American Journal of Mathematical and Computational Sciences
2020; 5(2): 9-16
http://www.aascit.org/journal/ajmcs

American Association for Science and Technology

Lie Symmetries and Invariant-Solutions of the Potential Korteweg-De Vries Equation

Faya Doumbo Kamano[1], Bakary Manga[2], Joel Tossa[3], Momo Bangoura[4]

[1]Department of Research, Distance Learning High Institute, Conakry, Guinea
[2]Department of Mathematics and Computer Sciences, University of Cheikh Anta Diop, Dakar, Senegal
[3]Institute of Mathematics and Physical Sciences, University of Abomey-Calavi, Porto-Novo, Benin
[4]Department of Mathematics, University of Gamal Abdel Nasser, Conakry, Guinea

Email address
fd.kamano@isfad-gn.org (F. D. Kamano), bakary.manga@ucad.edu.sn (B. Manga), joel.tossa@imsp-uac.org (J. Tossa), bangm59@yahoo.fr (M. Bangoura)

[*]Corresponding author

Citation
Faya Doumbo Kamano, Bakary Manga, Joel Tossa, Momo Bangoura. Lie Symmetries and Invariant-Solutions of the Potential Korteweg-De Vries Equation. *American Journal of Mathematical and Computational Sciences*. Vol. 5, No. 2, 2020, pp. 9-16.

Received: April 17, 2020; **Accepted:** June 6, 2020; **Published:** August 5, 2020

Abstract: The purpose of this paper is to investigate the nonlinear partial differential equation, known as potential Korteweg-de Vries (p-KdV) equation. We have implemented the Harrison technique that makes use of differential forms and Lie derivatives as tools to find the point symmetry algebra for the p-KdV equation. This approach allows us to obtain five infinitesimal generators of point symmetries. Fixing each generator of symmetries that we have found, we construct a complete set of functionally independent invariants, corresponding to the new independent and dependent variables. Using these new variables, called "similarity variables", the reduced equations have been constructed systematically, which leads to exact solutions that are group-invariant solutions for the p-KdV equation. The obtained solutions are of two types. The reduced equations from the generator of space and time translation groups are the first and the third order ordinary differential equations respectively and lead to the Travelling-invariant solutions. Then, the reduced equation from the generator of the Galilean boosts is the first order ordinary differential equation and leads to Galilean-invariant solutions. Under the generator of scaling symmetries, the potential KdV equation reduces to the third order ordinary differential equation, which does not admit symmetries. And then, there are no functionally independent invariants for that last equation, its solutions are essentially new functions not expressible in terms of standard special functions.

Keywords: Symmetries, Differential Forms, Lie Derivative, Korteweg-de Vries Equations, Invariant Solutions

1. Introduction

Lie symmetries of differential equations are one of the important concepts in the theory of differential equations and physics. Among others methods, Lie method is a firm one for finding symmetries of differential equations. This method was first applied to determine point symmetries (see [8] and [12]). In 1969-1970, B. Kent Harrison and Frank Estabrook devised a method to calculate symmetries of differential equations using differential forms and Cartan's formulation of differential equations [2]. They were simply trying to understand how the symmetries of Maxwell's equations could be found from the differential form version of those equations. Once they realized that the key to symmetries was the use of the Lie derivative, B. Kent Harrison applied the method to several others equations such as the one dimensional heat equation, the Short wave gas dynamic equation and the nonlinear Poisson equation (see [3] and [4]). Here we apply this method to the potential KdV equation, given as follows

$$u_{xxx} + a(u_x)^2 + u_t = 0, \qquad (1)$$

where $u(x,t)$ is a function of space x and time variable t; subscripts denoted partial derivatives; a is a real constant, with a is no vanishing.

The p-KdV equation is widely used in various branches of physics [1]. Exact travelling wave solutions to nonlinear evolution equations, particularly which appear in many physical structures in solitary wave theory such as solitons, kinks, peakons, and cuspons [14], draw considerable interest in recent years in revealing the mechanism of the complicated physical phenomena and dynamical processes modeled by these nonlinear evolution equations [7]. There has been an enormous interest on the study of soliton structures that propagate in nonlinear media without changing their profile by reason of their potential application in many physics areas. Their existence is a result of a delicate balance between linear dispersion and nonlinearity. The KdV equation has been the subject of extensive studies (see [1], [5-7] and [9, 10, 13, 14]).

The objective of this paper is to apply the differential form technique to investigate point symmetries of the p-KdV equation, which symmetries will be used to construct invariant solutions for the equation.

Note that in 2008, Mehdi Nadjafikhah and Seyed-Reza Hejazi applied the differential form approach to the standard *KdV* equation of the form $u_{xxx} + uu_x + u_t = 0$ (see [11]).

The remainder of this work is organized as follows. In section 2, we first determine the point symmetries of the potential KdV equation by using the Harrison approach. Our tools in this method are differential forms and Lie derivatives. In section 3, we recall the method to construct invariant solutions of a given system of differential equations. And finally, we provide invariant solutions of the equation (1) in section 4.

2. The Harrison Method Applied to the Potential Korteweg-De Vries Equation

The method proceeds as follow. We consider a set of partial differential equations, defined on a differential manifold M of n independent variables $x = (x_1,...,x_n) \in \mathrm{R}^n$ and m dependent variables $u = (u^1,...,u^m) \in \mathrm{R}^m$ ($n = 2$ and $m = 1$ in our special case.) Let $X = \mathrm{R}^2$, be the space representing the independent variables, and let $U = \mathrm{R}$, representing the space of dependent variable. We define the partial derivatives of the dependent variable as new variables (prolongation) in sufficient number to write the equation as second order equation, thus *prolonging* the manifold M to a manifold N = $\mathrm{M}^{(2)}$ of the *2nd jet-space* $X \times U^{(2)}$ of the manifold $X \times U$. The independent variables $(x,t) \in X$; the dependent variable $u \in U$ and $u^{(2)} = (\mathrm{u}, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) \in U^{(2)}$, thus $(x,t,u^{(2)}) = (\mathrm{t}, \mathrm{x}, \mathrm{u}, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) \in X \times U^{(2)}$. The space $\mathrm{M}^{(2)}$ is the corresponding 2^{nd} *prolongation* of the subspace $\mathrm{M} \subset X \times U$. Then we can construct a set of differential forms. We speak of the set of forms, representing the equations, as an ideal *I*. It is to be closed. Then for determining the invariance of the differential equations, we may construct the Lie derivative of the forms in the ideal *I*. Lie derivative of geometrical object, like tensors, are associated with symmetries of those objects. If the Lie derivative vanishes, then the vector *V* represents the direction of an infinitesimal symmetry transformation in the manifold. Here the Lie derivative will be denoted by L and the forms are our tensors. It is now simple to treat the invariance of a set of differential equations. A set of equations is invariant if a transformation leaves the equations still satisfied, provided that the original equations are satisfied. In the formalism we have introduced, this is easily stated: the Lie derivative of forms in the ideal must lie in the ideal: $\mathrm{L}_V\ I \subset I$. Then if the basis forms in the ideal are annulled, the transformed equations are also annulled. And this should therefore represent symmetries. In practice, this means simply that the Lie derivative of each of the (basis) forms in *I* is a linear combination of the forms in *I*. For further details on the method see ([2-4] and [11]).

In the sequel, we utilize this method to find the Lie point symmetries for the p-KdV equation of the form (1). First, write the equation (1) as a second order equation by defining a new variable $w = u_x$. Thus Eq.(1) becomes

$$\mathrm{w}_{xx} + aw^2 + u_t = 0 \tag{2}$$

Then, we construct a set of 1-forms on the manifold N as follow

Lemma 2.1. For Eq. (2), the required 1-forms are

$$\beta^1 = du - u_t dt - u_x dx$$

$$\beta^2 = du_t - u_{tt} dt - u_{tx} dx$$

$$\beta^3 = du_x - u_{tx} dt - u_{xx} dx$$

Proof. We consider the 8-dimensional manifold corresponding to the equation (2), with the coordinates $(\mathrm{t}, \mathrm{x}, \mathrm{u}, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$. Considering this equation, we have the following contact conditions

$$du = u_t dt + u_x\, dx,$$

$$du_t = u_{tt} dt + u_{tx} dx,$$

$$du_x = u_{tx} dt + u_{xx} dx,$$

which lead to the contact 1-forms of Lemma 2.1.

Then, we have to construct the forms of the ideal I by the following

Lemma 2.2. The ideal I consists of the following 2-forms:

$$\gamma^1 = (u_x u_{tt} - u_t u_{tx}) dx \wedge dt + u_{tx} dx \wedge du - u_x dx \wedge du_t + u_{tt} dt \wedge du - u_t dt \wedge u_t + du \wedge u_t; \tag{3}$$

$$\gamma^2 = (u_x u_{tx} - u_t u_{xx}) dx \wedge \mathrm{dt} + u_{xx} dx \wedge \mathrm{du} - u_x dx \wedge u_x + u_{tx} dt \wedge du - u_t dt \wedge u_x + du \wedge u_x; \tag{4}$$

$$\gamma^3 = ((u_{tx})^2 - u_{tt} u_{xx}) dx \wedge \mathrm{dt} + u_{xx} dx \wedge \mathrm{d}u_t - u_{tx} dx \wedge u_x + u_{tx} dt \wedge du_t - u_{tt} dt \wedge u_x + du_t \wedge u_x; \tag{5}$$

$$\gamma^4 = dx \wedge \mathrm{d}u_x + dt \wedge \mathrm{d}u_t\,; \tag{6}$$

$$\gamma^5 = dx \wedge \mathrm{d}u_{tx} + dt \wedge \mathrm{d}u_{tt}\,; \tag{7}$$

$$\gamma^6 = dx \wedge \mathrm{d}u_{xx} + dt \wedge \mathrm{d}u_{tx}\,; \tag{8}$$

$$\gamma^7 = au_x dt \wedge du - dx \wedge du + dt \wedge du_{xx} \,. \tag{9}$$

Proof. The proof of this lemma is straightforward. The forms $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5$ and γ^6 are obtained from *Lemma 2.1* as follows:

$$\gamma^1 = \beta^1 \wedge \beta^2,\ \gamma^2 = \beta^1 \wedge \beta^3,\ \gamma^3 = \beta^2 \wedge \beta^3,\ \gamma^4 = d\beta^1,\ \gamma^5 = d\beta^2,\ \gamma^6 = d\beta^3, \tag{10}$$

where $\wedge$ is the wedge product. And form γ^7 is obtained from Eq. (1) by noting that

$$u_{xxx} = \frac{du_{xx}}{dx}, u_x = \frac{du}{dx},\ u_t = \frac{du}{dt}. \tag{11}$$

Now, let

$$X = V^{\mathrm{t}} \frac{\partial}{\partial t} + V^{\mathrm{x}} \frac{\partial}{\partial x} + V^{\mathrm{u}} \frac{\partial}{\partial u}, \tag{12}$$

be a symmetry generator of the potential KdV equation (1), defined on the (t,x,u) space. The second prolongation of X is the vector field

$$V = V^{\mathrm{t}} \frac{\partial}{\partial t} + V^{\mathrm{x}} \frac{\partial}{\partial x} + V^{\mathrm{u}} \frac{\partial}{\partial u} + V^{u_t} \frac{\partial}{\partial u_t} + V^{u_x} \frac{\partial}{\partial u_x} + V^{u_{tt}} \frac{\partial}{\partial u_{tt}} + V^{u_{tx}} \frac{\partial}{\partial u_{tx}} + V^{u_{xx}} \frac{\partial}{\partial u_{xx}} \tag{13}$$

that acts on the manifold N, with the coordinates $(\mathrm{t}, \mathrm{x}, \mathrm{u}, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$, where the V^i ($i = 1,2,...,8$) are smooth functions to be determined in N. Write the Lie derivatives of forms in I as linear combinations of themselves as follow:

$$L_V \gamma^i = \textstyle\sum_{j=1}^{7} \lambda_j \gamma^j \tag{14}$$

where the γ^i are the forms of *Lemma 2.2* and the λ_j are 0-forms (functions), for $\mathrm{j} = 1, 2, \ldots, 7$

Proposition 2.1. Corresponding to the 2-form γ^1, the identity

$$L_V \gamma^1 = \textstyle\sum_{j=1}^{7} \lambda_j \gamma^j \tag{15}$$

is equivalent to the following system of partial differential equations:

$$V^{u_x} u_{tt} + u_x V^u_{,t} - u_x V^{u_t}_{,t} - \lambda_1 \left((u_x u_{tt} - u_t u_{tx})\right) - \lambda_2 \left((u_x u_{tt} - u_t u_{tx})\right) - \lambda_3 \left(u_{tx}^2 + u_{tt} u_{xx}\right) - V^{u_t} u_{tx}$$

$$-u_t V^{u_{tx}} + (u_x u_{tt} - u_t u_{tx}) V^x_{,x} - (u_x u_{tt} - u_t u_{tx}) V^t_{,t} + u_{tx} V^u_{,t} = 0$$

$$(u_x u_{tt} - u_t u_{tx}) V^t_{,u} + V^u_{,u_{tx}} + u_{tx} V^x_{,x} + u_{tx} V^u_{,u} - u_x V^{u_t}_{,u} + u_{tt} V^t_{,x} - \lambda_1 u_{tx} - \lambda_2 u_{xx} + \lambda_7 = 0$$

$$V^u_{,t} - (u_x u_{tt} - u_t u_{tx}) V^x_{,u} + u_{tx} V^x_{,t} + u_{tt} V^t_{,t} + u_{tt} V^u_{,u} - u_t V^{u_t}_{,u} - V^{u_t}_{,t} - \lambda_1 u_{tt} - \lambda_2 u_{tx} - \lambda_7 au = 0$$

$$(u_x u_{tt} - u_t u_{tx}) V^t_{,u_t} - V^{u_x} + u_{tx} V^u_{,u_t} - u_x V^x_{,x} - u_x V^{u_t}_{,u_t} - u_t V^t_{,x} + V^u_{,x} + \lambda_1 u_x + \lambda_3 u_{xx} = 0$$

$$-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_t} - u_x V^x_{,t} + u_{tt} V^u_{,u_t} - V^{u_t} - u_t V^t_{,t} - u_t V^{u_t}_{,u_t} + V^u_{,t} + \lambda_1 u_t - \lambda_3 u_{tx} - \lambda_4 = 0$$

$$-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_x} + u_{tt} V^u_{,u_t} - u_t V^{u_t}_{,u_x} + \lambda_2 u_t + \lambda_3 u_{tt} = 0$$

$$-u_{tx} V^x_{,u_t} - u_x V^x_{,u} - u_{tt} V^t_{,u_t} - u_t V^t_{,u} + V^u_{,u} + V^{u_t}_{,u_t} - \lambda_1 = 0$$

$$-u_{tx} V^x_{,u_x} - u_{tt} V^t_{,u_x} + V^{u_t}_{,u_x} - \lambda_2 = 0$$

$$(u_x u_{tt} - u_t u_{tx}) V^t_{,u_{tt}} + u_{tx} V^u_{,u_{tt}} - u_x V^{u_t}_{,u_{tt}} = 0$$

$$(u_x u_{tt} - u_t u_{tx}) V^t_{,u_x} + u_{tx} V^u_{,u_x} - u_x V^{u_t}_{,u_x} + \lambda_2 u_x + \lambda_3 u_{tx} - \lambda_4 = 0$$

$$(u_x u_{tt} - u_t u_{tx}) V^t_{,u_{tx}} + u_{tx} V^u_{,u_{tx}} - u_x V^{u_t}_{,u_{tx}} - \lambda_5 = 0$$

$$(u_x u_{tt} - u_t u_{tx}) V^t_{,u_{xx}} + u_{tx} V^u_{,u_{xx}} - u_x V^{u_t}_{,u_{xx}} - \lambda_6 = 0$$

$$-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_{tt}} + u_{tt} V^u_{,u_{tt}} - u_t V^{u_t}_{,u_{tt}} - \lambda_5 = 0$$

$$-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_{tx}} + u_{tt} V^u_{,u_{tx}} - u_t V^{u_t}_{,u_{tx}} - \lambda_6 = 0$$

$$-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_{xx}} + u_{tt} V^u_{,u_{xx}} - u_t V^{u_t}_{,u_{xx}} + \lambda_7 = 0$$

$$-u_{tx} V^x_{,u_{tt}} - u_{tt} V^t_{,u_{tt}} + V^{u_t}_{,u_{tt}} = 0$$

$$-u_{tx} V^x_{,u_{tx}} - u_{tt} V^t_{,u_{tx}} + V^{u_t}_{,u_{tx}} = 0$$

$$-u_{tx} V^x_{,u_{xx}} - u_{tt} V^t_{,u_{xx}} + V^{u_t}_{,u_{xx}} = 0$$

$$u_x V^x_{,u_x} + u_t V^t_{,u_x} - V^u_{,u_x} - \lambda_3 = 0$$

$$u_x V^x_{,u_{tx}} + u_t V^t_{,u_{tx}} - V^u_{,u_{tx}} = 0$$

$$u_t V^t_{,u_{tt}} + u_x V^x_{,u_{tt}} - V^u_{,u_{tt}} = 0$$

$$u_x V^x_{,u_{xx}} + u_t V^t_{,u_{xx}} - V^u_{,u_{xx}} = 0$$

Proof. First, expand the left-hand-side of (15) by using some simple features of Lie derivatives of differential forms:

$$L_V x^i = V^i,\ L_V\left(\omega_1 \wedge \omega_2\right) = L_V\left(\omega_1\right) \wedge \omega_2 + \omega_1 \wedge L_V(\omega_2),\ L_V dx^i = d(L_V x^i) = dV^i \tag{16}$$

Where x^i is a coordinate of N and V^i is a component of V. Expanding the dV^i in the resulting expression of $L_V\gamma^i$ by the usual chain rule (since the V^i are functions in N), using all eight variables, some terms drop out. This is due to the fact that $dt \wedge dt = 0$, $dx \wedge dx = 0$, etc., by the antisymmetry of 1-forms and leads to:

$$\mathrm{L}_V\gamma^1 = (V^{u_x} u_{tt} + u_x V^u_{,tt} - V^{u_t} u_{tx} - u_t V^{u_{tx}} + (u_x u_{tt} - u_t u_{tx}) V^x_{,x} + u_x u_{tt}) dx \wedge dt$$
$$-u_{tx} u_t V^t_{,t} dx \wedge dt + (u_{tx} V^u_{,t} - u_x V^{u_t}_{,t} - u_{tt} V^u_{,x} + u_t V^{u_t}_{,x}) dx \wedge dt$$
$$+((u_x u_{tt} - u_t u_{tx})) V^t_{,u} + V^u_{,u_{tx}})\, dx \wedge du + (u_{tx} V^x_{,x} + u_{tx} V^u_{,u} - u_x V^{u_t}_{,u}) dx \wedge du$$
$$+(u_{tt} V^t_{,x} - V^{u_{tx}}_{,x}) dx \wedge du + (V^u_{,t} - (u_x u_{tt} - u_t u_{tx}) V^x_{,u} + u_{tx} V^x_{,t}) dt \wedge du$$
$$+(u_{tt} V^t_{,t} - u_{tt} V^u_{,u} - u_t V^{u_t}_{,u} - V^{u_t}_{,t}) dt \wedge du + (u_x u_{tt} - u_t u_{tx}) V^t_{,u_{tt}} dx \wedge du_t$$
$$-(V^{u_x} + u_{tx} V^u_{,u_x} - u_x V^x_{,x} - u_x V^{u_t}_{,u_t} - u_t V^t_{,x} + V^u_{,x}) dx \wedge du_t + u_x u_{tt} V^t_{,u_x} dx \wedge du_x$$
$$+(-u_t u_{tx} V^t_{,u_x} + u_{tx} V^u_{,u_x} - u_x V^{u_t}_{,u_x})\, dx \wedge du_x - (u_x u_{tt} - u_t u_{tx}) V^x_{,u_t} dt \wedge du_t$$
$$+(-u_x V^x_{,t} + u_{tt} V^u_{,u_t} - V^{u_t} - u_t u_x V^t_{,t} - u_t V^{u_t}_{,u_t} + V^u_{,t})\, dt \wedge du_t - u_x u_{tt} V^x_{,u_x}\, dt \wedge du_x$$
$$+(u_t u_{tx} V^x_{,u_x} + u_{tt} V^u_{,u_x} - u_t V^{u_t}_{,u_x})\, dt \wedge du_x + (-u_{tx} V^x_{,u_t} - u_x V^x_{,u} - u_{tt} V^t_{,u_t}) du \wedge du_t$$

$$+(-u_t V^t_{,u} + V^u_{,u} + V^{u_t}_{,u_t})\, du \wedge du_t + (-u_{tx} V^x_{,u_x} - u_{tt} V^t_{,u_x} + V^{u_t}_{,u_x}) du \wedge du_x$$
$$+\left((u_x u_{tt} - u_t u_{tx}) V^t_{,u_{tt}} + u_{tx} V^u_{,u_{tt}} - u_x V^{u_t}_{,u_{tt}}\right) dx \wedge du_{tt} + u_x V^t_{,u_{tx}} dx \wedge du_{tx}$$
$$+(-u_t u_{tx} V^t_{,u_{tx}} + -u_{tx} V^u_{,u_{tx}} - u_x V^{u_t}_{,u_{tx}}) dx \wedge du_{tx} + (u_x u_{tt} - u_t u_{tx}) V^t_{,u_{xx}}\, dx \wedge du_{xx}$$
$$+(u_{tx} V^u_{,u_{xx}} - u_x V^{u_t}_{,u_{xx}})\, dx \wedge du_{xx} + (-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_{tt}} + u_{tt} V^u_{,u_{tt}})\, dt \wedge du_{tt}$$
$$-u_t V^{u_t}_{,u_{tt}}\, dt \wedge du_{tt} + (-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_{tx}} + u_{tt} V^u_{,u_{tx}} - u_t V^{u_t}_{,u_{tx}})\, dt \wedge du_{tx}$$
$$+(-(u_x u_{tt} - u_t u_{tx}) V^x_{,u_{xx}} + u_{tt} V^u_{,u_{xx}} - u_t V^{u_t}_{,u_{xx}}) dt \wedge du_{xx} - u_{tx} V^x_{,u_{tt}} du \wedge du_{tt}$$
$$+(-u_{tt} V^t_{,u_{tt}} + V^{u_t}_{,u_{tt}})\, du \wedge du_{tt} + (-u_{tx} V^x_{,u_{tx}} - u_{tt} V^t_{,u_{tx}} + V^{u_t}_{,u_{tx}})\, du \wedge du_{tx}$$
$$+(-u_{tx} V^x_{,u_{xx}} - u_{tt} V^t_{,u_{xx}} + V^{u_t}_{,u_{xx}}) du \wedge du_{xx} + (u_x V^x_{,u_x} + u_t V^t_{,u_x} - V^u_{,u_x}) du_t \wedge du_x$$
$$+(u_x V^x_{,u_{tx}} + u_t V^t_{,u_{tx}} - V^u_{,u_{tx}})\, du_t \wedge du_{tx} + (u_t V^t_{,u_{tt}} + u_x V^x_{,u_{tt}} - V^u_{,u_{tt}}) du_t \wedge du_{tt}$$
$$+(u_x V^x_{,u_{xx}} + u_t V^t_{,u_{xx}} - V^u_{,u_{xx}})\, du_t \wedge du_{xx}. \tag{17}$$

And the right-hand-side of (15) is of the form

$$\sum_{j=1}^{7} \lambda_j \gamma^j = (\lambda_1\, (u_x u_{tt} - u_t u_{tx}) + \lambda_2 (u_x u_{tx} - u_t u_{xx}) + \lambda_3 (u_{tx}^2 - u_{tt} u_{xx})) dx \wedge dt$$
$$+(\lambda_1 u_{tx} + \lambda_2 u_{xx} - \lambda_7) dx \wedge du + \lambda_1 u_{tt} + \lambda_2 u_{tx} + \lambda_7 au) dt \wedge du \tag{18}$$
$$+(-\lambda_1 u_x + \lambda_3 u_{xx}) dx \wedge du_t + (-\lambda_2 u_x - \lambda_3 u_{tx} + \lambda_4) dx \wedge du_x + (-\lambda_1 u_t + \lambda_3 u_{tx} + \lambda_4) dt \wedge du_t$$
$$+(-\lambda_2 u_t - \lambda_3 u_{tt}) dt \wedge du_x + \lambda_1 du \wedge du_t + \lambda_2 du \wedge du_x + \lambda_5 dx \wedge du_{tx}$$
$$+\lambda_6 dx \wedge du_{xx} + \lambda_5 dt \wedge du_{tt} + \lambda_6 dt \wedge du_{tx} - \lambda_7 dt \wedge du_{xx} + \lambda_3 du_t \wedge du_x$$

Equating the coefficients of basis 2-forms (*dx∧dt, dx ∧ du, dt∧du, du∧du$_t$, du$_t$ ∧du$_x$, etc.*) in both right and left-hand-side of System (15), we get the system of *Proposition (2.1).*

The (point) symmetries of the potential *KdV* equation are provided by the theorem billow:

Theorem 2.1. The symmetry Lie algebra of the potential KdV equation (2) is generated by the five vector fields

$$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial t}, X_3 = \frac{\partial}{\partial u},$$
$$X_4 = t\frac{\partial}{\partial x} + \frac{1}{2a} x \frac{\partial}{\partial u}, X_5 = \frac{1}{3} x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} \tag{19}$$
$$-\frac{1}{3} u \frac{\partial}{\partial u}, \qquad a \in R^*$$

Proof. Here, the reduced equation writes

$$\mathrm{w}_{xx} + aw^2 + u_t = 0, a \in R^*$$

Corresponding to this equation, the forms of the ideal *I* become

$$\overset{1}{\varphi} = \gamma^1, \overset{2}{\varphi} = \gamma^2, \overset{3}{\varphi} = \gamma^3, \overset{4}{\varphi} = \gamma^4, \overset{5}{\varphi} = \gamma^5, \gamma^6 = \overset{6}{\varphi}, \overset{7}{\varphi} = \gamma^7,$$

where $\gamma^1, \gamma^2, \gamma^3, \gamma^4, \gamma^5, \gamma^6$ and γ^7 are given by the *Lemma (2.2)*. And the symmetry condition (14) for the potential *KdV* equation (1) reads

$$\gamma^i = \lambda_i \gamma^i, \tag{20}$$

This leads to the system of partial differential equations of the form

$$-u_{tx} V^{u_t}_{,u_t} - u_{xx} V^{u_t}_{,u_x} + \cdots - u_{tx} V^{u_x}_{,u_x} - u_{tt} V^{u_x}_{,t} = 0$$

$$u_{tx}V^{u_x} - u_{xx}V^{u_t} - \ \dots + (u_x u_{tx} - u_t u_{xx})(V^x_{,x} + V^t_t) = 0$$

$$\dots$$

$$V^{u_{xx}}_{,u} - a u_x V^{u_x} + \ \dots - V^t_t = 0$$

$$u_{tx}V^x_{,x} - u_x V^{u_t}_{,u} - \dots - u_{tx}V^{u_t}_{,u_t} - u_{xx}V^{u_t}_{,u_x},$$

where commas indicate differentiation. From here onwards the calculations proceed in the standard way we arrive at the following solution of the system:

$$V^{\mathrm{t}} = C_1 t + C_2\,, V^{\mathrm{x}} = \tfrac{1}{3}xC_1 + C_3 t + C_4, V^{\mathrm{u}} = \tfrac{1}{3}uC_1 + \tfrac{C_3}{2a}x + C_5,$$

$$V^{u_t} = -\frac{4}{3}u_t C_1 - u_x C_3, V^{u_x} = -\frac{2}{3}u_x C_1 + \frac{C_3}{2a},$$

$$V^{u_{tt}} = -\frac{7}{3}u_{tt}C_1 - 2u_{tx}C_3, V^{u_{tx}} = -\frac{5}{3}u_{tx}C_1 - u_{xx}C_3, V^{u_{xx}} = -C_1 u_{xx},$$

for $a \in R^*$, where C_1, C_2, C_3, C_4 and C_5 are arbitrary constants, which gives the five vector fields

$$V_1 = \frac{\partial}{\partial x}, V_2 = \frac{\partial}{\partial t}, V_3 = \frac{\partial}{\partial u},$$

$$V_4 = t\frac{\partial}{\partial x} + \frac{1}{2a}x\frac{\partial}{\partial u} + \frac{1}{2a}\frac{\partial}{\partial u_x} - u_x\frac{\partial}{\partial u_t} - 2u_{tx}\frac{\partial}{\partial u_{tt}} - u_{xx}\frac{\partial}{\partial u_{tx}}, V_5 = \frac{1}{3}x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}$$

$$-\frac{1}{3}u\frac{\partial}{\partial u} - \frac{2u_x}{3}\frac{\partial}{\partial u_x} - \frac{4}{3}u_t\frac{\partial}{\partial u_t} - \frac{7}{3}u_{tt}\frac{\partial}{\partial u_{tt}} - \frac{5}{3}u_{tx}\frac{\partial}{\partial u_{tx}} - u_{xx}\frac{\partial}{\partial u_{xx}}.$$

Now, it is easy to see that the Lie bracket $[V_i, V_j]$ of vector fields lies in the vector space constructed by $\{V_1, V_2, V_3, V_4, V_5\}$. We have

$$[V_1, V_4] = \tfrac{1}{2a}V_3, [V_1, V_5] = \tfrac{1}{3}V_1, [V_2, V_4] = V_1, [V_2, V_5] = V_2, \ [V_3, V_5] = -\tfrac{1}{3}V_3,$$

$[V_4, V_5] = -\frac{1}{3}V_4$, ($[V_1, V_2], [V_1, V_3]$, $[V_2, V_3]$ and $[V_3, V_4]$) are the vanishing brackets. This means that the set of the vector fields $\{V_1, V_2, V_3, V_4, V_5\}$ makes a Lie algebra construction. The point parts of these vector fields correspond to the vector fields of the theorem 2.1. And it is easy to check that the third prolongations of these vectors vanish the potential KdV equation (1). Hence, $\{V_4\}$ makes a set of five parameter symmetry group for the potential KdV equation (1). And the theorem 2.1 follows. □

3. Invariant Solutions

In this section, we recall the general procedure for determining invariant solutions for any system of partial differential equations. To begin, let us consider an arbitrary system of l (nonlinear) partial differential equations (PDEs) of order k

$$S: \mathcal{R}_\nu(x, u^{(k)}) = 0, \nu = 1,\dots,l, \quad (21)$$

involving n independent variables $x = (x_1,\dots,x_n)$, m dependent variables $u = (u^1,\dots,u^m)$ and their partial derivatives of order up to k w.r.t the independent variables, and $\mathcal{R} = (\mathcal{R}_1,\dots, \mathcal{R}_l)$ is some mapping

$$\mathcal{R}: M^{(k)} \longrightarrow \mathbb{R}^l, (x, u^{(k)}) \longmapsto \mathcal{R}(x, u^{(k)}), \quad (22)$$

where $M^{(k)}$ is the k-jet space of a space $M \subset X \times U$ of the variables (x,u).

Let us write the Lie group of transformations associated to this system in the form

$$\tilde{x} = \phi(x, u, \epsilon), \ \tilde{u} = \gamma(x, u, \epsilon) \quad (23)$$

and suppose the general form of infinitesimal generator V of the group (19) be

$$V = \textstyle\sum_{i=1}^{n} \xi^i(x, u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{m} \eta_\alpha(x, u)\frac{\partial}{\partial u}. \quad (24)$$

The function $u = f(x)$, with components $u^\alpha = f^\alpha(x)$ ($\alpha = 1,\dots,m$), is said to be an *invariant solution* (see [12]) of (21) if $u^\alpha = f^\alpha(x)$ is an invariant surface of (19), and is a solution of (26), i.e., a solution is invariant if and only if:

$$\mathrm{V}\big(u^\alpha - f^\alpha(x)\big) = 0, \text{ for } \mathrm{u}^\alpha = f^\alpha(x), (\alpha = 1, \dots, m) \quad (25)$$

$$\mathrm{R}\big(\mathrm{x}, u^{(k)}\big) = 0 \quad (26)$$

The equations (21), called invariant surface conditions, have the form

$$\xi^1(x,u)\frac{\partial u^\alpha}{\partial x_1}+\cdots\xi^n(x,u)\frac{\partial u^\alpha}{\partial x_n}=\eta_\alpha(x,u), \alpha=1,\dots,m \quad (27)$$

and are solved by introducing the corresponding characteristic equations:

$$\frac{dx_1}{\xi^1(x,u)}=\cdots=\frac{dx_n}{\xi^n(x,u)}=\frac{du^1}{\eta_1(x,u)}=\cdots=\frac{du^m}{\eta_m(x,u)}. \quad (28)$$

This allows to express the solution $u = f(x)$ (that may be given in implicit form if some of the infinitesimals ξ^i depend on u) as

$$u^\alpha = \Psi^\alpha(J_1(x,u),\dots,J_{n-1}(x,u)),\ \alpha = 1,\dots,m. \quad (29)$$

By substituting (29) into (26), a reduced system of differential equations involving $n-1$ independent variables (called *similarity variables*) is obtained (See [8] and [12]) for more details.

4. Invariant Solutions of Equation (1)

The potential KdV equation is invariant under the five point symmetries presented in *Theorem 2.1.* For each one-parameter subgroup of the full symmetry group there will be a corresponding class of group-invariant solutions which will be determined from a reduced ordinary differential equation.

1) Reduction under X_1:

The vector $X_1 = \frac{\partial}{\partial x}$ generates the space translation group, with infinitesimals $\xi = 1, \tau = 0, \eta = o$. To obtain invariant solutions of equation (1) from this group, we have to solve the system of characteristic equations (28), which writes in the form

$$\frac{dt}{0}=\frac{dx}{1}=\frac{du}{0}. \quad (30)$$

The integration of (31) leads to the similarity variables

$$r = x,\ s = t,\ y(s) = u. \quad (31)$$

Taking $u = y(s)$, the equation (1) reduced to the following first ordinary differential equation

$$y' = 0, \quad (32)$$

which leads to the solution

$$y(s) = k, \quad (33)$$

for k arbitrary constant. In terms of original variables, we find that

$$u(x,t) = k, \quad (34)$$

with k arbitrary constant. Thus, the global invariant solution of the potential KdV equation under X_1 is an constant.

2) Reduction under X_2.

Here, equation (1) is invariant under the time translation group with infinitesimals $\xi = 0,\ \tau = 1,\ \eta = 0$. Corresponding to the system (28), we have to solve the equations

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du}{0}, \quad (35)$$

which leads to the following global invariants

$$v = t,\ w = x,\ h(w) = u \quad (36)$$

substituting the function $u = h(w)$ in (1), the potential KdV equation reduces to the following nonlinear third ordinary differential equation

$$h'^2 + h''' = 0 \quad (37)$$

This reduced equation is invariant under the group generated by the vector

$$V = \frac{\partial}{\partial w} - h\frac{\partial}{\partial h}, \quad (38)$$

with infinitesimals $\xi = 1,\ \eta = h$. Corresponding to these infinitesimals, the system (28) reads

$$\frac{dw}{1} = -\frac{dh}{h}. \quad (39)$$

Global invariants in terms of w and h are

$$z = \ln|w|,\ \varphi = wh, \text{ that is } w = e^{-z},\ h = \frac{-\varphi}{e^z}, \quad \text{for} \quad w \neq 0. \quad (40)$$

Substituting these expressions in to (37) we find the algebraic equation for the potential KdV equation to be

$$e^{-4z}\varphi(z)(\varphi(z) - 6) = 0 \quad (41)$$

Substituting back according to (36), we find that the symmetry X_2 yields to the *travelling wave solutions* for the equation (1) that are trivial solution $u(x,t) = 0$ for all $(x,t) \in R^2$ and depend only on x, $u(x,t) = \frac{6}{x}$ for all $(x,t) \in R^2$, with $x \neq 0$.

3) Reduction under X_4.

The infinitesimals for X_4 are $\xi = t,\ \tau = 0,\ \eta = \frac{x}{2a}$. Integrating the characteristic equations

$$\frac{dx}{1}=\frac{dt}{0}=\frac{du}{\frac{x}{2a}}, \quad (42)$$

obtains the similarity variables as follows

$$\alpha = \frac{x}{t},\quad \beta = t, \mu = \frac{x^2}{4at},\ that\ is\ x = \alpha\beta,\ and\ u = \frac{\alpha^2\beta}{4a} + \mu(\beta). \quad (43)$$

The reduced equation is simply

$$\mu' = 0, \quad (44)$$

which leads to the solution $\mu(\beta) = C$, for C arbitrary constant. In terms of the original variables, the general *Galilean-invariant solutions* to (1) is provided to be

$$u(x,t) = \frac{1}{4a}\frac{x^2}{t} + C, \; for \; a \neq 0 \; and \; t \neq 0 \qquad (45)$$

4) Reduction under X_5

Let us now deal with the symmetry generator X_5 which has infinitesimals of the form $\xi = \frac{x}{3}, \tau = t, \eta = \frac{-u}{3}$. The caracteristic equations write

$$\frac{dx}{\frac{x}{3}} = \frac{dt}{t} = \frac{du}{-\frac{u}{3}}, \qquad (46)$$

and the corresponding scale-invariant are

$$\begin{cases} v = 3\ln|x|, \gamma = \frac{t}{x^3}, q(\gamma) = xu, \\ that \; is \\ t = e^v, x = e^{\frac{v}{3}}, u = \frac{q(\gamma)}{e^{\frac{v}{3}}}. \end{cases} \qquad (47)$$

The reduced equation is the following nonlinear third ordinary differential equation

$$q''' = -\frac{5}{x}q'' + \frac{a}{3x}q'^2 + \left(\frac{2aq}{9x^2} - \frac{144}{27x^2} + \frac{1}{27x^3}\right)q' + \frac{a}{27x^3}q^2 - \frac{2}{9x^3}q \qquad (48)$$

Where a is a real-valued constant. Note that Eq. (48) has infinitesimals that are all zero

$$\xi(x,q) = 0, \eta(x,q) = 0 \qquad (49)$$

showing that this ordinary differential equation has no point symmetries and hence, no group-invariant solutions. The corresponding solutions of the p-KdV equation take the general form

$$u = \frac{h(\frac{t}{x^3})}{x}, \text{ for } \quad x \neq 0, \qquad (50)$$

5. Conclusion

We see that differential forms offer, in some ways, a more natural way of calculating symmetries of differential equations. With this technique we need only calculate prolongation coefficients up to *second* order and hence, obtain the prolongation of the generator *X*. This leads naturally to the point symmetry algebra in investigation. Using functionally independent invariants from the symmetries that we have found, the reduced equations have been constructed for each of generators of these symmetries. Exact solutions have been obtained for the p-KdV equation that are travelling-invariants and Galilean-invariants. Reduction under the generator of the scale-symmetry has not been completed because the infinitesimals of the corresponding reduced third ordinary differential equation are all zero.

Acknowledgements

The author thanks the Deutscher Ackademischer Austauschdienst (DAAD), which supported this work.

References

[1] Abdul-Majid Wazwaz, *Partial Differential Equations and Solitary Waves Theory*, Nonlinear Physical Sciences, Springer, USA, 2009.

[2] B. Kent Harrison, F. B. Estabrook, *Geometric Approach to Invariance Groups and Solution of Partial Differential Systems*, J. Math. Phys., 1971, V. 12, 653-666.

[3] B. Kent Harrison, *The differential Form Method for Finding Symmetries*, SIGMA, Vol. 1 (2005), Paper 001, 12 pages.

[4] B. Kent Harrison, *Differential Form Symmetry Analysis of Two Equations Cited by Fushchych*, Department of Physics and Astronomy, Brigham Young University, Provo, Utah 84602 U.S.A,, V. 1, 21-33 (1997).

[5] Biswas, A., Kumar, S., Krishnan, E. V., Ahmed, B., Strong, A., Johnson, S., Yildirim, A., *Topological solitons and other solutions to potential KdV equation.* Rom. Rep. Phys. 65 (4), 1125-1137 (2013).

[6] C. A. G Ìomez Sierra, M. Molati, M. P. Ramollo, *Exact Solutions of a Generalized KdV-mKdV Equation*, International Journal of Nonlinear Science, Vol. 13 (2012) No. 1, pp. 94-98.

[7] El-Wakil, S. A., Abdou, M. A., Elhanbaly, A., *solitons and periodic wave solutions for nonlinear evolution equations*. Phys. Lett. A 353 (1), 40â47 (2006).

[8] G. W. Bluman, and S. Kumei, *Symmetries and Differential Equations*, Springer Verlage, New York, 1989.

[9] Guo, Y., Wang, Y., *Weierstrass elliptic function solu- tions for a (N+1) dimensional potential KdV equation*. Appl. Math. Comput. 217 (20), 8080â8092 (2011).

[10] Houria Triki, Turgut Ak, Mehmet Ekici, Abdullah Sonmezoglu, Mohammad Mirzazadeh, Abdul Hamid Kara, Tugba Aydemir, *Some new exact wave solutions and conservation laws of potential Kortewegâde Vries equation* Nonlinear dynamic DOI 10.1007/s11071-017-3467-4, 2017.

[11] Mehdi Nadjafikhah and Seyed-Reza Hejazi, *Lie Symmetries and Solutions of KdV Equation*, Departmant of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran, International Mathematial Forum, 4 (4), 165-176 (2009).

[12] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, GTM, Vol. 107, Springer Verlage, New York, 1993.

[13] Wang, G.-W., Xu, T.-Z., Ebadi, G., Johnson, S., Strong, A. J., Biswas, A., *solitons, shock waves, and other solutions to potential KdV equation*. Nonlinear Dyn. 76 (2), 1059â1068 (2014).

[14] Wazwaz, A.-M., *New solitary wave solutions to the modified Kawahara equation*. Phys. Lett. A 360 (4â5), 588â592 (2007).