2)$$

provided the n numbers are positive.

Taking the arithmetic mean of the logarithms of the numbers and then taking the antilogarithm of the arithmetic mean obtained, the geometric can also be defined / formulated as

$$antilog\ \{\frac{1}{n}(log\ x_1 + log\ x_2 + \dots\dots\dots\dots + log\ x_n)\} \quad (3)$$

Thus, geometric mean can be thought of as the *antilog* of the arithmetic mean of the *logs* of the numbers.

Harmonic Mean

The Harmonic mean of the n numbers is defined by

$$\frac{1}{\frac{1}{n}(\frac{1}{x_1} + \frac{1}{x_2} + \dots\dots\dots + \frac{1}{x_n})} \quad (4)$$

or equivalently by

$$\{\frac{1}{n}(x_1^{-1} + x_2^{-1} + \dots\dots\dots\dots + x_n^{-1})\}^{-1} \quad (5)$$

provided the n numbers are all different from 0.

2.2. Some Other Existing Averages

Quadratic Mean (Also termed as Root Mean Square abbreviated as RMS):

Quadratic mean [11] is defined by

$$\sqrt{\left\{\frac{1}{n}\left(x_1^2 + x_2^2 + \dots\dots\dots + x_n^2\right)\right\}} \quad (6)$$

which in another form of expression is as

$$\{\frac{1}{n}(x_1^2 + x_2^2 + \dots\dots\dots\dots + x_n^2)\}^{1/2} \quad (7)$$

Cubic Mean:

Cubic mean [18] is defined by

$$\{\frac{1}{n}(x_1^3 + x_2^3 + \dots\dots\dots\dots + x_n^3)\}^{1/3} \quad (8)$$

Generalized p-Mean:

The expression of this mean is

$$\{\frac{1}{n}(x_1^p + x_2^p + \dots\dots\dots\dots + x_n^p\}^{1/p} \quad (9)$$

Note:

If $p = 1$, the generalized p-mean becomes the arithmetic mean.

If $p = 2$, the generalized p-mean becomes the quadratic mean.

If $p = -1$, the generalized p-mean becomes the harmonic mean.

3. Technique of Construction of Definition of Average

Kolmogorov constructed one definition/formulation of average known as the generalized *f*-mean which is also called Kolmogorov mean after Russian mathematician Andrey Kolmogorov [7, 8, 9, 13, 17, 19].

The generalized *f*-mean of

$$x_1,\ x_2, \dots\dots\dots\dots,\ x_n$$

denoted by $M_f(x_1,\ x_2, \dots\dots\dots\dots,\ x_n)$
is defined by

$$M_f(x_1,\ x_2, \dots\dots\dots\dots,\ x_n) = f^{-1}\ [(1/n)\{f(x_1) + f(x_2) + \dots\dots\dots\dots + f(x_n)\}] \quad (10)$$

where f is any invertible function.

The generalized *f*-mean satisfies the following properties:

Properties of the generalized f-mean:

1. Partitioning: The computation of the mean can be split into computations of equal sized sub-blocks i.e.

$$M_f(x_1, x_2, \dots\dots, x_{nk}) = M_f\{M_f(x_1, x_2, \dots\dots, x_k), M_f(x_{k+1}, x_{k+2}, \dots\dots, x_{2k}), \dots\dots, M_f(x_{(n-1)k+1}, x_{(n-1)k+2}, \dots\dots, x_{nk})\} \quad (11)$$

2. Subsets of elements can be averaged a priori, without altering the mean, given that the multiplicity of elements is maintained.

Thus, with $m = M_f(x_1, x_2, \dots\dots, x_k)$
it holds that

$$M_f(x_1, x_2, \dots\dots, x_k,\ x_{k+1}, x_{k+2}, \dots\dots, x_n) = M_f(m, m, \dots, m,\ x_{k+1}, x_{k+2}, \dots\dots, x_n) \quad (12)$$

3. The *f*-mean is invariant with respect to offsets and scaling of f i.e. for all real a & all non-zero real b,

$$g(t) = a + b\,f(t) \text{ implies } M_g(x) = M_f(x) \quad (13)$$

4. If f is monotonic, then M_f is also monotonic.

5. Any *f*-mean M_f of two variables has the mediality property namely

$$M_f[M_f\{(x, M_f(x,y)\}, M_f\{(y, M_f(x,y)\}] = M_f(x,y) \quad (14)$$

and the self-distributive property namely

$$M_f\{x, M_f(y,z)\} = M_f\{M_f(x,y), M_f(x,z)\} \quad (15)$$

Moreover, any of those properties is essentially sufficient to characterize quasi-arithmetic means.

6. Any *f*-mean M_f of two variables x & y has the balancing property namely

$$M_f[M_f\{(x, M_f(x,y)\}, M_f\{(y, M_f(x,y)\}] = M_f(x,y) \quad (16)$$

7. Under regularity conditions, a central limit theorem can be derived for the generalized *f*-mean, thus implying that for a large sample the expresion

$$\sqrt{n}.[\{ M_f(x_1, x_2, \ldots\ldots, x_n) - f^{-1}\{E_f(x_1, \ldots\ldots, x_n)\}],$$

where $E_f(x_1, x_2, \ldots\ldots\ldots, x_n)$ is the generalized *f*-expectation,
is approximately normal.

Remark:

This definition / formulation can be applied in searching for / constructing of a number of definitions / formulations for average.

The technique can be summarized in the following steps:

(1) Select suitable function which is invertible.
(2) Then find out the inverse function of the function selected.
(3) Then apply the function selected in the definition of *f*-Mean defined by equation (10).

4. Construction of Some Definitions/Formulations of Average from the Generalized *f*-Mean

Arithmetic Mean:

Let the invertible function *f*(.) be such that

$$f: x \rightarrow x$$

(i.e. *f* maps x to x)
i.e. $f(x) = x$
Here, $f^{-1} f(x) = f^{-1}(x)$
which implies, $I(x) = f^{-1}(x)$
where $I(.)$ is the identity map (function)
i.e. $I(x) = x$
Accordingly, $f^{-1}(x) = x$
Thus f^{-1} (.) is a function with

$$f^{-1:} x \rightarrow x$$

(i.e. f^{-1} also maps x to x)
Accordingly, if in the generalized *f*-mean, *f*(.) is selected as

$$f(x) = x$$

then the *f*-mean becomes

$$\frac{1}{n}(x_1 + x_2 + \ldots\ldots\ldots\ldots\ldots + x_n)$$

which is nothing but Pythagorean arithmetic mean defined by equation (1).

Geometric Mean:

Let the invertible function *f*(.) be such that

$$f: x \rightarrow \log x$$

(i.e. *f* maps x to $\log x$)
i.e. $f(x) = \log x$
Then $f^{-1}(\log x) = x$
i.e. $f^{-1}(\log x) = antilog\ (\log x)$
i.e. $f^{-1}(y) = antilog\ y$
Thus f^{-1} (.) is a function with

$$f^{-1:} x \rightarrow \text{antilog } x$$

(i.e. f^{-1} maps x to *antilog* x)
Accordingly, if in the generalized *f*-mean *f*(.) is selected as

$$f(x) = \log x$$

then the *f*-mean becomes

$$antilog\ \{\frac{1}{n}(log\ x_1 + log\ x_2 + \ldots\ldots\ldots\ldots\ldots + log\ x_n)\}$$

which is nothing but Pythagorean geometric mean defined by equation *(3) which is same as with that* defined by equation (2).

Harmonic Mean:

Let the invertible function *f*(.) be such that

$$f: x \rightarrow x^{-1}$$

(i.e. *f* maps x to x^{-1})
i.e. $f(x) = x^{-1}$
Then $f^{-1} = (x^{-1}) = x$
i.e. $f(y) = y^{-1}$ putting $y = x^{-1}$
Thus f^{-1} (.) is a function with

$$f^{-1:} x \rightarrow x^{-1}$$

(i.e. f^{-1} also maps x to x^{-1})
Accordingly, if in the generalized *f*-mean *f*(.) is selected as

$$f(x) = x^{-1}$$

then the *f*-mean becomes

$$\{\frac{1}{n}(x_1^{-1} + x_2^{-1} + \ldots\ldots\ldots\ldots\ldots + x_n^{-1})\}^{-1}$$

which is nothing but Pythagorean harmonic mean defined by equation (5).

Quadratic Mean:

Let the invertible function *f*(.) be such that

$$f: x \rightarrow x^2$$

(i.e. *f* maps x to x^2)
i.e. $f(x) = x^2$

Then $f^{-1}(x^2) = x$
i.e. $f^{-1}(x) = \sqrt{x}$
Thus f^{-1} (.) is a function with

$$f^{-1}: x \to \sqrt{x}$$

(i.e. f^{-1} maps x to $\sqrt{x}$)
Accordingly, if in the generalized f-mean f(.) is selected as

i.e. $f(x) = x^2$

then the f-mean becomes

$$\{\frac{1}{n}(x_1^2 + x_2^2 + \dots\dots\dots\dots. + x_n^2)\}^{1/2}$$

which is nothing but the quadratic mean defined by equation (6).

Square Root Mean:
Let the invertible function f(.) be such that

$$f: x \to \sqrt{x}$$

Then f^{-1} (.) is a function with

$$f^{-1}: x \to x^2$$

Accordingly, if in the generalized f-mean f(.) is selected as

$$f(x) = \sqrt{x}$$

then the f-mean becomes the squared mean defined by

$$\left\{\frac{1}{n}\left(\sqrt{x_1} + \sqrt{x_2} + \dots\dots\dots. + \sqrt{x_n}\right)\right\}^2 \quad (17)$$

This can also be a definition / formulation of average.
This definition / formulation of average can be termed as Square Root Mean.

Cubic Mean:
Let the invertible function f(.) be such that

$$f: x \to x^3$$

(i.e. f maps x to x^3)
i.e. $f(x) = x^3$
Then f^{-1} (.) is a function with

$$f^{-1}: x \to x^{1/3}$$

(i.e. f^{-1} maps x to $x^{1/3}$)
Accordingly, if in the generalized f-mean f(.) is selected as

$$f(x) = x^{1/3}$$

then the f-mean becomes

$$\{\frac{1}{n}(x_1^3 + x_2^3 + \dots\dots\dots\dots. + x_n^3)\}^{1/3}$$

which is the cubic mean defined by equation (8).

Cube Root Mean:
Let the invertible function f(.) be such that

$$f: x \to x^{1/3}$$

(i.e. f maps x to $x^{1/3}$)
i.e. $f(x) = x^{1/3}$
Then f^{-1} (.) is a function with

$$f^{-1}: x \to x^3$$

(i.e. f^{-1} maps x to x^3)
Accordingly, if in the generalized f-mean f(.) is selected as

$$f(x) = x^{1/3}$$

then the f-mean becomes

$$\{\frac{1}{n}(x_1^{1/3} + x_2^{1/3} + \dots\dots\dots\dots. + x_n^{1/3})\}^3 \quad (18)$$

This can also be a definition / formulation of average.
This definition / formulation of average can be termed as cube root mean.

Generalized p-Mean:
Let the invertible function f(.) be such that

$$f: x \to x^p$$

Then f^{-1} (.) is a function with

$$f^{-1}: x \to x^{1/p}$$

Accordingly, if in the generalized f-mean f(.) is selected as

$$f(x) = x^p$$

then the generalized f-mean becomes

$$\{\frac{1}{n}(x_1^p + x_2^p + \dots\dots\dots\dots. + x_n^p)\}^{1/p}$$

which is nothing but the Generalized p-Mean.

Generalized p^{th} Root Mean:
Let the invertible function f(.) be such that

$$f: x \to x^{1/p}$$

Then f^{-1} (.) is a function with

$$f^{-1}: x \to x^p$$

Accordingly, if in the generalized f-mean f(.) is selected as

$$f(x) = x^{1/p}$$

then the f-mean becomes

$$\{\frac{1}{n}(x_1^{1/p} + x_2^{1/p} + \dots\dots\dots\dots. + x_n^{1/p})\}^p$$

This can also be a definition / formulation of average.
This definition / formulation of average can be termed as Generalized p^{th} Root Mean.

e-Mean:
Let the invertible function f(.) be such that

$$f: x \to e^x$$

In this case, f^{-1} (.) will be a function with

$$f^{-1}: x \to ln\ e^x$$

Accordingly, if in the generalized f-mean $f(.)$ is selected as

$$f(x) = e^{x}$$

then the f-mean becomes

$$\log_e \{\frac{1}{n}(e^{x_1} + e^{x_2} + \dots\dots\dots\dots + e^{x_n}\} \quad (19)$$

This can also be an average as per the logic behind Technique.

This definition / formulation of average can be termed as e-Mean.

Note:

Selecting the function $f(.)$ as

$$f(x) = 10^{x}$$

in place of

$$f(x) = e^{x} = exp(x)$$

so that

$$f^{-1}(x) = log_{10} x$$

one average can similarly be constructed which can be termed as that 10 base exponent mean.

$$\log_{10} \{\frac{1}{n}(10^{x_1} + 10^{x_2} + \dots\dots\dots\dots + 10^{x_n}\} \quad (20)$$

Scale Mean or simply s-Mean:

Let the invertible function $f(.)$ be such that

$f: x \rightarrow s.x,$ for non-zero real number s

In this case, $f^{-1}(.)$ will be a function with

$$f^{-1:} x \rightarrow x/s$$

Accordingly, if in the generalized f-mean $f(.)$ is selected as

$$f(x) = s.x$$

then the f-mean becomes

$$\frac{1}{s}\{\frac{1}{n}(sx_1 + sx_2 + \dots\dots\dots\dots + sx_n)\} \quad (21)$$

This can also be definition/formulation of average.

This average can be termed as s-Mean.

Note:

Selecting the function $f(.)$ as

$f(x) = \frac{1}{s}.x,$ for non-zero real number s

one can obtain the f-mean as

$$s\{\frac{1}{n}(\frac{x_1}{s} + \frac{x_1}{s} + \dots\dots\dots\dots\dots + \frac{x_1}{s})\} \quad (22)$$

This can be termed as $\frac{1}{s}$ -Mean.

a -Shift Mean or simply a-Mean:

Let the invertible function $f(.)$ be such that

$f: x \rightarrow x - a,$ for some real a

In this case, $f^{-1}(.)$ will be a function with

$$f^{-1:} x \rightarrow x + ab$$

Accordingly, if in the generalized f-mean $f(.)$ is selected as

$$f(x) = x - a$$

then the f-mean becomes

$$\frac{1}{n}\{(x_1 - a) + (x_2 - a) + \dots\dots\dots\dots + (x_n - a)\} + a \quad (23)$$

This average can be termed as a-Mean.

Note:

Selecting the function $f(.)$ as

$f(x) = \frac{1}{s}(x - a),$ for real a non-zero real s

one can obtain that the f-mean comes down to be

$$s\{\frac{1}{n}(\frac{x_1 - a}{s} + \frac{x_2 - a}{s} + \dots\dots\dots\dots\dots + \frac{x_n - a}{s})\} + a \quad (24)$$

This can be termed as shift (a) - scale ($\frac{1}{s}$) -Mean.

5. Some Definitions/Formulations of Average of Variable

From the generalized f-mean due to Kolmogorov defined by equation (10), one can define the generalized f-mean of a function

$$g = g(.) = g(x) \quad (25)$$

of x *by*

$$f^{-1}[\frac{1}{n}\{f(g(x_1)) + f(g(x_2)) + \dots\dots\dots\dots + f(g(x_n))\}] \quad (26)$$

where f is any invertible function.

From this definition, one can obtain the definitions/formulations of various means as mentioned above, for a function of variable, as follows:

Arithmetic Mean:

Arithmetic Mean of $g(x)$ can be obtained as

$$\frac{1}{n}\{g(x_1) + g(x_2) + \dots\dots\dots\dots + g(x_n)\} \quad (27)$$

In particular,

Arithmetic Mean of x^2 *is*

$$\frac{1}{n}(x_1^2 + x_2^2 + \dots\dots\dots\dots + x_n^2) \quad (28)$$

Similarly, the Arithmetic Mean of |x| *is*

$$\frac{1}{n}(|x_1| + |x_2| + \dots\dots\dots\dots\dots\dots + |x_n|) \quad (29)$$

Arithmetic Mean of x^3 *is*

$$\frac{1}{n}(x_1^3 + x_2^3 + \dots\dots\dots\dots + x_n^3) \quad (30)$$

Arithmetic Mean of x^p *is*

$$\frac{1}{n}(x_1^p + x_2^p + \dots\dots\dots\dots + x_n^p) \quad (31)$$

Arithmetic Mean of $x^{1/p}$ *is*

$$\frac{1}{n}({x_1}^{1/p} + {x_2}^{1/p} + \ldots\ldots\ldots\ldots. + {x_n}^{1/p}) \quad (32)$$

Arithmetic Mean of e^x *is*

$$\frac{1}{n}(e^{x_1} + e^{x_2} + \ldots\ldots\ldots\ldots. + e^{x_n}) \quad (33)$$

Arithmetic Mean of log x *is*

$$\frac{1}{n}(\log x_1 + \log x_2 + \ldots\ldots\ldots\ldots\ldots\ldots. + \log x_n) \quad (34)$$

Geometric Mean:

From this definition given by equation (26), Geometric Mean of $g(x)$ can be obtained as

$$\{(g(x_1).g(x_2).g(x_3) \ldots\ldots\ldots\ldots g(x_n)\}^{1/n} \quad (35)$$

Harmonic Mean:

Similarly, Harmonic Mean of $g(x)$ can be obtained as

$$\frac{1}{\frac{1}{n}\{\frac{1}{g(x_1)} + \frac{1}{g(x_2)} + \ldots\ldots\ldots + \frac{1}{g(x_n)}\}} \quad (36)$$

Quadratic Mean:

Quadratic Mean of $g(x)$ can be obtained as

$$\{\frac{1}{n}({g_1}^2 + {g_2}^2 + \ldots\ldots\ldots\ldots. + {g_n}^2)\}^{1/2} \quad (37)$$

where $g_1 = g(x_1)$, $g_2 = g(x_2)$, ………, $g_n = g(x_n)$

Cubic Mean:

Cubic Mean of $g(x)$ can be obtained as

$$\{\frac{1}{n}({g_1}^3 + {g_2}^3 + \ldots\ldots\ldots\ldots. + {g_n}^3)\}^{1/3} \quad (38)$$

Generalized p Mean:

Generalized p Mean of $g(x)$ can be obtained as

$$\{\frac{1}{n}({g_1}^p + {g_2}^p + \ldots\ldots\ldots\ldots. + {g_n}^p)\}^{1/p} \quad (39)$$

Generalized p^{th} Root Mean:

Generalized p^{th} Root Mean of $g(x)$ can be obtained as

$$\{\frac{1}{n}({g_1}^{1/p} + {g_2}^{1/p} + \ldots\ldots\ldots\ldots. + {g_n}^{1/p})\}^p \quad (40)$$

e Mean:

e Mean of $g(x)$ can be obtained as

$$\log_e \{\frac{1}{n}(e^{g_1} + e^{g_2} + \ldots\ldots\ldots\ldots. + e^{g_n}\} \quad (41)$$

Scale *s* Mean or simply *s* Mean:

Scale *s* Mean or simply *s* Mean of $g(x)$ can be obtained as

$$\frac{1}{s}\{\frac{1}{n}(s\, g_1 + s\, g_2 + \ldots\ldots\ldots\ldots. + s\, g_n)\} \quad (42)$$

Scale $\frac{1}{s}$ Mean or simply $\frac{1}{s}$ -Mean:

Scale $\frac{1}{s}$ Mean or simply $\frac{1}{s}$ -Mean of $g(x)$ can be obtained as

$$s\,\{\frac{1}{n}(\frac{g_1}{s} + \frac{g_2}{s} + \ldots\ldots\ldots\ldots\ldots.. + \frac{g_n}{s})\} \quad (43)$$

a - Shift Mean or simply *a*-Mean:

a - Shift Mean or simply *a*-Mean of $g(x)$ can be obtained as

$$\frac{1}{n}\{(g_1 - a) + (g_2 - a) + \ldots\ldots\ldots\ldots + (g_n - a)\} + a \quad (44)$$

Shift (a) - scale ($\frac{1}{s}$) – Mean:

Shift (a) - scale ($\frac{1}{s}$) - Mean of $g(x)$ can be obtained as

$$s\,\{\frac{1}{n}(\frac{g_1 - a}{s} + \frac{g_2 - a}{s} + \ldots\ldots\ldots\ldots\ldots.. + \frac{g_n - a}{s})\} + a \quad (45)$$

6. Conclusion

One can conclude that the generalized *f*–mean due to Kolmogorov can be regarded as a source from where lots of definitions/formulations can be derived for various types of averages.

Different types of formulations of average are necessary for handling different types of data. That is why we need more and more formulations of average.

The types of average formulated here have been derived from the Generalized *f*-Mean due to Kolmogorov. However, this Generalized *f*-Mean, for generating means, is not sufficient to yield many types of averages to deal with many types of data. Thus, there is necessity of further study on searching for more and more techniques of defining / formulating of more types of averages.

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