Lie Symmetries and Invariant-Solutions of the Potential Korteweg-De Vries Equation

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Abstract: The purpose of this paper is to investigate the nonlinear partial differential equation, known as potential Korteweg-de Vries (p-KdV) equation. We have implemented the Harrison technique that makes use of differential forms and Lie derivatives as tools to find the point symmetry algebra for the p-KdV equation. This approach allows us to obtain five infinitesimal generators of point symmetries. Fixing each generator of symmetries that we have found, we construct a complete set of functionally independent invariants, corresponding to the new independent and dependent variables. Using these new variables, called “similarity variables”, the reduced equations have been constructed systematically, which leads to exact solutions that are group-invariant solutions for the p-KdV equation. The obtained solutions are of two types. The reduced equations from the generator of space and time translation groups are the first and the third order ordinary differential equations respectively and lead to the Travelling-invariant solutions. Then, the reduced equation from the generator of the Galilean boosts is the first order ordinary differential equation and leads to Galilean-invariant solutions. Under the generator of scaling symmetries, the potential KdV equation reduces to the third order ordinary differential equation, which does not admit symmetries. And then, there are no functionally independent invariants for that last equation, its solutions are essentially new functions not expressible in terms of standard special functions.

Keywords: Symmetries, Differential Forms, Lie Derivative, Korteweg-de Vries Equations, Invariant Solutions

1. Introduction

Lie symmetries of differential equations are one of the important concepts in the theory of differential equations and physics. Among others methods, Lie method is a firm one for finding symmetries of differential equations. This method was first applied to determine point symmetries (see [8] and [12]). In 1969-1970, B. Kent Harrison and Frank Estabrook devised a method to calculate symmetries of differential equations using differential forms and Cartan’s formulation of differential equations [2]. They were simply trying to understand how the symmetries of Maxwell’s equations could be found from the differential form version of those equations. Once they realized that the key to symmetries was the use of the Lie derivative, B. Kent Harrison applied the method to several others equations such as the one dimensional heat equation, the Short wave gas dynamic equation and the nonlinear Poisson equation (see [3] and [4]). Here we apply this method to the potential KdV equation, given as follows

\[ u_{xxx} + a(u_x)^2 + u_t = 0. \]  

(1)

where \( u(x,t) \) is a function of space \( x \) and time variable \( t \); subscripts denoted partial derivatives; \( a \) is a real constant, with \( a \) is no vanishing.

The p-KdV equation is widely used in various branches of physics [1]. Exact travelling wave solutions to nonlinear evolution equations, particularly which appear in many physical structures in solitary wave theory such as solitons, kinks, peakons, and cuspons [14], draw considerable interest
in recent years in revealing the mechanism of the complicated physical phenomena and dynamical processes modeled by these nonlinear evolution equations [7]. There has been an enormous interest on the study of soliton structures that propagate in nonlinear media without changing their profile by reason of their potential application in many physics areas. Their existence is a result of a delicate balance between linear dispersion and nonlinearity. The KdV partial differential equations, defined on a different manifold $M$ of independent variables $\gamma = (x, u, u_t, u_{xx}) \in \mathbb{R}^n$ in our special case.) Let $x, t, u, u_t, u_{xx} \in \mathbb{R}$ be independent variables, and let $U = \mathbb{R}$, representing the space of dependent variable. We define the partial derivatives of the dependent variable as new variables (prolongation) in sufficient number to write the equation as second order equation by defining a new variable $w = u_t$. Thus Eq.(1) becomes

$$w_{xx} + aw^2 + ut = 0 \quad (2)$$

Then, we construct a set of 1-forms on the manifold $N$ as follow

**Lemma 2.1.** For Eq. (2), the required 1-forms are

$$\beta^1 = du - u_t dt - u_x dx$$

$$\beta^2 = du_t - u_{tt} dt - u_{tx} dx$$

$$\beta^3 = du_x - u_{tx} dt - u_{xx} dx$$

**Proof.** We consider the 8-dimensional manifold corresponding to the equation (2), with the coordinates $(x, t, u, u_t, u_{tx}, u_{tt}, u_{xx}, u_{xx})$. Considering this equation, we have the following contact conditions

$$du = u_t dt + u_x dx,$$

$$du_t = u_{tt} dt + u_{tx} dx,$$

$$du_x = u_{tx} dt + u_{xx} dx,$$

which lead to the contact 1-forms of Lemma 2.1.

Then, we have to construct the forms of the ideal I by the following

**Lemma 2.2.** The ideal I consists of the following 2-forms:

$$\gamma^1 = (u_x u_t - u_x u_{tx}) dx \ dt + u_{tx} dx \ du - u_x dx \ du_t + u_{tt} dt \ du - u_t dt \ u_t + du \ u_t; \quad (3)$$

$$\gamma^2 = (u_x u_t - u_x u_{xx}) dx \ dt + u_{xx} dx \ du - u_x dx \ u_x + u_{tx} dt \ du - u_t dt \ u_x + du \ u_x; \quad (4)$$

$$\gamma^3 = ((u_{tx})^2 - u_t u_{xx}) dx \ dt + u_{xx} dx \ du_t - u_{tx} dx \ u_x + u_{tx} dt \ du_t - u_{tt} dt \ u_x + du \ u_t; \quad (5)$$

$$\gamma^4 = dx \ du_t + dt \ du_t; \quad (6)$$

$$\gamma^5 = dx \ du_{tx} + dt \ du_{tx}; \quad (7)$$

$$\gamma^6 = dx \ du_{xx} + dt \ du_{xx}; \quad (8)$$

It is to be closed. Then for determining the invariance of the differential equations, we may construct the Lie derivative of the forms in the ideal I. Lie derivative of geometrical object, like tensors, are associated with symmetries of those objects. If the Lie derivative vanishes, then the vector $V$ represents the direction of an infinitesimal symmetry transformation in the manifold. Here the Lie derivative will be denoted by $L$ and the forms are our tensors. It is now simple to treat the invariance of a set of differential equations. A set of equations is invariant if a transformation leaves the equations still satisfied, provided that the original equations are satisfied. In the formalism we have introduced, this is easily stated: the Lie derivative of forms in the ideal must lie in the ideal $I_v \subset I$. Then if the basis forms in the ideal are annulled, the transformed equations are also annulled. And this should therefore represent symmetries. In practice, this means simply that the Lie derivative of each of the (basis) forms in $I$ is a linear combination of the forms in $I$. For further details on the method see ([2-4] and [11]).

In the sequel, we utilize this method to find the Lie point symmetries for the p-KdV equation of the form (1). First, write the equation (1) as a second order equation by defining a new variable $w = u_t$. Thus Eq.(1) becomes

$$w_{xx} + aw^2 + ut = 0 \quad (2)$$

2. The Harrison Method Applied to the Potential Korteweg-De Vries Equation

The method proceeds as follow. We consider a set of partial differential equations, defined on a partial differential manifold $M$ of $n$ independent variables $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $m$ dependent variables $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ ($n = 2$ and $m = 1$ in our special case.) Let $X = \mathbb{R}^2$, be the space representing the independent variables, and let $U = \mathbb{R}$, representing the space of dependent variable. We define the partial derivatives of the dependent variable as new variables (prolongation) in sufficient number to write the equation as second order equation, thus prolonging the manifold M to a manifold $N = M^{(2)\text{jet}}$ of the 2nd jet-space $X \times U^{(2)}$ of the manifold $X \times U$. The independent variables $(x, t) \in X$, the dependent variable $u \in U$ and $u^{(2)} = (u, u_x, u_{xx}, u_{tx}, u_{tt}, u_{xx}) \in U^{(2)}$, thus $(x, t, u^{(2)}) = (x, t, u, u_x, u_{xx}, u_{tx}, u_{tt}, u_{xx}) \in X \times U^{(2)}$. The space $M^{(2)}$ is the corresponding 2nd prolongation of the subspace $M \subset X \times U$. Then we can construct a set of differential forms. We speak of the set of forms, representing the equations, as an ideal $I$.
\[ \gamma^7 = au_x dt \ du_x dx \ du^+ dt \ du_{xx}. \]  
(9)

Proof. The proof of this lemma is straightforward. The forms \( \gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \) and \( \gamma^6 \) are obtained from Lemma 2.1 as follows:
\[ \gamma^1 = \beta^2 \wedge \beta^3, \quad \gamma^2 = \beta^2 \wedge \beta^4, \quad \gamma^3 = \beta^2 \wedge \beta^5, \quad \gamma^4 = d \beta^2, \quad \gamma^5 = d \beta^3, \quad \gamma^6 = d \beta^5, \]  
(10)
where \( \wedge \) is the wedge product. And form \( \gamma^7 \) is obtained from Eq. (1) by noting that
\[ u_{x,xx} = \frac{du_{xx}}{dx}, \quad u_x = \frac{du}{dx}, \quad u_t = \frac{du}{dt} \]  
(11)

Now, let
\[ X = V_t \frac{\partial}{\partial t} + V_x \frac{\partial}{\partial x} + V_u \frac{\partial}{\partial u} \]  
(12)
be a symmetry generator of the potential KdV equation (1), defined on the \((t, x, u)\) space. The second prolongation of \( X \) is the vector field
\[ \tilde{V} = V_t \frac{\partial}{\partial t} + V_x \frac{\partial}{\partial x} + V_u \frac{\partial}{\partial u} + V_{u,t} \frac{\partial}{\partial u,t} + V_{u,x} \frac{\partial}{\partial u,x} + V_{u,xx} \frac{\partial}{\partial u,xx} \]  
(13)
that acts on the manifold \( N \), with the coordinates \((t, x, u, u_t, u_x, u_{xx})\), where the \( V^j (j = 1, 2, \ldots, 8) \) are smooth functions to be determined in \( N \). Write the Lie derivatives of forms in \( I \) as linear combinations of themselves as follow:
\[ L_V \gamma^i = \sum_{j=1}^7 \lambda_j \gamma^j \]  
(14)
where the \( \gamma^j \) are the forms of Lemma 2.2 and the \( \lambda_j \) are 0-forms (functions), for \( j = 1, 2, \ldots, 7 \).

Proposition 2.1. Corresponding to the 2-form \( \gamma^j \), the identity
\[ L_V \gamma^i = \sum_{j=1}^7 \lambda_j \gamma^j \]  
(15)
is equivalent to the following system of partial differential equations:
\[ \begin{align*}
V^{uu} u_{tt} + u_x V^u_{tt} - u_x V^u_t t - &\lambda_1 (u_x u_{tt} - u_t u_{tx}) - \lambda_2 (u_x u_{tt} - u_t u_{tx}) - \lambda_3 (u_x^2 + u_{tt} u_{xx}) - V^{uu} u_{tx} \\
- &u_t V^{ux} + (u_x u_{tt} - u_t u_{tx}) V^x_t - (u_x u_{tt} - u_t u_{tx}) V^x_t + u_{tt} V^x_t = 0 \\
( &u_x u_{tt} - u_t u_{tx}) V^u_t + V^u_x + u_{tx} V^x_u + u_{tx} V^u_t - u_x V^u_t + u_{tt} V^x_t - \lambda_3 u_{tt} - \lambda_2 u_{xx} + \lambda_7 = 0 \\
V^u_t - ( &u_x u_{tt} - u_t u_{tx}) V^u_t + u_{tx} V^x_u - u_{tx} V^u_t + u_{tt} V^x_t - u_x V^u_t - V^u_t - \lambda_1 u_t - \lambda_2 u_x - \lambda_7 a u = 0 \\
( &u_x u_{tt} - u_t u_{tx}) V^x_t - V^{ux} + u_{tx} V^u_t - u_x V^x_u - u_x V^u_t - u_t V^x_t + V^x_u + \lambda_1 u_x + \lambda_3 u_{xx} = 0 \\
- &u_x u_{tt} - u_t u_{tx}) V^x_t - u_x V^x_u + u_{tx} V^u_t - V^{ux} - u_t V^x_t - u_t V^u_t + V^x_t + \lambda_1 u_t - \lambda_3 u_{xx} - \lambda_4 = 0 \\
- &u_x u_{tt} - u_t u_{tx}) V^x_t + u_{tt} V^u_t - u_x V^u_t + \lambda_2 u_t + \lambda_3 u_{tt} = 0 \\
- &u_x V^x u_{tt} - u_x V^x u_{tt} - u_t V^x u_{tt} + u_x V^u_t + V^u_t - \lambda_1 = 0 \\
-u_x V^x_{u_t} - u_x V^x_{u_t} - u_t V^x_{u_t} + V^u_t - \lambda_2 = 0 \\
- &u_x V^x_{u_t} - u_x V^x_{u_t} + V^u_t - \lambda_2 = 0 \\
(u_x u_{tt} - u_t u_{tx}) V^x_{u_t} + u_{tx} V^u_{u_t} - u_x V^u_{u_t} = 0 \\
(u_x u_{tt} - u_t u_{tx}) V^x_{u_t} + u_{tx} V^u_{u_t} - u_x V^u_{u_t} + \lambda_2 u_x + \lambda_3 u_{tx} - \lambda_4 = 0
\end{align*} \]
\[ (u_x u_{tt} - u_t u_{tx}) V_{ttx}^t + u_{tx} V_{ttx}^u - u_x V_{ttx}^u - \lambda_5 = 0 \]

\[ (u_x u_{tt} - u_t u_{tx}) V_{txx}^t + u_{tx} V_{txx}^u - u_x V_{txx}^u - \lambda_6 = 0 \]

\[-(u_x u_{tt} - u_t u_{tx}) V_{x}^t + u_{tt} V_{x}^u - u_t V_{x}^u - \lambda_5 = 0 \]

\[-(u_x u_{tt} - u_t u_{tx}) V_{x}^t + u_{tt} V_{x}^u - u_t V_{x}^u - \lambda_6 = 0 \]

\[-(u_x u_{tt} - u_t u_{tx}) V_{x}^t + u_{tt} V_{x}^u - u_t V_{x}^u + \lambda_7 = 0 \]

\[-u_{tx} V_{utt}^x - u_{tt} V_{ttt}^x + V_{utt}^u = 0 \]

\[-u_{tx} V_{utt}^x - u_{tt} V_{ttt}^x + V_{utt}^u = 0 \]

\[-u_{tx} V_{utt}^x - u_{tt} V_{ttt}^x + V_{utt}^u = 0 \]

\[ u_x V_{x}^x + u_t V_{x}^t - V_{x}^u - \lambda_3 = 0 \]

\[ u_x V_{x}^x + u_t V_{x}^t - V_{x}^u = 0 \]

\[ u_x V_{x}^x + u_t V_{x}^t - V_{x}^u = 0 \]

\[ u_x V_{x}^x + u_t V_{x}^t - V_{x}^u = 0 \]

\[ u_x V_{x}^x + u_t V_{x}^t - V_{x}^u = 0 \]

**Proof.** First, expand the left-hand-side of (15) by using some simple features of Lie derivatives of differential forms:

\[ L_V x^i = V^i, \quad L_V \left( \omega_1, \quad \omega_2 \right) = L_V \left( \omega_1 \right) \quad \omega_2 + \omega_1 \quad L_V \left( \omega_2 \right), \quad L_V dx^i = d(L_V x^i) = dV^i \]  

(16)

Where \( x^i \) is a coordinate of \( N \) and \( V^i \) is a component of \( V \). Expanding the \( dV^i \) in the resulting expression of \( L_V y^i \) by the usual chain rule (since the \( V^i \) are functions in \( N \)), using all eight variables, some terms drop out. This is due to the fact that \( dt \quad dt = 0, \quad dx \quad dx = 0, \quad etc., \quad by \quad the \quad antisymmetry \quad of \quad 1 \)-forms and leads to:

\[ L_V y^1 = (V^{ux} u_{tt} + u_x V^{ux}_{tt} - V^{ux} u_{tx} - u_x V^{ux} + (u_x u_{tt} - u_t u_{tx}) V_x^x + u_x u_{tt}) dx \quad dt \]

\[ -u_{tx} u_t V_x^t dx \land dt + (u_{tx} V_x^u - u_x V_{ttx}^x + u_t V_{ttx}^t) dx \land dt \]

\[ +((u_x u_{tt} - u_t u_{tx})) V_x^t + V_{ltt}^x \land dt \land du + (u_{tx} V_x^x + u_{tx} V_{ttx}^x - u_x V_{ttx}^x) dx \land du \]

\[ +(u_{tt} V_x^t - (u_x u_{tt} - u_t u_{tx}) V_x^x + u_{tx} V_{ttx}^x) dt \land du \]

\[ +(u_{tt} V_x^t - u_{tt} V_{ttt}^t + V_{tt}^t \land dt \land du + (u_{tt} V_{ttt}^t - u_{tt} V_{ttt}^t) V_x^t dt \land du_x \]

\[ -(V^{ux} + u_x V^{ux}_{tt} - u_x V^{ux}_{tt} - u_t V_x^t + V_x^t) dx \land du_t + u_x u_{tt} V_x^t dx \land du_x \]

\[ +(-u_{tt} V_x^t + u_{tx} V_{ttx}^x - u_x V_{ttx}^x) dx \land du_x - (u_x u_{tt} - u_t u_{tx}) V_x^t dt \land du_x \]

\[ +(-u_x V_x^x + u_x V_{txx}^x - V^{ux} - u_x V_x^x - u_t V_x^t + V_x^t) dt \land du_t \]

\[ +(u_{tt} V_x^t + u_{tx} V_{ttx}^x - u_x V_{ttx}^x) dt \land du_t + (u_{tt} V_{ttx}^x - u_x V_{ttx}^x) dt \land du_x \]

\[ +(-u_x V_x^x + u_t V_{ttx}^t - V^{ux} - u_x V_x^t - u_t V_x^t + V_x^t) dt \land du_x \]

\[ +(u_{tt} V_x^t + u_{tx} V_{ttx}^x - u_x V_{ttx}^x) dt \land du_x + (-u_{tx} V_x^t - u_x V_{ttx}^x - u_t V_{ttx}^t) dt \land du_t \]
\[ (+u_t V_{tt}^t + V_{tt}^t + V_{tt}^t) \, du \wedge du_t + (-u_{tx} V_{tx}^x - u_{tt} V_{tx}^x + V_{txt}^x) \, du \wedge du_x + \]
\[ + \left( (u_x u_{tt} - u_t u_{tx}) V_{tt}^t + u_{tx} V_{tt}^t - u_x V_{tt}^t \right) \, dx \wedge du_t + u_x V_{tt}^t \, dx \wedge du_t + \]
\[ + (u_{tx} V_{tt}^t - u_x V_{tt}^t - u_t V_{tt}^t) \, dx \wedge du_{tx} + (u_x u_{tt} - u_t u_{tx}) V_{tt}^t \, dx \wedge du_{xx} + \]
\[ + (-u_t u_{tx} V_{tx}^t + u_{tx} V_{tt}^t - u_x V_{tt}^t) \, dt \wedge du_t + \]
\[ + (-u_{tx} V_{tt}^t - u_t V_{tt}^t - u_x V_{tt}^t) \, du \wedge du_{tx} + (-u_{tx} V_{tt}^t - u_t V_{tt}^t + u_{tt} V_{tt}^t) \, du \wedge du_{tt} + \]
\[ + (-u_{tx} V_{tt}^t + V_{tt}^t) \, du \wedge du_x + (u_x V_{tt}^t + u_t V_{tt}^t - V_{tt}^t) \, du \wedge du_x + \]
\[ + (u_x V_{tt}^t + u_t V_{tt}^t - V_{tt}^t) \, du \wedge du_{tx} + (u_t V_{tt}^t + u_{tt} V_{tt}^t - V_{tt}^t) \, du \wedge du_{tt} + \]
\[ + (u_x V_{tt}^t + u_t V_{tt}^t - V_{tt}^t) \, du \wedge du_{xx} + u_t V_{tt}^t \, du \wedge du_{xx} + \]
\[ = 0. \quad (17) \]

And the right-hand-side of (15) is of the form

\[ \sum_{j=1}^7 \lambda_j \phi^j = (\lambda_1 (u_x u_{tt} - u_t u_{tx}) + \lambda_2 (u_x u_{tx} - u_t u_{xx}) + \lambda_3 (u_x^2 - u_{tx} u_{xx})) \, dx \wedge dt + \]
\[ + (\lambda_1 u_{tx} + \lambda_2 u_{xx} - \lambda_7) \, dx \wedge du + \lambda_1 u_{tt} + \lambda_2 u_{tx} + \lambda_7 \, du \wedge dt + \]
\[ + (-\lambda_1 u_x + \lambda_3 u_{xx}) \, dx \wedge du_t + (-\lambda_2 u_x - \lambda_3 u_{xx}) \, dx \wedge du_x + (-\lambda_1 u_t + \lambda_3 u_{xx} + \lambda_4) \, dt \wedge du_t + \]
\[ + (-\lambda_2 u_t - \lambda_3 u_{tt}) \, dt \wedge du_x + \lambda_1 du \wedge du_t + \lambda_2 du \wedge du_x + \lambda_5 \, dx \wedge du_{tt} + \]
\[ + \lambda_6 \, dx \wedge du_{xx} + \lambda_5 \, dt \wedge du_{tt} + \lambda_6 \, dt \wedge du_{xx} + \lambda_3 du \wedge du_{xx} + \lambda_7 du \wedge du_{xx} \]
\[ = 0. \quad (18) \]

Equating the coefficients of basis 2-forms \((dx \wedge dt, dx \wedge du, dt \wedge du, du \wedge dx, du \wedge du, etc.)\) in both right and left-hand-side of System (15), we get the system of Proposition (2.1).

The (point) symmetries of the potential KdV equation are provided by the theorem below:

**Theorem 2.1.** The symmetry Lie algebra of the potential KdV equation (2) is generated by the five vector fields

\[ X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial t}, X_3 = \frac{\partial}{\partial u}, \]
\[ X_4 = t \frac{\partial}{\partial x} + \frac{1}{2} x \frac{\partial}{\partial u}, X_5 = \frac{1}{3} x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} \quad (19) \]

**Proof.** Here, the reduced equation writes

\[ w_{xx} + aw^2 + u_t = 0, a \in R^* \]

Corresponding to this equation, the forms of the ideal \(I\) become

\[ \phi^1 = \gamma^1, \phi^2 = \gamma^2, \phi^3 = \gamma^3, \phi^4 = \gamma^4, \phi^5 = \gamma^5, \phi^6 = \phi^7 = \gamma^7, \]

where \(\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \gamma^6, \gamma^7\) are given by the Lemma (2.2). And the symmetry condition (14) for the potential KdV equation (1) reads

\[ \gamma^1 = \lambda_1 \gamma^1. \quad (20) \]

This leads to the system of partial differential equations of the form

\[ -u_{tx} V_{tt}^t + u_{xx} V_{tx}^t + \cdots - u_{tx} V_{tt}^t + u_{tt} V_{tt}^t = 0 \]
Now, it is easy to see that the Lie bracket set of five parameter symmetry group for the potential KdV vanish the potential KdV equation (1). Hence, 

\[ \{ \begin{array}{l}
\quad \text{easy to check that the third prolongations of these vectors correspond to the vector fields of theorem 2.1. And it is construction. The point parts of these vector fields have}
\quad \text{variables}\end{array} \] 

\[ \text{up to } k \text{ w.r.t the independent variables, and } \mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_l) \text{ is some mapping} \]

\[ \begin{align*}
\text{where commas indicate differentiation. From here onwards the calculations proceed in the standard way we arrive at the following solution of the system:} \\
V^1 &= C_1t + C_2, V^x = \frac{1}{3}x + C_3, V^u = \frac{1}{3}u + C_4, \\
V^{u_t} &= -\frac{4}{3}u_t + C_5, V^{u_x} = \frac{2}{3}u_x + C_6, \\
V^{u_{tt}} &= -\frac{7}{3}u_{tt} + C_7, V^{u_{xx}} = \frac{5}{3}u_{xx} + C_8,
\end{align*} \]

for \( \alpha \in \mathbb{R}^l \), where \( C_1, C_2, C_3, C_4, C_5 \) are arbitrary constants, which gives the five vector fields

\[ \begin{align*}
V_1 &= \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial t}, V_3 = \frac{\partial}{\partial u}, \\
V_4 &= t \frac{\partial}{\partial x} + \frac{1}{2a}x \frac{\partial}{\partial t} + \frac{1}{2a}u_x \frac{\partial}{\partial u} - u_x \frac{\partial}{\partial u_t} - 2u_{tx} \frac{\partial}{\partial u_{tt}} - u_{xx} \frac{\partial}{\partial u_{tx}} V_5 = \frac{1}{3}x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} - \frac{1}{3}u \frac{\partial}{\partial u} - \frac{2}{3}u_x \frac{\partial}{\partial u_x} - \frac{4}{3}u_t \frac{\partial}{\partial u_t} - \frac{7}{3}u_{tt} \frac{\partial}{\partial u_{tt}} - \frac{5}{3}u_{xx} \frac{\partial}{\partial u_{xx}} - u_{xx} \frac{\partial}{\partial u_{xx}}
\end{align*} \]

Now, it is easy to see that the Lie bracket \([V_i, V_j]\) of vector fields lies in the vector space constructed by \([V_1, V_2, V_3, V_4, V_5]\). We have

\[ [V_4, V_5] = -\frac{1}{3}V_4, \quad ([V_1, V_2], [V_1, V_3], [V_2, V_3] \text{ and } [V_3, V_4]) \text{ are the vanishing brackets. This means that the set of the vector fields } [V_1, V_2, V_3, V_4, V_5] \text{ makes a Lie algebra construction. The point parts of these vector fields correspond to the vector fields of the theorem 2.1. And it is easy to check that the third prolongations of these vectors vanish the potential KdV equation (1). Hence, \{V_4, V_5\} makes a set of five parameter symmetry group for the potential KdV equation (1). And the theorem 2.1 follows.} \]

3. Invariant Solutions

In this section, we recall the general procedure for determining invariant solutions for any system of partial differential equations. To begin, let us consider an arbitrary system of \( l \) (nonlinear) partial differential equations (PDEs) of order \( k \)

\[ S: \mathcal{R}(x, u^{(l)}) = 0, \forall = 1, \ldots, l \] \hspace{1cm} (21)

involving \( n \) independent variables \( x = (x_1, \ldots, x_n) \), \( m \) dependent variables \( u = (u_1, \ldots, u^m) \) and their partial derivatives of order up to \( k \) w.r.t the independent variables, and \( \mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_l) \) is some mapping

\[ \mathcal{R}: M^{(k)} \rightarrow \mathbb{R}^l, (x, u^{(k)}) \rightarrow \mathcal{R}(x, u^{(k)}), \] \hspace{1cm} (22)

where \( M^{(k)} \) is the \( k \)-jet space of a space \( M \subset X \times U \) of the variables \( (x, u) \).

Let us write the Lie group of transformations associated to this system in the form

\[ \tilde{x} = \phi(x, u, \epsilon), \quad \tilde{u} = \gamma(x, u, \epsilon) \] \hspace{1cm} (23)

and suppose the general form of infinitesimal generator \( V \) of the group (19) be

\[ V = \sum_{i=1}^{n} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{a=1}^{m} \eta_a(x, u) \frac{\partial}{\partial u^a} \] \hspace{1cm} (24)

The function \( u = f(x) \), with components \( u^a = f^a(x) (\alpha = 1, \ldots, m) \), is said to be an invariant solution (see [12]) of (21) if \( u^a = f^a(x) \) is an invariant surface of (19), and is a solution of (26), i.e., a solution is invariant if and only if:

\[ V(u^a - f^a(x)) = 0, \quad \text{for } u^a = f^a(x), (\alpha = 1, \ldots, m) \] \hspace{1cm} (25)

\[ R(x, u^{(k)}) = 0 \] \hspace{1cm} (26)

The equations (21), called invariant surface conditions, have the form
\( \xi^1(x,u) \frac{\partial u}{\partial x_1} + \cdots + \xi^n(x,u) \frac{\partial u}{\partial x_n} = \eta \), \( \alpha = 1, \ldots, m \) \hspace{1cm} (27)

and are solved by introducing the corresponding characteristic equations:

\[ \frac{dx_1}{\xi^1(x,u)} = \cdots = \frac{dx_n}{\xi^n(x,u)} = \frac{dt}{\eta_1(x,u)} = \cdots = \frac{du}{\eta_m(x,u)} \] \hspace{1cm} (28)

This allows to express the solution \( u = f(x) \) (that may be given in implicit form if some of the infinitesimals \( \xi \) depend on \( u \)) as

\[ u^n = \Psi^0(J_1(x,u), \ldots, J_{n-1}(x,u)), \alpha = 1, \ldots, m. \] \hspace{1cm} (29)

By substituting (29) into (26), a reduced system of differential equations involving \( n-1 \) independent variables (called similarity variables) is obtained (See [8] and [12]) for more details.

4. Invariant Solutions of Equation (1)

The potential KdV equation is invariant under the five point symmetries presented in Theorem 2.1. For each one-parameter subgroup of the full symmetry group there will be a corresponding class of group-invariant solutions which will be determined from a reduced ordinary differential equation.

1) Reduction under \( X_1 \):

The vector \( X_1 = \frac{\partial}{\partial x} \) generates the space translation group, with infinitesimals \( \xi = 1, \tau = 0, \eta = 0 \). To obtain invariant solutions of equation (1) from this group, we have to solve the system of characteristic equations (28), which writes in the form

\[ \frac{dt}{a} = \frac{dx}{1} = \frac{du}{0}. \] \hspace{1cm} (30)

The integration of (31) leads to the similarity variables

\[ r = x, s = t, y(s) = u. \] \hspace{1cm} (31)

Taking \( u = y(s) \), the equation (1) reduced to the following first ordinary differential equation

\[ y' = 0. \] \hspace{1cm} (32)

which leads to the solution

\[ y(s) = k, \] \hspace{1cm} (33)

for \( k \) arbitrary constant. In terms of original variables, we find that

\[ u(x,t) = k, \] \hspace{1cm} (34)

with \( k \) arbitrary constant. Thus, the global invariant solution of the potential KdV equation under \( X_1 \) is an constant.

2) Reduction under \( X_2 \):

Here, equation (1) is invariant under the time translation group with infinitesimals \( \xi = 0, \tau = 1, \eta = 0 \). Corresponding to the system (28), we have to solve the equations

\[ \frac{dx}{a} = \frac{dt}{1} = \frac{du}{0}. \] \hspace{1cm} (35)

which leads to the following global invariants

\[ v = t, w = x, h(w) = u \] \hspace{1cm} (36)

substituting the function \( u = h(w) \) in (1), the potential KdV equation reduces to the following nonlinear third ordinary differential equation

\[ h'' + h''' = 0 \] \hspace{1cm} (37)

This reduced equation is invariant under the group generated by the vector

\[ V = \frac{\partial}{\partial w} - h \frac{\partial}{\partial h}, \] \hspace{1cm} (38)

with infinitesimals \( \xi = 1, \eta = h \). Corresponding to these infinitesimals, the system (28) reads

\[ \frac{dw}{1} = \frac{dh}{h}. \] \hspace{1cm} (39)

Global invariants in terms of \( w \) and \( h \) are

\[ z = \ln|w|, \varphi = wh, \text{that is} \ w = e^{-z}, h = \frac{\varphi}{e^z}, \text{for } w \neq 0. \] \hspace{1cm} (40)

3) Reduction under \( X_3 \):

The infinitesimals for \( X_4 \) are \( \xi = t, \tau = 0, \eta = \frac{x}{2a} \). Integrating the characteristic equations

\[ \frac{dx}{1} = \frac{dt}{0} = \frac{du}{2a}. \] \hspace{1cm} (42)

obtains the similarity variables as follows

\[ \alpha = \frac{2}{t}, \beta = t, \mu = \frac{x^2}{4at}, \text{that is} \ x = \alpha \beta, \text{and } u = \frac{a^2}{4a} + \mu(\beta). \] \hspace{1cm} (43)

The reduced equation is simply

\[ \mu' = 0. \] \hspace{1cm} (44)

which leads to the solution \( \mu(\beta) = C \), for \( C \) arbitrary constant. In terms of the original variables, the general Galilean-invariant solutions to (1) is provided to be
\[ u(x, t) = \frac{1}{4a} \frac{x^2}{t} + C, \text{ for } a \neq 0 \text{ and } t \neq 0 \]  

(45)

4) Reduction under \( X_s \)

Let us now deal with the symmetry generator \( X_s \) which has infinitesimals of the form \( \xi = \frac{a}{3} t, \gamma = t, \eta = -\frac{a}{3} u \). The characteristic equations write

\[
\frac{dx}{\xi} = \frac{dt}{\gamma} = \frac{du}{\eta} = \frac{\frac{a}{3} x}{t} \tag{46}
\]

Where \( a \) is a real-valued constant. Note that Eq. (48) has infinitesimals that are all zero

\[ \xi(x, q) = 0, \eta(x, q) = 0 \]  

(49)

showing that this ordinary differential equation has no point symmetries and hence, no group-invariant solutions. The corresponding solutions of the p-KdV equation take the general form

\[ u = \frac{h(x)}{x}, \text{ for } x \neq 0. \]  

(50)

5. Conclusion

We see that differential forms offer, in some ways, a more natural way of calculating symmetries of differential equations. With this technique we need only calculate prolongation coefficients up to second order and hence, obtain the prolongation of the generator \( X \). This leads naturally to the point symmetry algebra in investigation. Using functionally independent invariants from the symmetries that we have found, the reduced equations have been constructed for each of generators of these symmetries. Exact solutions have been obtained for the p-KdV equation that are travelling-invariants and Galilean-invariants. Reduction under the generator of the scale-symmetry has not been completed because the infinitesimals of the corresponding reduced third ordinary differential equation are all zero.

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References


