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Synthesis of Optimal Control Program of Spacecraft Attitude Taking into Account Energy of Rotation

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Abstract: The solving the original dynamical control problem of optimal reorientation from a state of rest to a state of rest is investigated. The control function is torque vector. The case, when control is limited and the used functional takes into account kinetic rotation energy and time of maneuver, is studied in detail. For designing the optimal control program, the quaternion method and the Pontryagin's maximum principle are used. Analytic solution of the proposed problem is presented basing on the differential equation connecting the angular velocity vector and quaternion of spacecraft attitude. It is shown that the chosen criterion of quality provides a turn of a spacecraft with rotation energy which do not exceed the required value. This property of the proposed control increases safety of flight. All key expressions and equations are written in quaternion form which is convenient for onboard realization and implementation. Analytical formulas were written for duration of acceleration and braking. For specific cases of spacecraft's configurations (dynamically symmetric and spheric-symmetrical spacecraft as particular cases), complete solution of optimal control problem in closed form is given. Numerical example and results of mathematical simulation for spacecraft motion under optimal control are demonstrated. This data supplements the made theoretical descriptions, and illustrates reorientation process in visual form.

Keywords: Rotation Maneuver, Quaternion of Attitude, Optimal Control Problem, Criterion of Quality, Maximum Principle

1. Introduction

The optimal control problem of transfering the spacecraft into the required angular position was solved. Kinematics of motion is described by quaternion models [1]. Many authors investigated the optimal solutions to problems of controlling a spacecraft's angular position [1-25]. The solutions which correspond to rotation around a motionless axis are known [1-6], and rotation maneuvers around the principal central axe were studied in detail [3, 4]. Time-optimal maneuvers is topical [3, 4, 7-13]. Specific solutions are obtained for axi-symmetric spacecraft [13-15]. For example, some authors did replacement of variables, formulated equivalent boundary-value problem of maximum principle and reduced an initial control problem to reorientation problem for spheric-symmetrical body [14]; special control regime of rotation was studied also [15]. Attitude control of the spacecrafts with inertial actuators has specific features [16-19], and the patented method is known [20]. An analytical solution to the optimal reorientation problem in a closed form, if it were found, would be of great practical interest, because it allows the finished laws of the programmed control and the optimal trajectory of spacecraft motion to be applied onboard of a spacecraft [8, 9].

Finding and studying the optimal control program for spacecraft reorientation (with respect to rotation energy) is topic and subject of this research. Principal difference of the presented work consists in use of new minimized index which combines the duration of maneuver and integral of kinetic rotation energy. Minimization of the adopted index of quality is very important problem in practice of spacecraft flight. The main results are: it was shown that two-impulse control is optimum (with one or two switching), and, in many cases, spacecraft rotates by inertia between acceleration and braking; for optimal solution, estimations of the relative growth of the index of quality due to the limited controlling moment (with respect to ideal rotation when torque is unbounded) were done.

2. Statement of Optimization Problem

Angular motion of the spacecraft as rigid body is described by dynamic equations [4]:

$$\begin{aligned} &J_1\dot{\omega}_1+\left(J_3-J_2\right)\omega_2\omega_3=M_1,\\ &J_2\dot{\omega}_2+\left(J_1-J_3\right)\omega_1\omega_3=M_2,\\ &J_3\dot{\omega}_3+\left(J_2-J_1\right)\omega_1\omega_2=M_3 \end{aligned} \tag{1}$$

where J_i are the principal central moments of inertia, M_i are projections of torque M onto the axises of the basis E formed by the principal central axes of spacecraft's inertia, ω_i are projections of the spacecraft's absolute angular velocity vector $\boldsymbol{\omega}$ onto the axises of the basis E ($i=\overline{1,3}$). To describe spatial motion of a spacecraft, the quaternions are used [1]. Angular position of the body coordinate system is defined relative to the reference basis I which is inertial coordinate system (as we assume). Motion of the basis E relative to the reference basis I is determined by the quaternion Λ [1]. Therefore, the following kinematic equation is true:

$$2\dot{\Lambda}=\Lambda\circ\boldsymbol{\omega} \tag{2}$$

For simplicity, it is assumed that the quaternion Λ specifying the current attitude is the normalized quaternion ($\|\Lambda\|=1$). The spacecraft motion control relative to its center of mass is done by change of the torque M (external or internal, if attitude control is done with use of inertial actuators, i.e. powered gyroscopes). Let us assume that region of admissible values for the vector M is described by the inequality

$$M_1^2/J_1+M_2^2/J_2+M_3^2/J_3\le u_0^2 \tag{3}$$

where $u_0>0$ is some positive value specifying power of actuators of spacecraft attitude system. In many practical modes of reorientation, initial state satisfies condition $\boldsymbol{\omega}(0)=0$ and final angular velocity must be absent $\boldsymbol{\omega}(T)=0$ (these cases occur very frequently, especially if attitude control is done relative to inertial coordinate system). The angular positions of the initial and final spacecraft attitude with respect to the reference basis I are given by the quaternions Λ_{in} and Λ_f, respectively. The boundary conditions are:

$$\Lambda(0)=\Lambda_{in},\ \boldsymbol{\omega}(0)=0 \tag{4}$$

$$\Lambda(T)=\Lambda_f,\ \boldsymbol{\omega}(T)=0 \tag{5}$$

where T is the time of ending the reorientation process, and the quaternions Λ_{in} and Λ_f which specify the position of spacecraft's axes at the initial and final moments of time have arbitrary predefined values satisfying the condition $\|\Lambda_{in}\|=\|\Lambda_f\|=1$. For optimization of rotation control, quadratic criterion of quality (together with time factor) is used [26]. Effectiveness of control is estimated by the index

$$G=\int_0^T(J_1\omega_1^2+J_2\omega_2^2+J_3\omega_3^2)dt+k_0T \tag{6}$$

where $k_0>0$ is a constant positive coefficient.

The reorientation optimal control problem is formulated as follows: spacecraft must transfer from the state (4) into the state (5) according to the equations (1), (2) and restriction (3) with minimal value of the functional (6) (time T, when the spacecraft reorientation maneuver should end, is not fixed). The assumed criterion of quality allows us to determine the energetically advantageous angular motion trajectory along which the spacecraft will turn from its initial position Λ_{in} into the required final angular position Λ_f and find the corresponding control mode. Also, the chosen criterion of quality provides turn of a spacecraft with the bounded rotation energy.

3. Solution Procedure of the Optimal Control Problem

It is considered, angular velocity projections ω_i are controllable variables (for minimization of index (6)). In spite of the fact that value of the functional (6) does not explicitly depend on the controlling moment M (expression (6) does not contain M_i), the proposed problem of optimal control is a dynamic rotation problem [1], where the moments M_i serve as control functions. The restriction for phase variable Λ is insignificant because it is fulfilled at any motion about the center of mass); the norm $\|\Lambda\|$ of the attitude quaternion Λ is constant due to equation (2), $\|\Lambda\|=$ const [1].

For solving the posed problem (1)- (6), the Pontryagin's maximum principle is used [27]. Let ϕ_i be the conjugate variables that correspond to the angular velocities ω_i. Since the minimized index (6) does not include the position coordinates, the universal variables r_i can be used ($I=\overline{1,3}$) [21]. The Hamiltonian H for the problem (1)- (6) is:

$$\begin{aligned} H=\ &\varphi_1(M_1+(J_2-J_3)\,\omega_2\,\omega_3)/J_1+\varphi_2(M_2+(J_3-J_1)\,\omega_1\,\omega_3)/J_2+\varphi_3(M_3+\\ &(J_1-J_2)\,\omega_1\,\omega_2)/J_3+\omega_1r_1+\omega_2r_2+\omega_3r_3-J_1\omega_1^2-J_2\omega_2^2-J_3\omega_3^2-k_0 \end{aligned} \tag{7}$$

where r_i are [21]

$$r_1=(\lambda_0\psi_1+\lambda_3\psi_2-\lambda_1\psi_0-\lambda_2\psi_3)/2,$$

$$r_2=(\lambda_0\psi_2+\lambda_1\psi_3-\lambda_2\psi_0-\lambda_3\psi_1)/2,$$

$$r_3=(\lambda_0\psi_3+\lambda_2\psi_1-\lambda_3\psi_0-\lambda_1\psi_2)/2$$

ψ_j are the conjugate variables that correspond to the components of the quaternion λ_j ($j=\overline{0,3}$).

The function H does not take into account the constraint

$\|\Lambda\| = 1$ since $\|\Lambda\ (0)\| = 1$. For the universal variables r_i, we have [21]:

$$\dot{r}_1 = \omega_3 r_2 - \omega_2 r_3\ ,\ \dot{r}_2 = \omega_1 r_3 - \omega_3 r_1\ ,\ \dot{r}_3 = \omega_2 r_1 - \omega_1 r_2 \qquad (8)$$

Change in vector r formed by the universal variables r_i is given by the solution of the equation

$$\dot{\mathbf{r}} = -\boldsymbol{\omega}\times\mathbf{r}$$

(the symbol × denotes the vector product of two vectors). The vector r is motionless relative to the inertial basis I, and $|\mathrm{r}| = \text{const} \neq 0$ [21]. The equations for conjugate variables φ_i have the form

$$\dot{\phi}_i = -\frac{\partial H}{\partial \omega_i}$$

Therefore, the conjugate system of equations is

$$\begin{aligned}\dot{\phi}_1 &= 2J_1\omega_1 - \omega_3 n_2\phi_2 - \omega_2 n_3\phi_3 - r_1\\ \dot{\phi}_2 &= 2J_2\omega_2 - \omega_3 n_1\phi_1 - \omega_1 n_3\phi_3 - r_2\\ \dot{\phi}_3 &= 2J_3\omega_3 - \omega_2 n_1\phi_1 - \omega_1 n_2\phi_2 - r_3\end{aligned} \qquad (9)$$

where $n_1 = (J_2 - J_3) / J_1$, $n_2 = (J_3 - J_1) / J_2$, $n_3 = (J_1 - J_2) / J_3$ are the constant coefficients.

Thus, the problem of finding an optimal control is reduced to solving the system of equations of spacecraft's angular motion (1), (2), and equations (8), (9) under the condition that the control itself is chosen by maximizing the Hamiltonian. The optimal function r (t) is related with Λ (t) by the formula

$$\mathbf{r} = \tilde{\Lambda}\circ\mathbf{c}_E\circ\Lambda\text{, where } \mathrm{c}_E = \Lambda_{\text{in}}\circ\mathbf{r}(0)\circ\tilde{\Lambda}_{\text{in}} = \text{const}$$

The direction of vector c_E depends on the initial and final positions. In order for the spacecraft to have the required attitude at the right-hand end $\Lambda(T) = \Lambda_{\mathrm{f}}$, the vector r (0) should be determined by the corresponding solution of equation (2). The system of differential equations (8), (9), together with the maximality condition of the Hamiltonian H, are necessary conditions of optimality. The maximum conditions of the Hamiltonian H determine sought solution M (t). Boundary positions Λ (0) and Λ (T) determine the solutions Λ (t) and r (t). The boundary problem of the maximum principle is to find the value of the vector r (0) for which the solution of system of differential equations (1), (2), (8), (9) together with simultaneous maximization of the Hamiltonian H, at every current moment of time, satisfies reorientation conditions (4), (5).

To find the control function M (t) (the optimal control program) and the optimal vector r, the conditions of maximum for Hamiltonian H must be formalized. Let us rewrite the function H in the form

$$H = \varphi_1 M_1 / J_1 + \varphi_2 M_2 / J_2 + \varphi_3 M_3 / J_3 + H_{\text{inv}}$$

where H_{inv} does not explicitly depend on the control functions M_i. Let φ be the vector with components ϕ_i . If $\varphi \neq 0$, the maximum of the function H for the controls M_i (t) under restriction (3) is achieved when

$$M_i = \frac{u_0\phi_i}{\sqrt{\phi_1^2 / J_1 + \phi_2^2 / J_2 + \phi_3^2 / J_3}} \qquad (10)$$

(the case $\varphi = 0$, in which the Hamiltonian does not explicitly depend on the control M, requires additional consideration). Further we will demonstrate that M = 0 if $\dot{\boldsymbol{\phi}} = 0$ (because $\varphi = 0$ if $\dot{\boldsymbol{\phi}} = 0$). The optimal solution is determined by the closed system of equations (1), (2), and (8)- (10) considering the conditions (4) and (5). Due to the fact that $|\mathrm{r}| = \text{const} = |\mathrm{r}\ (0)| \neq 0$, for simplicity, the normalized vector $\mathrm{p} = \mathrm{r} / |\mathrm{r}|$ is used, $|\mathrm{p}| = 1$. For the vector p, we have $\dot{\mathbf{p}} = -\boldsymbol{\omega}\times\mathbf{p}$, or

$$\begin{aligned}\dot{p}_1 &= \omega_3 p_2 - \omega_2 p_3,\\ \dot{p}_2 &= \omega_1 p_3 - \omega_3 p_1,\\ \dot{p}_3 &= \omega_2 p_1 - \omega_1 p_2\end{aligned} \qquad (11)$$

where p_i are the components of the vector p. Note that $r_i = |\mathrm{r}\ (0)|p_i$. Solution to the system of equations (1) and (8)- (10) (under the requirement ω (0) = ω (T) = 0) has the form

$$\phi_i = a(t)p_i \qquad (12)$$

$$J_i\omega_i = b\ p_i \qquad (13)$$

where b is a scalar value; a (t) is scalar function of time with $\dot{a} \leq 0$ ($b \geq 0$ for optimal motion ω (t)).

After substituting solution (12), (13) into the system (9) and considering the equations (11) for the derivatives $\dot{p}_i$, we obtain the identity expressions if $\dot{a}$ p = $(2b - r_0)$ p, where $r_0 = |\mathrm{r}\ (0)|$. Therefore, the optimal functions a (t) and b (t) satisfy the condition $\dot{a} = 2b - r_0$ (since $|\mathrm{p}| \neq 0$), from which two features follow: $\dot{a}$ (0) = $\dot{a}$ (T) = $-r_0$ and b (0) = b (T) = 0 (due to the requirement ω (0) = ω (T) = 0). At initial instant t = 0, a (0)>0; otherwise M·p<0 and b<0 due to the equations (1), (10), (12), and $\dot{a}$ <0, a<0 for any t>0. However, in such a scenario (when a (0)≤0), the switching is absent (since b≤0 and $\dot{a}$ <0); the torque M acts in one direction, accelerating the spacecraft until ω→∞. Accordingly, at end of reorientation maneuver, the condition M·L<0 is necessary (and M·p<0 also) and a (T)<0. The scalar function a (t) is the continuous function of time. Therefore, moment of time when a (t) = 0 exists. If $\dot{a}$ = 0 then a (t) = 0, $\dot{\boldsymbol{\phi}} = 0$, φ = 0 (otherwise the value a (t) does not change a sign, i.e. a (t)>0 and M·L>0 during interval of control [0, T], but such rotation does not satisfy the condition **ω** (T) = 0); L is angular momentum of a spacecraft (the symbol · denotes the scalar product of vectors).

From (1) and (13), $\dot{b}$ = M·p. If a (t)>0, we have acceleration process. If a (t)<0, we have braking; $\ddot{a}$ >0 for acceleration phase [0, t_1], $\ddot{a}$ <0 for braking [t_2, T] (since $\ddot{a}$ =

$2\dot{b}$). For functions a (t) and b (t), we have the following properties: a (T-t) = -a (t) and b (T-t) = b (t).

The equalities (12), (13) are satisfied together. For spin-up, the optimal controlling moment M can be calculated by the formula

$$M_i = u_0 J_i \omega_i \Big/ \sqrt{J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2} \quad (14)$$

Optimal torque M and angular momentum L are parallel during acceleration phase. Differentiation of left and right parts of the equalities (14) gives the following differential equations:

$$\begin{aligned} \dot{M}_1 &= \omega_3 M_2 - \omega_2 M_3, \\ \dot{M}_2 &= \omega_1 M_3 - \omega_3 M_1, \\ \dot{M}_3 &= \omega_2 M_1 - \omega_1 M_2 \end{aligned} \quad (15)$$

(angular accelerations $\dot{\omega}_i$ are taken from dynamic equations (1)). Rewrite last equations in vector form

$$\dot{\mathbf{M}} = -\boldsymbol{\omega} \times \mathbf{M}$$

The obtained differential equation means immobility of the vector M relative to inertial coordinate system. As consequence, |M| = const during acceleration stage. For optimal braking, the torque M is

$$M_i = -u_0 J_i \omega_i \Big/ \sqrt{J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2} \quad (16)$$

After differentiation of equalities (16) we obtain differential equations (15) from which the property |M| = const appears for the entire braking stage. Thus, equality |M| = const is satisfied for optimal rotation during acceleration and braking phases (direction of vector M is not changed relative to inertial basis I); i.e. within acceleration and braking segments, optimal torque M is the fixed vector relative to inertial coordinate system. If relations (13) are fulfilled, then

$$p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3 = \text{const}$$

To be certain of this, let us differentiate the left-hand part of the given equality with respect to time considering the equations (11) for p_i and dependences (13) for components ω_i of angular velocity.

$$p_1\dot{p}_1/J_1 + p_2\dot{p}_2/J_2 + p_3\dot{p}_3/J_3 = \omega_1 p_2 p_3/J_3 - \omega_1 p_2 p_3/J_2 + \omega_1 p_2 p_3/J_2 - \omega_2 p_1 p_3/J_3 + \omega_2 p_1 p_3/J_3 - \omega_1 p_2 p_3/J_3 \equiv 0$$

For optimal solution (12), the dependences (10) can be rewritten in the form

$$M_i = \frac{u_0 \text{sign} a(t) p_i}{\sqrt{p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3}}$$

Thus, when a (t) ≠ 0 and φ ≠ 0, the statement |M| = const ≠ 0 is true, and, therefore, $\dot{b}$ = const, as well. The torque M satisfies the condition (3). Therefore, we can write the following relation

M = $m_0$sign$a$ (t) p, where $m_0 = u_0 / C$,

$$\sqrt{p_1^2(0)/J_1 + p_2^2(0)/J_2 + p_3^2(0)/J_3}$$

For time interval when a (t) = const = 0, the system (9) is transformed to the equations $2J_i\omega_i - r_i = 0$, and the relations

$$\omega_i = r_i / 2J_i \quad (17)$$

are satisfied. Let us find the controlling moments within time interval $t_1<t<t_2$, during the rotation with a (t) = const = 0. Substitute the functions ω_i (t) computed by the expressions (17) into dynamic equations (1) with taking into account the fact |r| = const ≠ 0. As result, all components M_i are $M_i = 0$. Between acceleration phase and braking M = 0 and b (t) = r_0 / 2. This follows from the analysis of equations (17) that show a relation between angular momentum L and the vector r of universal variables. The fact that L = r / 2 and |L| = const, keeping in mind the immobility of vector r in the inertial basis I, implies that the spacecraft's angular momentum vector is constant relative to inertial coordinate system; kinetic energy E_k is constant also ($\dot{E}_k = \boldsymbol{\omega} \cdot \mathbf{M}$ = 0 because M = 0).

From formula (13), we see that b = |L| and direction of angular momentum relative to the inertial coordinate system is constant. The equations (13) clearly demonstrate that the vector p is the unit vector of the spacecraft's angular momentum vector L. Equations (11), together with equalities (13), form a closed system of equations which determine unique properties of optimal motion; the optimal reorientation (in the sense of minimizing the index (6)) is performed along the "trajectory of free motion" (the concept of "trajectory of free motion" was described earlier [12]). The optimal function b (t) is a non-negative piecewise-linear function of time: b (t) = 0 for t = 0 and t = T; at $t = t_1$ and $t = t_2$, b (t) = r_0 / 2. Duration of maneuver T is equal to $T = t_2+t_1$ (since |M| = m_0 for acceleration segment and segment of braking, and $t_1 = r_0 / 2m_0$) Optimal motion is determined by the system of equations (11), (13), (2) with the conditions (4), (5) for solution Λ (t).

Let us find the proportion between the angular kinetic energy E_k and angular momentum L during optimal reorientation. Kinetic energy E_k and the value b are related by expression

$$E_k = b^2 (p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3)/2$$

Therefore, the proportion:

$$E_k / |\mathbf{L}|^2 = (p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3)/2 = \text{const}$$

For segments when a (t) ≠ 0 and M ≠ 0, we have $|\mathbf{M}|^2 = u_0^2 |\mathbf{L}|^2 / (2E_k)$ (see (14), (16)), i.e.

$E_k / |\mathbf{L}|^2 = u_0^2 / (2|\mathbf{M}|^2) = \text{const}$ (because |M| = const = m_0 if $a(t) \neq 0$). If $a(t) = 0$, then M = 0, |L| = const and E_k = const. The quantities |L| and E_k are the continuous functions, therefore, E_k / $|L|^2$ is continuous function of time, and it is constant within all three segments of control, hence, this proportion $E_k / |L|^2$ = const within entire interval $0 \le t \le T$. It is key property of optimal motion for the criterion (6).

The Hamiltonian H is independent of time in explicit form, i.e. $\partial H / \partial t = 0$. Therefore, H = const inside the entire interval of control $0 \le t \le T$ [28]. From the formula (7), the equality

$$\omega_1 r_1 + \omega_2 r_2 + \omega_3 r_3 - J_1\omega_1^2 - J_2\omega_2^2 - J_3\omega_3^2 - k_0 = \text{const}$$

is obtained for time interval when φ = 0. After substitution the values ω_i calculated by the equations (17) in this equality (taking into account the equalities $r_i = r_0 p_i$), the conditions:

$$r_1^2/J_1 + r_2^2/J_2 + r_3^2/J_3 = \text{const}\text{ , and } p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3 = \text{const}$$

are satisfied for the times when $a(t) = 0$ because |r| = const (at the segments of acceleration and braking, the above mentioned conditions are satisfied automatically, as it follows from the equations (14), (15), (16)).

Thus, the problem of constructing the optimal control is reduced to finding such vector p (0) that as a result of spacecraft motion, according to the equations (2), (11), and (13) with initial conditions (4), the equalities (5) will be satisfied. It is virtually impossible to find a general solution of this system of equations. A difficulty is to find the vectors p (0) and p (T) which are related by the dependence

$$\Lambda_\text{f} \circ \mathbf{p}(T) \circ \tilde{\Lambda}_\text{f} = \Lambda_\text{in} \circ \mathbf{p}(0) \circ \tilde{\Lambda}_\text{in} \tag{18}$$

The time of ending the reorientation process is not fixed, therefore $H(T) = 0$; the Hamiltonian H is independent of time in explicit form, hence, $H = 0$ inside the entire interval of control $0 \le t \le T$ [28]. Maximum value of kinetic energy E_k and modulus of angular momentum (and values $a(0)$ and r_0) are determined by condition $H = 0$. At instant $t = 0$, angular velocity ω is zero, the function H is equal to

$$m_0 a(0)(p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3) - k_0 = 0$$

Hence, the value $a(0)$ for optimal function $a(t)$ is $a(0) = k_0 / (u_0 C)$. Accordingly, $a(T)$ is $a(T) = -k_0 / (u_0 C)$. At instants when $a(t) = 0$, the function H is

$$r_0 b(p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3) - b^2(p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3) - k_0 = 0$$

and $b = r_0 / 2$ if $\dot{a} = 0$ (this follows from (17)). From last equation, we find the optimal value $r_0 = 2\sqrt{k_0} / C$. It is obvious that modulus of angular momentum |L| has maximal value $L_{\max}$ between acceleration and braking. Thus, $L_{\max}$ is determined unambiguously $L_{\max} = \sqrt{k_0} / C$. The found magnitude $L_{\max}$ corresponds to the maximal kinetic rotation energy $E_{\max} = k_0 / 2$. Respectively, $t_1 = \sqrt{k_0} / u_0$ if phase of uncontroled motion (when M = 0) is not absent. Note, the vectors $\boldsymbol{\omega}$ and p are related as

$$\omega_i = \frac{\sqrt{2E_k}}{\sqrt{p_1^2/J_1 + p_2^2/J_2 + p_3^2/J_3}} \cdot \frac{p_i}{J_i} \tag{19}$$

The task of the onboard control system for realization of optimal control is to impart the calculated angular velocity to the spacecraft at time moment $t = 0$ and to suppress kinetic energy to zero at time moment $t = T$, when $\Lambda(t) = \Lambda_\text{f}$ (after the spacecraft reaches its final position Λ_f). From the moment of reaching the necessary initial angular velocity ω_cal and until the reorientation is finished, when the spacecraft will be in the neighborhood of the required position Λ_f, there is no torque M; the spacecraft performs uncontroled rotation (M = 0), i.e. free motion. Creating the initial angular velocity and damping the final rotation happens in an impulse (as fast as the spacecraft's actuators will allow). Between the impulsive imparting of angular momentum and the impulsive suppressing of angular momentum

$$J_1^2\omega_1^2 + J_2^2\omega_2^2 + J_3^2\omega_3^2 = \text{const} = \text{k}_0 / \text{C}^2,$$

$$J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2 = \text{const} = \text{k}_0 \tag{20}$$

Topicality of the solved problem consists in the fact that by minimizing index (6) the energy spent to perform spacecraft reorientation from position Λ_in into position Λ_f is bounded and maneuver duration T is minimum. Indeed, if consider the function $f_0 = \sqrt{J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2}$, the integral

$$Q = \int_0^T f_0(t)dt$$

is minimum for motion according to the law (11), (13) [23]. Narrow-mindedness of rotation energy was proven earlier. Since f_0 is piecewise-linear function of time, duration of optimal maneuver is

$$T = Q / \sqrt{k_0} + \sqrt{k_0} / u_0 \tag{21}$$

The first term is duration of so-called kinematic control (or ideal maneuver) when $u_0 \to \infty$, $t_1 \to 0$, and the equalities (17) are satisfied within entire interval of time $0<t<T$. This duration is minimum because Q is minimal possible value since optimal maneuver (in sense (6)) satisfies equations (11), (13). Second term is duration of braking under restriction (3) (when $u_0<\infty$ and $u_0 \neq 0$). This time is minimal for control (16) [22]. Hence, the value (21) is minimal possible value of reorientation's time with restriction (3) and condition $E_k \le k_0$ / 2 for kinetic energy E_k during turn maneuver. We can show that the found control (12), (13) is indeed optimum (since the functions φ_i and ω_i calculated by the formulas (12), (13) are single solution of the system (1), (9), (10), (11) if ω (0) = ω (T) = 0 and $r_i = r_0 p_i$).

4. Optimal Program of Spacecraft Rotation for Special Cases

We assume that the control non-limited by any restrictions is ideal mode (in this case, $u_0 \to \infty$ and $t_1 \to 0$, the braking is momentary process also). In ideal motion optimal with respect to criterion (6), the spacecraft's reorientation is carried out with zero controlling moment M = 0. Constructing the optimal reorientation regime with minimal value (6) is non-trivial task. For the optimal reorientation problem (constructing the optimal programmed motion **ω** (t)), it is crucial to find the initial vector p (0) and the corresponding angular velocity **ω** (0+) (the angular velocity **ω** (0+) is calculated by formulas (19)). The vector p (0) depends on reorientation parameters $\Lambda_t = \tilde{\Lambda}_{in} \circ \Lambda_f$ and the spacecraft characteristics J_1, J_2, J_3. For arbitrary values $J_1 \neq J_2 \neq J_3$, it is hard to find the solution of the considered problem of spacecraft's three-dimensional reorientation for arbitrary values Λ_{in} and Λ_f because the vectors p (0) and p (T) are related by (18). Analytical solution of the system of equations (2), (11), and (19) exists for dynamically spherical and dynamically symmetric bodies only.

For a spherically symmetric spacecraft (when $J_1 = J_2 = J_3$), the solution p (t), **ω** (t) have elementary form: p (t) = const and **ω** (t) = const, or in detail

$$p_i = v_i \Big/ \sqrt{v_1 + v_2 + v_3}\text{ , and } \omega_i = \frac{2v_i \arccos\ v_0}{T\sqrt{v_1 + v_2 + v_3}}$$

where v_0, v_1, v_2, v_3 are components of the reorientation quaternion $\Lambda_t = \tilde{\Lambda}_{in} \circ \Lambda_f$.

For a dynamically symmetric spacecraft (for example, when $J_2 = J_3$), the optimal control problem can be solved completely also. For this distribution of mass, the following differential equations

$$J_2\dot{\omega}_2 = (J_3 - J_1)\omega_1\omega_3,\ J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2$$

are satisfied under condition $\omega_1 = \text{const}$.

Last system of differential equations describes the oscillator (with the parameter ω_1 = const), for which ω_2 and ω_3 are harmonic functions of time. Therefore, p_1 = const = p_{10} and harmonic oscillations of the functions p_2 and p_3. In this special case, the optimal motion is the simultaneous rotation of the spacecraft as a rigid body around its axial axis OX and around spacecraft's angular momentum L which is constant in the inertial space and which constitutes a certain constant angle ϑ with the spacecraft's axial axis. Angular velocities with respect to OX and p axises have a constant ratio (as is shown above, the vectors L and p are parallel). The solution of system (2), (11), (19), necessary for solving the control problem, is regular precession. For the regular precession case

$$\Lambda_f = \Lambda_{in} \circ e^{\mathbf{p}_0\beta/2} \circ e^{\mathbf{e}_1\alpha/2}$$

where p_0 = p (0); e_1 is the unit vector of the spacecraft's axial axis; α is the spacecraft's rotation angle around its axial axis; β is the spacecraft's rotation angle around the vector p, e is the quaternion exponential [1]. It is assumed that $|\alpha| \le \pi$, $0 \le \beta \le \pi$.

For a dynamically symmetric spacecraft with moments of inertia $J_1 \neq J_2 = J_3$, the solution p (t) is written as follows:

$$p_1 = p_{10} = \text{const} = \cos\vartheta,\ p_2 = p_{20}\cos\kappa + p_{30}\sin\kappa,\ p_3 = -p_{20}\sin\kappa + p_{30}\cos\kappa \tag{22}$$

where

$$\kappa = \frac{J_3 - J_1}{J_2}\int_0^t \omega_1(t)dt.$$

In this case, the dependences (22), together with equalities (19), form a solution of the system of equations (2), (11) under condition (13). At the same time, the vector p also generates a cone around the axial axis OX in the body-fixed coordinate system. The specific value of p_0 is determined exclusively by the requirement that, according to equations (2), (11), (19), boundary conditions (4) and (5) must be satisfied. In this type of control, the spacecraft's angular momentum preserves a constant direction in the inertial reference basis I, while the axially symmetric body moves along a "conic trajectory". For moving from position Λ_{in} into position Λ_f, a spacecraft rotates simultaneously around the vector c_E, which is constant relative to the inertial basis I, by the angle β, and around its own longitudinal axis by the angle α. Using the mathematical formalism of quaternions to describe rotations of rigid body about the center of mass, relations reflecting a dependence between the values p_0, α, and β are written. The dependence of parameters p_0, α, and β on the boundary angular positions Λ_{in} and Λ_f is given by the following system of equations:

$$\cos\frac{\beta}{2}\cos\frac{\alpha}{2} - p_{10}\sin\frac{\beta}{2}\sin\frac{\alpha}{2} = v_0,\ p_{20}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} + p_{30}\sin\frac{\beta}{2}\sin\frac{\alpha}{2} = v_2,\ \cos\frac{\beta}{2}\sin\frac{\alpha}{2} + p_{10}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} = v_1$$

$$-p_{20}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} + p_{30}\sin\frac{\beta}{2}\cos\frac{\alpha}{2} = v_3,\ \alpha = \frac{J_2 - J_1}{J_1}p_{10}\beta$$

For a known reorientation time *T*, angular rotation velocities around the *OX* and p axises are equal to $\dot{\alpha} = \alpha / T$, and $\dot{\beta} = \beta / T$ (for ideal mode $T = Q / \sqrt{k_0}$). The magnitude of angular momentum during optimal rotation is |L| = $J_2 \beta$ / *T*. The programmed values of controllable functions ω_i (projections of the angular velocity vector $\boldsymbol{\omega}$) have the following form:

$$\omega_1 = \dot{\alpha} + \dot{\beta} p_{10},$$

$$\omega_2 = \dot{\beta}\sqrt{1 - p_{10}^2}\sin(\dot{\alpha}t + \sigma_0),$$

$$\omega_3 = \dot{\beta}\sqrt{1 - p_{10}^2}\cos(\dot{\alpha}t + \sigma_0)$$

where $\sigma_0 = \operatorname{arctg}\ (p_{20} / p_{30})$.

Notice, optimal values p_0, α, and β corresponding to solution of last system of five transcendent equations and which correspond to free motion from position Λ_{in} into position Λ_{f} can be determined with using the device [30].

For a non-symmetric spacecraft (when $J_1 \neq J_2 \neq J_3$), the system (2), (11), (19) can be solved by numerical methods only (e.g., using the method of successive approximations or iterations methods with consecutive approach to true solution). To find the vector p_0, it is necessary the solving the boundary problem $\Lambda(0) = \Lambda_{\text{in}}$, $\Lambda(T) = \Lambda_{\text{f}}$, taking into account the equations (1), (2) imposed upon the motion, in which M_i = 0. As a result, the value of the angular velocity vector at the initial time moment $\boldsymbol{\omega}_{\text{cal}}$, for which the spacecraft is moved by its free rotation with respect to the center of mass (M = 0) from the state $\Lambda(0) = \Lambda_{\text{in}}$, $\boldsymbol{\omega}$ (0) = $\boldsymbol{\omega}_{\text{cal}}$ into the state Λ (*T*) = Λ_{f}, will be found ($\boldsymbol{\omega}$ (*T*) is arbitrary here). In particular, the method of solving the boundary problem and determining the vector p_0 was described in detail in article [12]. The value of the vector p_0 relates to $\boldsymbol{\omega}_{\text{cal}}$ as

$$p_{i0} = \frac{J_i \omega_{i\,\text{cal}}}{\sqrt{(J_1\omega_{1\text{cal}})^2 + (J_2\omega_{2\text{cal}})^2 + (J_3\omega_{3\text{cal}})^2}}$$

The known algorithms presented in patent [24] and system [25] can be used for finding calculated values $\boldsymbol{\omega}_{\text{cal}}$ and p_0 also. These algorithms [12, 24, 25] are reliable and provide asymptotic approaching for sought value p_0. Other calculation schemes [31-33] can be useful only in some specific cases.

If the moment M is limited, some non-zero time τ is required for imparting the required angular momentum to the spacecraft and for suppressing the existing angular momentum to zero. A restriction on the magnitude of feasible controlling moment leads to the appearance of intervals with non-zero duration when spacecraft increases and decreases its angular velocity.

5. Constructing the Optimal Program of Motion Under Restrictions on the Controlling Moment for Main Types of Control

In many practical tasks, reorientation is made in situation when initial state satisfies condition $\boldsymbol{\omega}$ (0) = 0 and final angular velocity must be absent $\boldsymbol{\omega}$ (*T*) = 0 (these cases occur very frequently, especially if attitude control is done relative to inertial coordinate system). It is obvious, in moments of time *t* = 0 and *t* = *T*, angular velocity calculated according to the formula (19), corresponding to nominal program of optimal rotation maneuver (when $E_k = k_0 / 2$), is not equal to zero. Therefore, segments of acceleration and braking at the beginning and the ending of reorientation maneuver are inevitable. For the optimal motion, spacecraft reorientation from one angular position Λ_{in} to another position Λ_{f} is done by impulsive imparting the necessary angular velocity (the nominal value of the angular momentum vector) to the spacecraft, rotation of the spacecraft with the constant kinetic energy and modulus of angular momentum, and short-term (impulse) reduction of the rotation energy to zero. For reorientation maneuver, very important characteristic is integral

$$S = \int_0^T |\mathbf{L}(t)|\, dt \tag{23}$$

The value of characteristic *S* is determined only by the rotation conditions Λ_{in}, Λ_{f}, and the spacecraft's principal central moments of inertia J_1, J_2, J_3. If time of reaching the calculated angular velocity which is equal to

$$\omega_{i\ \text{nom}} = \frac{\sqrt{k_0}}{J_i C} p_{i0}$$

and duration of suppressing the angular velocity to zero are infinitesimal, then duration of reorientation is $T = SC / \sqrt{k_0}$ because modulus of angular momentum during uncontroled motion (between acceleration and braking) is |L| = $\sqrt{k_0} / C$, where the integral (23) is calculated by formula

$$S = t_{\text{pr}} \sqrt{J_1^2 \omega_{1\text{cal}}^2 + J_2^2 \omega_{2\text{cal}}^2 + J_3^2 \omega_{3\text{cal}}^2}$$

where t_{pr} is the predicted time of achieving the condition $\Lambda(t_{\text{pr}}) = \Lambda_{\text{f}}$ during free rotation from the position $\Lambda(0) = \Lambda_{\text{in}}$ with initial angular velocity $\boldsymbol{\omega}$ (0) = $\boldsymbol{\omega}_{\text{cal}} \neq 0$ (according to the equations (2), (1) in which all values M_i = 0). Note, the value *S* and the vector p_0, which satisfy optimal motion, are computed together. Remind

$$C = \sqrt{p_{10}^2 / J_1 + p_{20}^2 / J_2 + p_{30}^2 / J_3}$$

For a spherically symmetric spacecraft and for a dynamically symmetric spacecraft, key characteristics and the constants of control law are determined straightforwardly, without integration of motion equations (1), (2). For a spherically symmetric spacecraft, the integral S is

$$S = 2J_1 \arccos \nu_0$$

Optimal modulus of angular momentum during uncontroled rotation and kinetic energy are

$$L_{\text{opt}} = 2J_1 \arccos \nu_0 / T, \; E_k = 2J_1 \arccos^2 \nu_0 / T^2$$

For an axially symmetric spacecraft (when $J_2 = J_3$), the integral S, the optimal modulus of angular momentum and kinetic energy during uncontroled rotation are

$$S = J_2\beta, \; L_{\text{opt}} = J_2\beta / T,$$
$$E_k = J_2^2\beta^2(\cos^2\vartheta / J_1 + \sin^2\vartheta / J_2) / (2T^2)$$

For zero boundary conditions ω (0) = ω (T) = 0, in general case, maneuver includes two phases during which magnitude of the torque M is maximal possible: acceleration and braking, and phase of uncontroled motion at which the equations (20) are satisfied. This type of control is called control with two points of switching (t_1 is moment of time when kinetic energy E_k achieves level $k_0 / 2$, and t_2 is moment of beginning of braking). A detailed analysis of the main reorientation stages: acceleration, braking, and uncontroled spacecraft rotation with the constant kinetic energy and angular momentum (relative to inertial coordinate system), shows that all three stages have a common property, namely, spacecraft rotates along the "trajectory of free motion" for which direction of angular momentum remains constant in inertial coordinate system. Taking into account that during the entire reorientation (on the entire time interval [0, T]) the torque M is parallel to the angular momentum vector L (i.e., the controlling moment acts in the same direction as L, or in the opposite direction, or is equal to zero), we can conclude that M×L = 0, and, therefore, there are no reasons for a rotation of the angular momentum L in inertial coordinate system. Also, the integral (23) does not depend on the character of variation of the function |L (t)| if angular motion satisfies the system of equations (11), (13) [23], and S is equal to the value (23) for ideal mode (for motion according to (11), (17), (20)).

However, there are such situations (under certain values S, u_0, k_0) when the stage of uncontroled motion is absent and M ≠ 0 within the entire interval of control $0 \le t \le T$ (braking replaces acceleration at once). This type of control is called as control with one switching.

If $S > k_0 / (u_0C)$, then $t_2 > t_1$ ($t_2 \ne t_1$), time of optimal reorientation is $T = SC / \sqrt{k_0} + \sqrt{k_0} / u_0$; $\dot{a}\ (T/2) = 0$.

If $S = k_0 / (u_0C)$, then $t_2 = t_1$, time of optimal reorientation is $T = 2\sqrt{k_0} / u_0$; the derivative $\dot{a}\ (T/2) = 0$.

If $S < k_0 / (u_0C)$, then we have control with one switching when maximal energy of rotation $E_{\max} < k_0 / 2$, and duration of optimal reorientation is $T = 2\sqrt{SC / u_0}$; point of switching is $t_0 = \sqrt{SC / u_0}$, maximal energy of rotation is $E_{\max} = u_0SC / 2$, maximal modulus of angular momentum is $L_{\max} = \sqrt{u_0S / C}$, therefore, the derivative $\dot{a}$ (T / 2)<0 (i.e. $\dot{a}$<0 on the entire interval of time $0 \le t \le T$).

For spacecraft reorientation with limited control, key property of optimal motion remains valid is independent of number of switching, proportion $\rho = E_k / |L|^2$ for kinetic energy E_k and angular momentum L is constant on the entire interval of time $0 \le t \le T$, independently of duration of acceleration and braking (independently of presence or absence of the uncontroled stage with M = 0). As consequence, modulus of torque M is identical for acceleration and braking, and it is equal to same magnitude

$$m_0 = u_0 \big/ \sqrt{2\rho} = u_0 / C$$

In section 3, it was demonstrated that kinetic energy $E_k = k_0 / 2$ if $\dot{a} = 0$. It is obvious, $E_k < k_0 / 2$ at acceleration and braking. Hence, $E_k \le k_0 / 2$ during the entire interval of time [0, T] (if $S < k_0 / (u_0C)$, then $E_{\max} = u_0SC / 2 < k_0 / 2$ also). Thus, for optimal control (in sense (6)) the property $E_{\max} \le k_0 / 2$ is satisfied.

The optimal control functions M_i and angular velocities ω_i change according to the following laws:

$$M_i = 0.5m_0\,[\text{sign}\,(t_0 - t) + \text{sign}\,(t_2 - t)]\,p_i \qquad (24)$$

$$J_i\omega_i = 0.5m_0\,(T - |t - t_0| - |t - t_2|)\,p_i \qquad (25)$$

where $t_0 = \sqrt{\min(k_0, u_0SC)} \big/ u_0$; t_2 = max ($SC\sqrt{1/k_0}$, $\sqrt{SC / u_0}$); $T = t_0 + t_2$; t_0 is moment of acceleration ending; t_2 is moment of the beginning of braking.

For the less kinetic energy of rotation, duration of braking is less. If $t_2 > t_0$ (i.e. when $S > k_0 / (u_0C)$), then we have control with two points of switching when phase of rotation with M = 0 (between acceleration and braking) is not absent, $t_0 = t_1$ also. If $t_0 = t_2$ (i.e. when $S \le k_0 / (u_0C)$), then moments of time when M = 0, are absent, we have control with one switching (braking follows acceleration at once).

The condition $|M| \le m_0$ is satisfied within the entire interval of control. Optimal torque M is parallel to motionless line relative to inertial coordinate system, i.e. M = m (t) p. The scalar function m (t) is specified as $m(t) = M_1p_1 + M_2p_2 + M_3p_3$. Control function m (t) is three-positional relay or two-positional relay if optimum is control with one switching; m (t) can be written in the following form: $m(t) = m_0$ if $|L| < L_{\text{opt}}$ and $t < T/2$; $m(t) = -m_0$ if $t \ge T - \tau$; $m(t) = 0$ if $\tau \le t < T - \tau$ (it is obvious that situation $m(t) = 0$ is absent if $\tau = T/2$, because the condition $\tau \le t < T - \tau$ is not satisfied). Here, $L_{\text{opt}} = m_0\tau$, and L_{opt} is modulus of angular momentum at time moment $t = T/2$ (or during free rotation if phase of uncontroled motion takes place); τ is duration of acceleration (braking). Note that spacecraft's angular momentum satisfies the inequality $|L| \le L_{\text{opt}}$ for any time t.

If optimal control program has two points of switching L_{opt}

= $\sqrt{k_0}\,/\,C$; if optimum is control with one switching L_{opt} = $\sqrt{u_0 S / C}$. For optimal control, spacecraft acceleration continues until angular momentum is equal to the target level

$$\mathbf{L}_{tag} = L_{opt} \circ \tilde{\Lambda} \circ \Lambda_{in} \circ \mathbf{p}(0) \circ \tilde{\Lambda}_{in} \circ \Lambda$$

Thus, it is proven the following conclusion: spacecraft's reorientation occurs with the minimal value of the index (6) if and only if the spacecraft rotates according to the law (11), (24), (25). If allow a steplike change of the angular velocity vector $\boldsymbol{\omega}$, then the proposed optimal control problem (the kinematic reorientation problem) can be considered solved: equations (2), (11), and (19) completely define the necessary motion $\boldsymbol{\omega}$ (t), the main moment of forces is zero (i.e., the spacecraft's rotation is an Euler–Poinsot motion of rigid body [29]) within interval $0<t<T$ in which $E_k = k_0 / 2$.

If $\sqrt{k_0}\,/\,u_0$ is much less than $SC/\sqrt{k_0}$, the beginning of braking will be determined from the fact that the angular momentum magnitude |L (t)| changes linearly when angular velocity $\boldsymbol{\omega}$ is reduced to zero. During braking, the modulus of the controlling moment is constant, and the time moment from which braking will be started is specified by the following condition:

$$4\arcsin \frac{K\sqrt{\delta_2^2+\delta_3^2}}{\sqrt{(J_2\omega_2)^2+(J_3\omega_3)^2}} = \frac{K^2\sqrt{\omega_2^2+\omega_3^2}}{m_0\sqrt{(J_2\omega_2)^2+(J_3\omega_3)^2}}$$

where m_0 is the maximal controlling moment magnitude that can be provided by the actuators of spacecraft's attitude control system; δ_j are the components of the discrepancy quaternion $\tilde{\Lambda}(t) \circ \Lambda_f$; K = |L (t)| is the current magnitude of the spacecraft's angular momentum. The said condition for finding the start moment of braking phase allows the onboard control system to form a signal of angular velocity reduction based on the information on the current spacecraft position and measurements of angular velocity. Use of this condition increases the precision of reorientation into final position Λ_f.

The assumed criterion of quality supports motion of a spacecraft with the bounded kinetic energy of rotation during reorientation maneuver. For case $\tau<<T$, the construction "acceleration of rotation, the uncontroled rotation, damping of rotation" is optimum for optimal control problem (1)- (6). Let us assess the relative growth in functional G due to the nonzero time it takes to gain and suppress the angular momentum. The value G is compared relative to the value G_{imp} which corresponds to the value (6) for ideal mode of reorientation (when $t_1 \to 0$ and $\tau \to 0$). For spacecraft rotation along the "trajectory of free motion" wich satisfies the system of equations (11), (13), the value (6) is

$$G = C^2 L_{max}^2 (T_{imp} - \tau / 3) + k_0 (L_{max}^2 T_{imp} + \tau) \qquad (26)$$

because the modulus of angular momentum changes according to the linear law during optimal acceleration and braking, where L_{max} is the magnitude of angular momentum at time moment $t = T / 2$; $T_{imp} = S / L_{max}$; $\tau = L_{max} / m_0$. Time of reorientation end is $T = T_{imp}+\tau$.

Optimum is such value of L_{max} for which the value G is minimal. The condition $T>2\tau$ must be satisfied. Hence, $\tau<T_{imp}$, and L_{max} satisfies the condition $L_{max}< \sqrt{u_0 S / C}$. The value G is absolute minimum if $\tau = 0$ (this value is G_{imp} = $2SC\sqrt{k_0}$). When $\tau \neq 0$, the value (26) is minimum when L_{max} = $\sqrt{k_0}\,/\,C$. Therefore, L_{opt} = $\sqrt{k_0}\,/\,C$ is optimal value of the parameter L_{max} for program of optimal reorientation if $\tau \neq 0$ (if $\sqrt{k_0}\,/\,u_0<T_{imp} = SC\,/\,\sqrt{k_0}$ or $u_0SC>k_0$). If $\tau>0$, the value (6) is

$$G = 2SC\sqrt{k_0} + 2k_0\sqrt{k_0}\,/\,(3u_0)\,,\ T = SC/\sqrt{k_0} + \sqrt{k_0}\,/\,u_0$$

The relative growths in functional G and duration of reorientation T are

$$\frac{\Delta G}{G_{imp}} = \frac{k_0}{3u_0 SC}\,,\ \frac{\Delta T}{T_{imp}} = \frac{k_0}{u_0 SC} \qquad (27)$$

where G_{imp} and T_{imp} are value (6) and duration T for impulse control (when $\tau \to 0$).

The time τ changes from zero to $T / 2$. When duration of acceleration and braking increases, the functions (27) increase everywhere within the range $0 \leq \tau < T / 2$ (note, $\tau = \sqrt{k_0}\,/\,u_0$). The minimal value corresponds to the case $\tau \to 0$. If $\tau = 0$, we obtain ideal maneuver. For control with one switching

$$G = 2SC\sqrt{u_0 SC}\,/\,3 + 2k_0\sqrt{SC/u_0}\,,\ T = 2\sqrt{SC/u_0}$$

Thus, key results are the following: optimal control program of spacecraft reorientation was found; it was demonstrated that two-impulse control when spacecraft rotates by inertia between acceleration and braking is optimum in general case; for optimal solution, estimations of the relative growth in the functional of quality due to the limited controlling moment were done. Other characteristic properties of the obtained optimal motion are determined also. For a dynamic symmetric spacecraft, a complete solution of the reorientation problem in closed form is presented; optimal values of control law parameters can be found by the device [30]. The obtained control method is differs from all other known solutions. Main difference consists in new form of the minimized index which allows to turn a spacecraft with the bounded rotation energy (maneuver time is minimized also). This useful quality is advantage of the presented control mode because it significantly saves the controlling resources.

6. Example and Results of Mathematical Modeling

Let us present a numerical solution of optimal control problem for spacecraft reorientation with minimal value of the integral (6). As an example, let us consider spacecraft reorientation for 180 degree from initial position Λ_{in}, when body axes coincide with the axes of reference basis I, into the target position Λ_f. It is assumed that initial and final angular velocities are zero, $\boldsymbol{\omega}$ (0) = $\boldsymbol{\omega}$ (T) = 0. Values of the elements of quaternion Λ_f that characterizes the target attitude of a spacecraft are:

$$\lambda_0 = 0,\ \lambda_1 = 0.707107,\ \lambda_2 = 0.59,\ \lambda_3 = 0.39$$

Let us find the optimal control program for angular velocity $\boldsymbol{\omega}$ (t) for transferring the spacecraft from the state Λ (0) = Λ_{in}, $\boldsymbol{\omega}$ (0) = 0 to the state Λ (T) = Λ_f, $\boldsymbol{\omega}$ (T) = 0. The constant u_0 which characterizes power of actuators is u_0 = 0.2N kg$^{-1/2}$. The inertial characteristics of a spacecraft have the values:

$$J_1 = 77543.7\text{kg m}^2,\ J_2 = 228466.1\text{kg m}^2,\ J_3 = 175682.5\text{kg m}^2$$

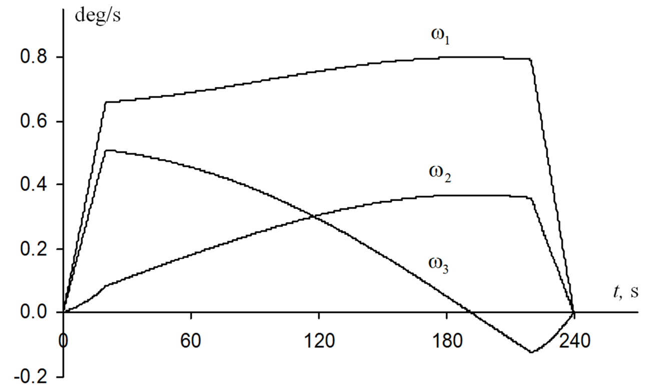

Figure 1. *Changing the angular velocities during optimal reorientation maneuver.*

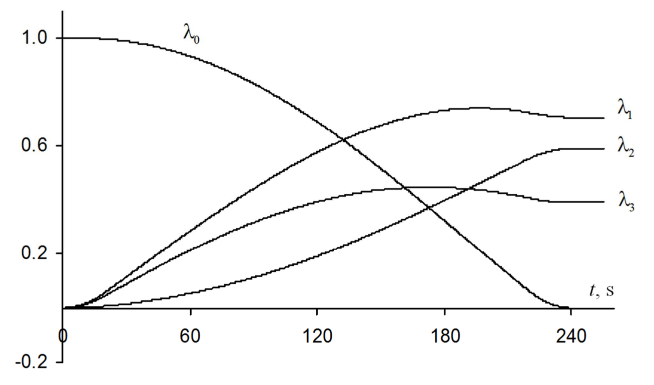

Figure 2. *Changing the components of orientation quaternion Λ (t) during optimal reorientation.*

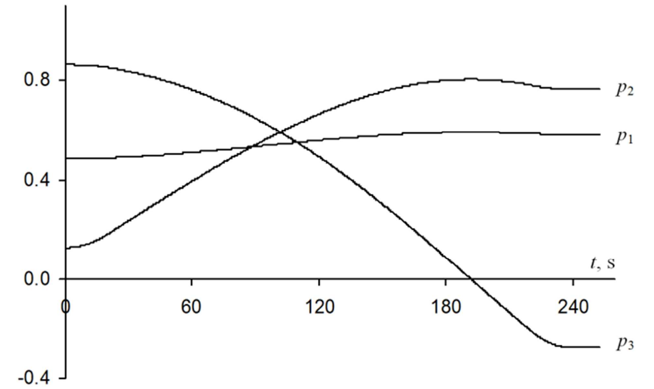

Figure 3. *Changing the components of unit vector p in time under optimal control.*

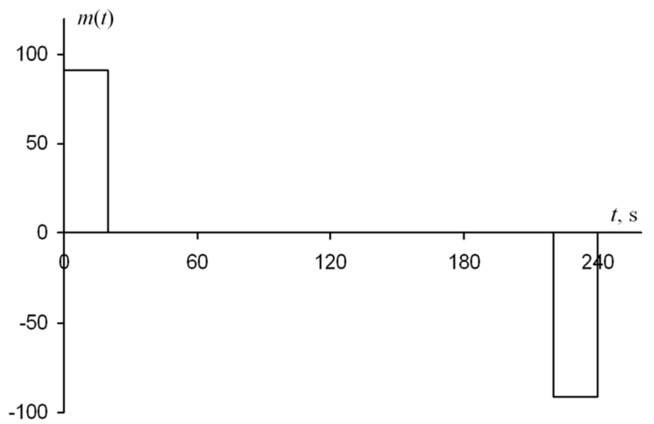

Figure 4. *Changing the scalar function m (t) for optimal control.*

As a result of solving the kinematic problem of spacecraft's reorientation from position Λ (0) = Λ_{in} into position Λ (T) = Λ_f (the optimal reorientation problem in the impulse setting), the calculated value of the vector p_0 = {0.485149; 0.126100; 0.865292} and integral S = 401564 N m s^2 were obtained. According to these calculated values, the initial angular velocity is equal to $\boldsymbol{\omega}_{cal}$ = {0.599785 ∘ / s; 0.052913 ∘ / s; 0.472173 ∘ / s}. Iterations method guaranteeing successive approach to true value p_0 was used [12] (in most cases, this method provides asymptotic approaching). The maximal value of the controlling moment is m_0 = 91 N m. Let us find optimal control program if the coefficient k_0 is k_0 = 20 joules. The obtained values S, p_0 (and C also), and u_0, k_0 show that S>k_0 / (u_0C) and optimal program is control with phase of the uncontroled rotation. The durations of acceleration and braking are the same and equal τ = 22 s, the angular momentum magnitude within stage of rotation by inertia is L_{opt} = 2002 N m s. Optimal changing the controlling moment M is described by the law

$$\mathrm{M} = u_0\,[\mathrm{sign}\,(\sqrt{k_0}\,/\,u_0 - t\,) + \mathrm{sign}\,(\,SC/\sqrt{k_0} - t\,)]\,\tilde{\Lambda}\circ\Lambda_{in}\circ\,p_0\circ\tilde{\Lambda}_{in}\,\circ\Lambda\,/\,(2C)$$

Results of the mathematical modeling of the reorientation process under optimal control are given on Figures 1, 2, 3, and 4. The duration of reorientation was T = 240 s. Figure 1 shows the character of changing the angular velocities in the spacecraft-related system of coordinates ω_1 (t), ω_2 (t), ω_3 (t) with respect to time. At the stage between acceleration and braking, the spacecraft rotates with a constant energy E_k = 10 joules. The value of functional (6), which characterizes the cost-efficiency of the rotation trajectory Λ (t), $\boldsymbol{\omega}$ (t) after spacecraft's angular motion from position Λ_{in} into position Λ_f, has been equal to G = 8200 J s. Figure 2 shows the graphs of changes in the components of quaternion Λ (t) that defines the current spacecraft position in the process of the rotation maneuver: λ_0 (t), λ_1 (t), λ_2 (t), λ_3 (t). Figure 3 shows the dynamics of components p_1 (t), p_2 (t), p_3 (t) of unit vector p in time. The variables λ_j and p_i are dimensionless quantities. It is characteristic that the change in the projection p_1 is very small in comparison with changes in the projections p_2 and p_3 (the angular velocity component ω_1 also changes a lot less on the interval of free rotation than angular velocity components ω_2 and ω_3). This confirms the fact that the OX axis is longitudinal axis. Unlike variables ω_i, variables p_i and λ_j are smooth functions of time. Finally, Figure 4 shows the behaviour of scalar function m (t). It is well visible, the change in the function m (t) has relay character.

7. Conclusion

The optimal control problem for spatial reorientation of a spacecraft from a position of rest to a position of rest is considered. The optimization has been performed for case when rotation energy integral should be minimized together with turn duration. Finding the optimal mode of spacecraft reorientation with a minimal value of energy's "expenditure" is quite topical. An analytic solution of the proposed problem is presented, formal equations and computational expressions for constructing the optimal reorientation program were obtained. To solve the formulated problem, the maximum principle is applied basing on universal variables [21], and use of quaternions significantly simplifies computational procedures and reduces the computational costs of control algorithm, which makes it suitable for onboard realization. The main characteristic properties of optimal motion and the type of trajectory, which is optimal with respect to the chosen criterion, were determined. The reorientation problem has been solved completely in dynamic statement.

In this research, new control method of spacecraft attitude is obtained; the used criterion of quality is new and has special form what is principal difference from the known works. The designed method of spacecraft's motion control was described in detail. The solved problem is very topical since the designed control algorithm of reorientation maneuver guarantees a motion with rotation energy not exceeding the required value which is determined by coefficients of the minimized functional. Importance and significance of the executed investigations consist in the fact that the chosen criterion of quality bounds energy of rotation and minimizes reorientation time under this condition. It is proved that two-impulse control when spacecraft rotates by inertia between acceleration and braking is optimum in general case, and the proposed mode of reorientation is best relative to the known solutions. Control with one switching is special case of optimal rotation (it is critical variant when phase of uncontroled rotation is absent, acceleration and braking are the adjoining phases). Presence of ready formulas, for synthesis of optimal motion program during a slew maneuver, does the carried out research as practically significant and suitable for direct use in practice of spaceflights.

It was demonstrated that ideal optimal solution is two-impulse control when spacecraft rotates by inertia between jump-like acceleration and jump-like braking. If the controlling torque is limited, analytical formulas were written for duration of acceleration and braking, and turn's time also. It is shown that direction of spacecraft's angular momentum is constant in the inertial coordinate system within the entire reorientation interval, and the spacecraft rotates along the "trajectory of free motion". A procedure for implementing the control mode is described. We estimated how the duration of gaining and suppression of angular momentum influences energy costs and turn's time. Expressions for computing the temporal characteristics of the reorientation maneuver and the condition for finding the deceleration start moment based on factual kinematic motion parameters with use of terminal control principles are presented, it leads to high orientation precision. Example and results of mathematical modeling for spacecraft motion under optimal control are given. The obtained results demonstrate that the designed control method of spacecraft's three-dimensional reorientation is feasible in practice.

Notice, some particular cases of spacecraft maneuver are known in recent publications [34-39]. But rotation energy is not taken into account in [34-39]. In [35, 36, 38], time of reorientation is minimized, and conical motions were considered only, and solution is received only for axially symmetric spacecraft [36]. In a paper [37], combination of time and norm of angular momentum, instead of rotation energy, is minimized; the controlling moment M $\neq$ 0 and the vectors M and L are perpendicular in interval of nominal rotation between acceleration and braking (we recall that the vector M is parallel to the angular momentum L or equal to zero for the proposed control); such maneuver [37] is not optimum in energy sense or fuel consumption because M $\neq$ 0 within entire interval of time [0, T]. The works [35, 38] consider a relay control of a turn in orbit plane when final state is gravitationally stable position. Note, the solutions [35, 36, 38] are not applicable for the general case of three-dimensional turn of arbitrary spacecraft; the work [39] describes synthesis of terminal reorientation control only for the spacecraft which moves along a circular orbit. But the method designed in present article is universal control, it does not depend on a ratio (proportion) of moments of inertia or final position of a spacecraft. The universality of the designed control method is proved by the following factors: it does not depend from actuators type; mass and size of a spacecraft; configuration and distribution of spacecraft's masses; altitude of working orbit (and from others, for example, from periodicity of reorientation, angle of a turn). Importance of the proposed mode of reorientation consists not only in energy aspects but in security sense because rotation with energy not exceeding the given value allows us to stop rotation of a spacecraft within known duration (it is very topical in different critical situations).

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