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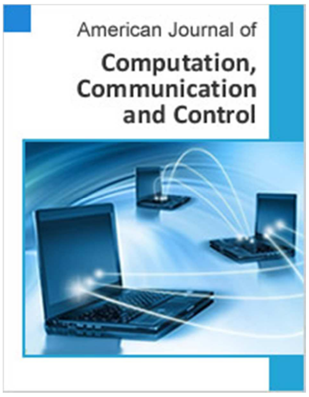

Record Values from Size-Biased Half Normal Distribution: Properties and Recurrence Relations for the Single and Product Moments

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Abstract

In this paper the size-biased form of half normal distribution is considered for upper record values that can be used only for non-negative values. Some properties of the UR-SHND as cdf, mean, variance, skewness, kurtosis, mode, Shannon entropy, survival function, hazard function have been derived. The joint pdf of the UR-SHND is developed and covariance of the joint upper record values is derived. It is concluded that there is positive correlation between upper record values from SHND. Some recurrence relations for the single and product moments are developed. These relations can be used to derive the moments of the UR-SHND in a simple recursive manner.

Keywords

pdf,
cdf,
SHND,
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SHANNON Entropy,
Recurrence Relations,
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1. Introduction

Record values are the largest or smallest values obtained from a sequence of random variables. The term was first introduced by K. N. Chandler in [12]. Chandler [12] give mathematical term to record times and records statistics. This theory is closely related to order statistics especially extreme order statistics. Everybody knows that, records are associated with record breaking events. Common examples are: extreme weather events such as the occurrence of lowest or highest temperatures; inimitable performances in sports; financial crisis like the major stock market break downs; records of global warming and climate change; occurrence of cyclones and floods; even as the record breaking events continue to enjoy media attention. There is also an increased research interest in the statistical study of record events, lower and upper record values have been derived from various continuous probability distributions. The name Ahsanullah is considered milestone in the record theory. Ahsanullah [2], [3], [4]; Balakrishan and Ahsanullah [8], Ahsanullah and Kirmani [1]; Kirmani and Beg [17], discussed record values from various distributions.

Nagaraja [20] record values and extreme value distributions, Basak [9] lower record values from exponential distribution, Awad and Raqab [7] prediction intervals for the future record values from exponential distribution: comparative study, Alzaid and Ahsanullah [5] discussed a characterization of the Gumbel distribution based on record values, Sultan [25] record values from modified Weibull distribution, Shahbaz et al [24] distribution of bivariate concomitants of records, Seo and Kim [23] statistical inference on Weibull distribution using record values, Kumar et al [18] introduced inference of exponentiated moment exponential distribution based on lower record values. The probability density function of the upper record values

$$f_n(x) = \frac{[R(x)]^{n-1}}{\Gamma(n)} f(x), -\infty < x < \infty.) \tag{1}$$

where $H(x) = -lnF(x), 0 < F(x) < 1.$

The joint pdf of nth and mth lower record values is

$$f_{n,m}(x,y) = \frac{[R(x)]^{m-1} r(x) [R(y) - R(x)]^{n-m-1}}{\Gamma(m)\Gamma(n-m)} f(y),, n > m, -\infty < x < \infty.) \tag{2}$$

where

$$r(x) = \frac{f(x)}{[1 - F(x)]}$$

The half-normal distribution is one of the widely used probability distribution for non-negative data modeling, specifically, to describe the lifetime process under fatigue. The half-normal distribution has been used to model data in diverse fields of applications as, Castro et al [11] introduced epsilon half-normal model and developed properties and inference, Cooray and Ananda [13], Nogales & Pérez [21] unbiased estimation for the general half-normal distribution, Arthur Pewsey [6] large sample inference for the general half normal distribution, Wiper et al [26] discussed Bayesian inference for the half-normal and half-t distributions, Lu et al [19] studied acceptance sampling plans for half-normal distribution when the lifetime experiment is truncated at a preassigned time, Khan and Islam [15] considered the problem of strength of a manufactured item against an array of stresses following half-normal distribution, Khan and Islam [16] evaluated the maintenance performance of the system when time is continuous and consider half-normal failure lifetime model as well as repair time model, a detail can be found in Byers [10].

The pdf of the half normal distribution

$$f(x) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-x^2/2\sigma^2}, x > 0 \tag{3}$$

Where $\sigma > 0$ is scale parameter.

The cdf of the half normal distribution is

$$F(x) = erf\left(\frac{x}{\sigma\sqrt{\pi}}\right) \tag{4}$$

Where erf is the error function. The mean of the half normal distribution

$$Mean = E(X) = \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \tag{5}$$

The concept of weighted distributions can be traced to the study of Fisher [14]. In extending the basic ideas of Fisher, Patil and Rao [22] saw the need for a unifying concept and identified various sampling situations that can be modeled by what he called weighted distributions. When an investigator records an observation by nature according to a certain stochastic model, the recorded observation will not have the original distribution unless every observation is given an equal chance of being recorded. For example, suppose that the original observation X has $f(x)$ as the pdf that the probability of recording the observation x is $0 < w(x) < 1$, then the pdf of X^w, the recorded observation is

$$f_w(x) = \frac{w(x)f(x)}{E[w(x)]} \tag{6}$$

It can be size biased for $w(x) = x$ as

$$f_w(x) = \frac{xf(x)}{E[X]} \tag{7}$$

By using eq. (3) and (5) in eq. (7) the pdf of the size-biased half normal distribution (SHND)

$$f(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, \quad x > 0 \tag{8}$$

The cdf of the SHND

$$F(x) = 1 - e^{-x^2/2\sigma^2} \tag{9}$$

2. Upper Record Values from the Size-Biased Normal Distribution

Using eq. (8), (9) and (1), the probability density function (pdf) of the lower record values from the size-biased half normal distribution (UR-SHND)

$$f_n(x) = \frac{x^{2n-1} e^{-x^2/2\sigma^2}}{2^{n-1}\sigma^{2n}\,\Gamma n}, \sigma > 0, n > 1, x > 0 \tag{10}$$

For n = 1 the pdf in eq. (10) is parent distribution, named size-biased normal distribution in eq. (8).

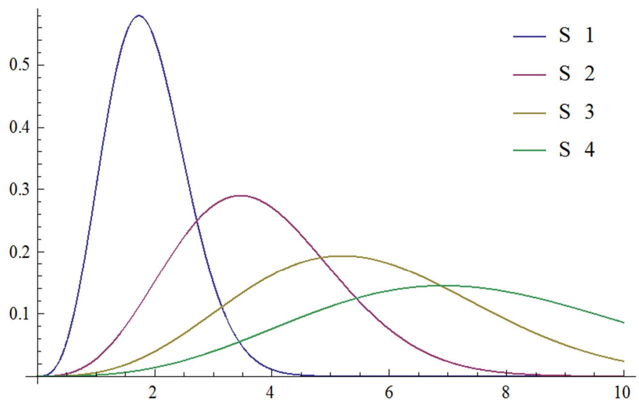

Figure 1. *Pdf graphs of UR-SHND for n = 2 and different values of σ i.e. (S).*

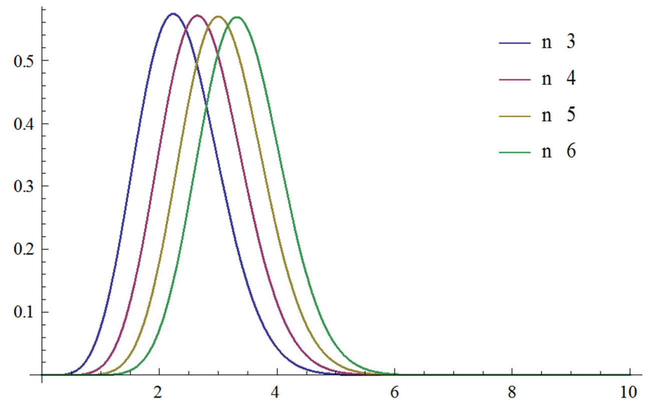

Figure 2. *Pdf graphs of UR-SND for $\sigma = 1$ and different values of n.*

The cumulative distribution function (cdf) of the UR-SHND

$$F_n(x) = \frac{1}{\Gamma(n)}\gamma\left(n, \frac{x^2}{2\sigma^2}\right) \quad (11)$$

where $\gamma\left(n, \frac{x^2}{2\sigma^2}\right)$ is lower incomplete gamma function.

The rth moments of the UR-SHND

$$\mu'_{r(n)} = \frac{\sigma^r 2^{r/2}\Gamma\left(\frac{r}{2}+n\right)}{\Gamma(n)} \quad (12)$$

For r = 1, 2, 3, 4 the first four moments of the UR-SHND

$$\mu'_{1(n)} = \frac{\sigma\sqrt{2}\Gamma(n+0.5)}{\Gamma(n)}, \; \mu'_{2(n)} = 2n\sigma^2, \quad (13)$$

$$\mu'_{3(n)} = \frac{\sigma^3 2^{3/2}\Gamma(1.5+n)}{\Gamma(n)}, \quad \mu'_{4(n)} = 4n(n+1)\sigma^4. \quad (14)$$

Mean and variance of the UR-SHND

$$\text{Mean} = \mu'_{1(n)} = \frac{\sigma\sqrt{2}\Gamma(n+0.5)}{\Gamma(n)}, \quad (15)$$

$$\text{Variance} = \mu_{2(n)} = 2\sigma^2\left[n - \left(\frac{\Gamma(n+0.5)}{\Gamma(n)}\right)^2\right] \quad (16)$$

Coefficient of skewness and kurtosis of the UR-SHND

$$\beta_1 = \frac{\left[(\Gamma n)^2\Gamma(1.5+n) - 3n(\Gamma n)^2\Gamma(0.5+n) - 2\left(\Gamma(0.5+n)\right)^3\right]^2}{\left[n(\Gamma n)^2 - \left(\Gamma(0.5+n)\right)^2\right]^3} \quad (17)$$

$$\beta_2 = \frac{\left[n(n+1)(\Gamma n)^4 - 4(\Gamma n)^2\Gamma(1.5+n)\Gamma(0.5+n) - 6n(\Gamma n)^2\left(\Gamma(0.5+n)\right)^2 - 3\left(\Gamma(0.5+n)\right)\right]^4}{\left[n(\Gamma n)^2 - \left(\Gamma(0.5+n)\right)^2\right]^2} \quad (18)$$

From Eq. (17) and (18) it can be seen that coefficient of skewness and kurtosis are independent from the scale parameter σ. The mode of UR-SHND is exist for $n = 2, i.e.$

$$mode = \sqrt{\frac{2n-1}{\sigma^2}} \quad (19)$$

The Shannon Entropy of the UR-SHND is

$$H(x) = \frac{1}{\Gamma(n)}\left[\Gamma(n)\left\{n + ln(\Gamma n) + 2nln(\sigma) + (n-1)ln(2) - (2n-1)\sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\Gamma\left(n+\frac{i-1}{2}\right)(2\sigma^2)^{(i-1)/2}\right\}\right] \quad (20)$$

The survival and hazard functions of the UR-SHND

$$S_n(x) = \frac{\Gamma\left(n, \frac{x^2}{2\sigma^2}\right)}{\Gamma(n)} \quad (21)$$

Where $\Gamma(a,x) = \int_x^{\infty} t^{a-1}e^{-t}$ is the upper incomplete gamma function.

$$h_n(x) = \frac{x^{2n-1}e^{-x^2/2\sigma^2}}{2^{n-1}\sigma^{2n}\Gamma\left(n, \frac{x^2}{2\sigma^2}\right)} \quad (22)$$

Some additional properties of the UR-SHND

$$\frac{f(x+1,\sigma^2)}{f(x)} = \left(1+\frac{1}{x}\right)^{2n-1} e^{-\frac{(1+x)}{2\sigma^2}} \quad (23)$$

Equation (23) is a decreasing function in $\boldsymbol{x}$, therefore Ur-SHND is unimodal has an increasing failure/hazard rate.

$$\mu - \mu_2 = \frac{\sigma\sqrt{2}\Gamma(n)\Gamma(n+0.5) - 2\sigma^2\left[n\Gamma(n) - \left(\Gamma(n+0.5)\right)^2\right]}{(\Gamma n)^2} \quad (24)$$

It follows that $\mu < (=)(>)\mu_2$ for different values of $\sigma^2 and\, n$, That is, the UR-SHND is over dispersed (equi-dispersed) (under-dispersed).

Table 1. *Dispersion of the values from mean.*

n	μ / σ^2			
	1	**2**	**3**	**4**
2	$1.41\mu_2$	$1.72\mu_2$	$1.85\mu_2$	$1.90\mu_2$
3	$9.74\mu_2$	$16.73\mu_2$	$23.27\mu_2$	$29.57\mu_2$
4	$53.55\mu_2$	$97.46\mu_2$	$139.78\mu_2$	$181.29\mu_2$
5	$292.34\mu_2$	$541.31\mu_2$	$783.16\mu_2$	$1021.30\mu_2$

From table 1 it can be seen that $\mu > \mu_2$ for different values of the $\sigma^2 and\, n$. It means that the UR-SHND is over dispersed and as the values of $\sigma^2 and\, n$ increase the amount of dispersion is decreasing.

3. Joint Upper Record Values from the Size-Biased Half Normal Distribution

Sing eq. (8) and (9) in eq. (2), the joint pdf of the UR-SHND

$$f_{n,m}(x,y)=\frac{1}{\Gamma(m)\Gamma(n-m)2^{n-2}\sigma^{2n}}x^{2m-1}y\left(y^2-x^2\right)^{n-m-1}e^{-y^2/2\sigma^2},n>m,\quad 0<x<y<\infty \tag{25}$$

To find covariance between upper record values the joint expectation of (25)

$$E\left[X_{U(n)}X_{U(m)}\right]=\int_0^\infty\int_0^\infty xyf_{n,m}(x,y)\,dxdy$$

$$E\left[X_{U(n)}X_{U(m)}\right]=\frac{2\sigma^2 n\Gamma(n)\Gamma(m+0.5)}{\Gamma(m)\Gamma(n+0.5)} \tag{26}$$

And

$$E\left(X_{U(m)}\right)=\frac{\sigma\sqrt{2}\Gamma(m+0.5)}{\Gamma(m)},\ E\left(X_{U(n)}\right)=\frac{\sigma\sqrt{2}\Gamma(n+0.5)}{\Gamma(n)} \tag{27}$$

$$Cov\left(X_{U(n)}X_{U(m)}\right)=\frac{2\sigma^2\Gamma(m+0.5)}{\Gamma(m)}\left[\frac{n\Gamma(n)}{\Gamma(n+0.5)}-\frac{\Gamma(n+0.5)}{\Gamma(n)}\right] \tag{28}$$

Table 2. *Covariance's between upper record values.*

n \ m	Covariance 2	3	5	9
3	$0.382\sigma^2$	-----------	---------	------------
4	$0.332\sigma^2$	$0.414\sigma^2$	---------	------------
10	$0.210\sigma^2$	$0.263\sigma^2$	$0.345\sigma^2$	$0.468\sigma^2$

From table 2 it can be seen that there is positive/direct correlation between the upper record values and correlation between consecutive values $\{(n,m)$: (3,2), (4,3), (10,9)$\}$ is high comparatively.

4. Recurrence Relations for Moments

In this section some recurrence relations satisfied by the single and product moments of the upper record values from the SHND are given. These relations can be used to obtain all single and product moments of record values form the SND in simple recursive manner. It is found that the size-biased normal distribution having the following relationship between pdf and cdf,

$$f(x)=\frac{x}{\sigma^2}\left[1-F(x)\right] \tag{29}$$

Theorem 4.1: for $n>1\ and\ r=0,1,2,3,\ldots..$

$$E\left(X_{U(n-1)}\right)^{r+2}=E\left(X_{U(n)}\right)^{r+2}-\sigma^2(r+2)E\left(X_{U(n)}\right)^r \tag{30}$$

Proof: Let $X_{U(n)}$ from UR-SHND in eq. (10) and, for $n>1\ and\ r=0,1,2,3,\ldots..$

$$E\left(X_{U(n)}\right)^r=\frac{1}{\Gamma(n)}\int_0^\infty x^r\left[-\ln(1-F(x))\right]^{n-1}f(x)dx$$

$$E\left(X_{U(n)}\right)^r=\frac{1}{\sigma^2(r+2)(n-1)!}\int_0^\infty x^{r+2}\left[-\ln(1-F(x))\right]^{n-1}f(x)dx- \tag{31}$$

$$\frac{1}{\sigma^2(r+2)(n-2)!}\int_0^\infty x^{r+2}\left[-\ln(1-F(x))\right]^{n-2}f(x)dx$$

The relationship in eq. (30) is obtained by simplifying and rewriting eq. (31).

Theorem 4.2: For $i<m<n-2\ and\ r,s=0,1,2,3\ \ldots\ldots$

$$E\left[X_{U(m)}^{r}X_{U(n-1)}^{s+1}\right]=E\left[X_{U(m)}^{r}X_{U(n)}^{s+2}\right]-\sigma^{2}(s+2)E\left[X_{U(m)}^{r}X_{U(n)}^{s}\right] \tag{32}$$

Proof: Let $X_{U(n)}$ and $X_{U(n)}$ from UR-SHND in eq. (25) and, for $i<m<n-2$ and $r,s=0,1,2,3\ldots\ldots$

$$E\left(X_{U(m)}^{r}X_{U(n)}^{s}\right)=E\left(x^{r}y^{s}\right)$$

$$E\left[X_{U(n)}^{r}X_{U(m)}^{s}\right]=\frac{1}{\Gamma(m)\Gamma(n-m)}\int_{0}^{\infty}x^{r}\left[-\ln\left(1-F(x)\right)\right]^{m-1}\frac{f(x)}{1-F(x)}I(x)\,dx \tag{33}$$

Where $I(x)=\int_{0}^{\infty}y^{s}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-1}f(y)\,dy$

$$I(x)=\frac{1}{\sigma^{2}}\int_{0}^{\infty}y^{s+1}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-1}\left[1-F(y)\right]dy$$

$$\begin{aligned}I(x)=&\frac{1}{\sigma^{2}(s+2)}\int_{0}^{\infty}y^{s+2}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-1}f(y)\,dy-\\&\frac{(n-m-1)}{\sigma^{2}(s+2)}\int_{0}^{\infty}y^{s+2}\left[-\ln\left(1-F(y)\right)+\ln\left(1-F(x)\right)\right]^{n-m-2}f(y)\,dy\end{aligned} \tag{34}$$

Substituting eq. (34) in the eq. (33), simplifying and rewriting it, the relationship in eq. (32) is obtained.

5. Conclusion

Record values applied in many areas of real life situations and record values on various continuous distribution have been discusses. In this paper the size-biased form of half normal distribution is considered for upper record values that can be used only for non-negative values. Some statistical properties of the UR-SHND including pdf graphs, cdf, mean, variance, skewness, kurtosis, mode, Shannon entropy, survival function, hazard function have been developed. The joint pdf of the UR-SHND is introduced and covariance is derived. It is concluded that there is positive correlation between upper record values from SHND. Some recurrence relations for the single and product moments are developed. These relations can be used to derive the moments of the UR-SHND in a simple recursive manner.

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