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Generalized f_G -Mean: Derivation of Various Formulations of Average

Dhritikesh Chakrabarty

Department of Statistics, Handique Girls' College, Gauhati University, Guwahati, India

Email address
dhritikesh.c@rediffmail.com, dhritikeshchakrabarty@gmail.com

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Abstract: Average is a basic concept and averaging is a basic technique/tool behind the most of the measures associated to characteristics of data. A number of definitions/formulations of average have been developed so far. These definitions/formulations have been found suitable to be used in different situations. There are still many situations where there are scarcities of definitions/formulations of average to deal with the respective situations. It has been found possible to compose general definition of average so that the definitions/formulations of average, already developed, can be obtained/derived from it as well as more new definitions/formulations of average can be obtained/derived from it. One general definition of average, termed in this paper as Generalized f_G -Mean, has been introduced with a view to obtain different definitions/formulations of average. Applying the technique, some definitions/formulations have been derived for various types of averages. This paper describes this general definition and the derivation of a number of definitions/formulations of average.

Keywords: Average, Generalized f_G -Mean, Existing Means, Derivations

1. Introduction

Average [1, 2] is basic concept and averaging is a basic technique/tool behind the most of the measures associated to characteristics of data. A number of definitions/formulations of averages had primarily been developed by Pythagoras [3, 4, 5] who constructed the definitions / formulations of the three most common averages namely Arithmetic Mean, Geometric Mean & Harmonic Mean which are known as Pythagorean means [6, 7].

Let

$$x_1,\ x_2, \ldots\ldots\ldots\ldots\ldots,\ x_n$$

be n numbers of a list and/or the values assumed by a variable x.

Some of the definitions /formulations of averages, already constructed [6, 7, 8, 9, 10, 11], are as follows:

$$\text{Arithmetic Mean} = \frac{1}{n}(x_1 + x_2 + \ldots\ldots\ldots\ldots\ldots + x_n) \quad (1)$$

$$\text{Geometric Mean} = (x_1 x_2 x_3 \ldots\ldots\ldots x_n)^{1/n} \quad (2)$$

provided the n numbers are non-negative.

$$\text{Harmonic Mean} = \{\frac{1}{n}(x_1^{-1} + x_2^{-1} + \ldots\ldots + x_n^{-1})\}^{-1} \quad (3)$$

provided the n numbers are all different from 0.

$$\text{Quadratic Mean} = \{\frac{1}{n}(x_1^{2} + x_2^{2} + \ldots\ldots\ldots\ldots\ldots + x_n^{2})\}^{1/2} \quad (4)$$

$$\text{Square Root Mean} = \{\frac{1}{n}(x_1^{1/2} + x_2^{1/2} + \ldots\ldots\ldots\ldots\ldots + x_n^{1/2})\}^{2} \quad (5)$$

$$\text{Cubic Mean} = \{\frac{1}{n}(x_1^{3} + x_2^{3} + \ldots\ldots\ldots\ldots\ldots + x_n^{3})\}^{1/3} \quad (6)$$

$$\text{Cube Root Mean} = \{\frac{1}{n}(x_1^{1/3} + x_2^{1/3} + \ldots\ldots\ldots\ldots\ldots + x_n^{1/3})\}^{3} \quad (7)$$

$$\text{Generalized } p\text{-Mean} = \{\frac{1}{n}(x_1^{p} + x_2^{p} + \ldots\ldots\ldots\ldots\ldots + x_n^{p}\}^{1/p} \quad (8)$$

$$p^{\text{th}} \text{ Root Mean} = \{\frac{1}{n}({x_1}^{1/p} + {x_2}^{1/p} + \ldots\ldots\ldots\ldots. + {x_n}^{1/p})\}^p \quad (9)$$

$$e\text{-Mean} = log_e\{\frac{1}{n}(e^{x_1} + e^{x_2} + \ldots\ldots\ldots\ldots. + e^{x_n}\} \quad (10)$$

$$\text{Scale s-Mean} = \frac{1}{s}\{\frac{1}{n}(sx_1 + sx_2 + \ldots\ldots\ldots\ldots. + sx_n)\} \quad (11)$$

$$\text{Shift } a\text{-Mean} = \frac{1}{n}\{(x_1 - a) + (x_2 - a) + \ldots\ldots. + (x_n - a)\} + a \quad (12)$$

$$\text{shift } a\text{– Inverse Scale s -Mean} = s\{\frac{1}{n}(\frac{x_1 - a}{s} + \frac{x_2 - a}{s} + \ldots\ldots\ldots + \frac{x_n - a}{s})\} + a \quad (13)$$

These definitions/formulations have been found suitable to be used in different situations. There are still many situations where there are scarcities of definitions/formulations of average to deal with the respective situations. It has been found possible to compose general definition of average so that the definitions/formulations of average, already developed, can be obtained/derived from it as well as more new definitions/formulations of average can be obtained/derived from it. Kolmogorov constructed one definition/formulation of average known as the Generalized *f* -Mean [12-17].

The Generalized *f* - Mean of

$$x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_n,$$

is defined by

$$f^{-1}[\frac{1}{n}\{f(x_1) + f(x_2) + \ldots\ldots\ldots\ldots. + f(x_n)\}] \quad (14)$$

where *f* is an invertible function.

It has been shown that the Generalized *f* - Mean can be used to derive the definitions/formulations of the existing means and also of new means [9, 10, 11, 18].

However, the Generalized *f* -Mean due to Kolmogorov has also been found to be not suitable for some situations. Recently, one more generalized definition of average has been constructed which has been termed as Generalized f_H – Mean [18]. In this paper, attempt has been made on the construction of one more general definition of average in a similar manner as the construction of the Generalized *f* - Mean by Kolmogorov. The definition constructed here has been termed as Generalized f_G -Mean. It has been shown that the existing definitions/formulations of average can be obtained from the Generalized f_G -Mean. This paper describes this general definition of average and the derivation of the existing definitions/formulations of average.

2. Generalized f_G - Mean: One Technique of Definition of Average

In constructing the generalized *f*-mean (also known as Kolmogorov mean) of

$$x_1, x_2, \ldots\ldots\ldots\ldots, x_n$$

the arithmetic mean of the functional values

$$f(x_1), f(x_2), \ldots\ldots\ldots\ldots, f(x_n)$$

is taken first and then the inverse functional value of the arithmetic obtained is taken to obtain the generalized *f*-mean defined by

$$f^{-1}[\frac{1}{n}\{f(x_1) + f(x_2) + \ldots\ldots\ldots\ldots. + f(x_n)\}]$$

Here, *f* is an invertible function.

Let us now take the geometric mean of the functional values

$$f(x_1), f(x_2), \ldots\ldots\ldots\ldots, f(x_n)$$

and then the inverse functional value of the geometric mean obtained.

In doing this we will obtain the formulation

$$f^{-1}[\{f(x_1)f(x_2)\ldots\ldots.. f(x_n)\}^{1/n}$$

This can be regarded one generalized definition of average.

Let this average be termed as the Generalized f_G - Mean in this article.

Thus, the Generalized f_G - Mean of

$$x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_n$$

denoted by $f_G(x_1, x_2, \ldots\ldots.., x_n)$ can be defined as

$$f_G(x_1, x_2, \ldots\ldots.., x_n) = f^{-1}[\{f(x_1)f(x_2)\ldots\ldots.. f(x_n)\}^{1/n} \quad (15)$$

where *f* is any invertible function.

3. Properties of Generalized f_G - Mean

The Generalized f_G - Mean satisfies the following properties:

1. Partitioning: The computation of the Generalized f_G - Mean can be split into computations of equal *k* - sized sub-blocks i.e.

$$f_G(x_1, x_2, \ldots\ldots\ldots., x_{nk}) = f_G\{f_G(x_1, x_2, \ldots\ldots, x_k), f_G(x_{k+1}, x_{k+2}, \ldots\ldots, x_{2k}), \ldots\ldots\ldots, f_G(x_{(n-1)k+1}, x_{(n-1)k+2}, \ldots\ldots, x_{nk})\}$$

Proof: We have

$$f_G\,(x_1, x_2, \ldots\ldots, x_k) = f^{-1}[\{f(x_1)f(x_2)\ldots\ldots\ f(x_k)\}^{1/k},\ f_G\,(x_{k+1}, x_{k+2}, \ldots\ldots, x_{2k}) = f^{-1}[\{f(x_{k+1})f(x_{k+2})\ldots\ldots\ f(x_{2k})\}^{1/k},$$

$$f_G\,(x_{(n-1)k+1}, x_{(n-1)k+2}, \ldots\ldots, x_{nk}) = f^{-1}\left[f(x_{(n-1)k+1}).f(x_{(n-1)k+2})\ldots\ldots\ f(x_{nk})\right\}^{1/k}$$

Accordingly,

$$f_G\,\{f_G\,(x_1, x_2, \ldots\ldots, x_k), f_G\,(x_{k+1}, x_{k+2}, \ldots\ldots, x_{2k}), \ldots\ldots\ldots, f_G\,(x_{(n-1)k+1}, x_{(n-1)k+2}, \ldots\ldots, x_{nk})\}$$

$$= f^{-1}[\{f(x_1)f(x_2)\ldots\ldots\ f(x_{nk})\}^{1/nk}$$

$$= f_G\,(x_1, x_2, \ldots\ldots\ldots, x_{nk})$$

2. Subsets of elements can be averaged a priori, without altering the mean, given that the multiplicity of elements is maintained.

Thus, with

$$m = f_G\,(x_1, x_2, \ldots\ldots, x_k)$$

it holds that

$$f_G\,(x_1, x_2, \ldots\ldots, x_k,\ x_{k+1}, x_{k+2}, \ldots\ldots, x_n) = f_G\,(m, m, \ldots, m,\ x_{k+1}, x_{k+2}, \ldots\ldots, x_n)$$

Proof: We have

$$m = f_G\,(x_1, x_2, \ldots\ldots, x_k) = \{f(x_1)f(x_2)\ldots\ldots\ f(x_k)\}^{1/k}$$

Therefore,

$$f(m) = \mathrm{f}\, f^{-1}[\{f(x_1)f(x_2)\ldots\ldots\ f(x_k)\}^{1/k}]$$

$$= \{f(x_1)f(x_2)\ldots\ldots\ f(x_k)\}^{1/k}$$

Hence,

$$f_G\,(m, m, \ldots, m,\ x_{k+1}, x_{k+2}, \ldots\ldots, x_n)$$

$$= f^{-1}[\{\,f(m)\,f(m)\ldots\ldots f(m)f(x_{k+1})f(x_{k+2})\ldots\ldots\ f(x_n)\}^{1/n}]$$

$$= f^{-1}[\{f(x_1)f(x_2)\ldots\ldots\ f(x_k)f(x_{k+1})f(x_{k+2})\ldots\ldots\ f(x_n)\}^{1/n}]$$

$$= f_G\,(x_1, x_2, \ldots\ldots, x_k,\ x_{k+1}, x_{k+2}, \ldots\ldots, x_n)$$

3. The Generalized f_G - Mean is invariant with respect to scaling of f i.e. for all real b,

$$\phi(t) = b\,f(t) \text{ implies } \phi_G\,(x) = f_G\,(x)$$

Proof: We have

$$\phi_G\,(x_1, x_2, \ldots\ldots, x_n) = \phi^{-1}[\{\phi(x_1).\phi(x_2)\ldots\ldots\ \phi(x_n)\}^{1/n}]$$

$$= \phi^{-1}[\{\phi(x_1).\phi(x_2)\ldots\ldots\ \phi(x_n)\}^{1/n}]$$

Now,

$$\phi(t) = b\,f(t) \Rightarrow t = \phi^{-1}\{\,b\,f(t)\,\} \Rightarrow t = \phi^{-1}\{\,b\,f(t)\}$$

$$\Rightarrow \phi^{-1}(y) = f^{-1}(\tfrac{y}{b}), \text{ putting } y = b\,f(t) \Rightarrow t = f^{-1}(\tfrac{y}{b})$$

$$\Rightarrow \phi^{-1}: x \rightarrow f^{-1}(\tfrac{x}{b}) \text{ i.e. } p\,\phi^{-1} \text{ maps } x \text{ to } f^{-1}(\tfrac{x}{b})$$

Thus

$$\phi_G\,(x_1, x_2, \ldots\ldots, x_n) = \phi^{-1}[\{\phi(x_1).\phi(x_2)\ldots\ldots\ \phi(x_n)\}^{1/n}]$$

$$= \phi^{-1}[\{bf(x_1).bf(x_2)\ldots\ldots\ bf(x_n)\}^{1/n}]$$

$$= f^{-1}\left[\frac{\{bf(x_1).bf(x_2)..............bf(x_n)\}^{1/n}}{b}\right]$$

$$= f^{-1}[\{f(x_1)f(x_2)........ f(x_n)\}^{1/n}]$$

$$= f_G\,(x_1, x_2, \ldots\ldots, x_n)$$

4. If f is monotonic, then f_G is monotonic.
Proof: We have

$$f_G\,(x_1, x_2, \ldots\ldots, x_n) = f^{-1}[\{f(x_1)f(x_2)........ f(x_n)\}^{1/n}]$$

If f is monotonic then

$$\text{either } f(x_1) < f(x_2) < \ldots\ldots\ldots\ldots\ldots < f(x_n)$$

$$\text{or } f(x_1) > f(x_2) > \ldots\ldots\ldots\ldots\ldots > f(x_n)$$

This implies,

$$\text{either } f^{-1}[\{f(x_1)f(x_2)........ f(x_k)\}^{1/k} < f^{-1}[\{f(x_1)f(x_2)........ f(x_k)\}^{1/(k+1)}$$

$$\text{or } f^{-1}[\{f(x_1)f(x_2)........ f(x_k)\}^{1/k} > f^{-1}[\{f(x_1)f(x_2)........ f(x_k)\}^{1/(k+1)}$$

for all k,
which implies,

$$\text{either } f_G\,(x_1, x_2, \ldots\ldots, x_k) < f_G\,(x_1, x_2, \ldots\ldots, x_{k+1})$$

$$\text{or } f_G\,(x_1, x_2, \ldots\ldots, x_k) < f_G\,(x_1, x_2, \ldots\ldots, x_{k+1})$$

Hence, f_G is monotonic..
5. Generalized f_G - Mean of two variables has the mediality property namely

$$f_G\,[f_G\,\{(x, f_G\,(x, y)\}, f_G\,\{(y, f_G\,(x, y)\}] = f_G\,(x, y)$$

Proof: We have

$$f_G\,(x, y) = f^{-1}\{f(x).f(y)\}^{1/2}$$

Thus,

$$f_G\,\{(x, f_G\,(x, y)\} = f^{-1}[f(x).ff^{-1}\{f(x).f(y)\}^{\frac{1}{2}}]^{1/2}$$

$$= f^{-1}[f(x).\{f(x).f(y)\}^{\frac{1}{2}}]^{1/2}$$

Similarly,

$$f_G\,\{(y, f_G\,(x, y)\} = f^{-1}[f(y).\{f(x).f(y)\}^{\frac{1}{2}}]^{1/2}$$

Now,

$$ff_G\,\{(x, f_G\,(x, y)\} = ff^{-1}[\{f(x).\{f(x).f(y)\}^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$= [f(x).\{f(x).f(y)\}^{\frac{1}{2}}]^{1/2}$$

Similarly,

$$ff_G\,\{(y, f_G\,(x, y)\} = f\,f^{-1}[\{f(y).\{f(x).f(y)\}^{\frac{1}{2}}]^{1/2}$$

$$= [f(y).\{f(x).f(y)\}^{\frac{1}{2}}]^{1/2}$$

Therefore,

$$f_G\,[f_G\,\{(x, f_G\,(x, y)\}, f_G\,\{(y, f_G\,(x, y)\}]$$

$$= f^{-1}[\{f(x).f(y)\}^{\frac{1}{2}}]$$

$$= f_G(x, y)$$

6. Generalized f_G - Mean of two variables has the self-distributive property namely

$$f_G\{x, f_G(y, z)\} = f_G\{f_G(x, y), f_G(x, z)\}$$

Proof: We have

$$f_G(x, y) = f^{-1}\{f(x).f(y)\}^{1/2}$$

Accordingly,

$$f_G(x, z) = f^{-1}\{f(x).f(z)\}^{1/2} \& f_G(y, z) = f^{-1}\{f(y).f(z)\}^{1/2}$$

Now,

$$f_G\{f_G(x, y), f_G(x, z)\} = f^{-1}[ff^{-1}\{f(x)f(y)\}^{\frac{1}{2}}.ff^{-1}\{f(x)f(z)\}^{\frac{1}{2}}]^{1/2}$$

$$= f^{-1}[\{f(x)f(y)\}^{\frac{1}{2}}.\{f(x)f(z)\}^{\frac{1}{2}}]^{1/2}$$

$$= f^{-1}[f(x).\{f(y)\}^{1/2}.\{f(z)\}^{1/2}]^{1/2}$$

Thus,

$$f_G\{x, f_G(y, z)\} = f^{-1}\{f(x).ff_G(y, z)\}^{1/2}$$

$$= f^{-1}[f(x).ff^{-1}\{f(y)f(z)\}^{\frac{1}{2}}]^{1/2}$$

$$= f^{-1}[f(x).\{f(y)\}^{1/2}.\{f(z)\}^{1/2}]^{1/2}$$

$$= f_G\{f_G(x, y), f_G(x, z)\}$$

7. f_G - Mean of two variables x & y has the balancing property namely

$$f_G[f_G\{x, f_G(x, y)\}, f_G\{(y, f_G(x, y)\}] = f_G(x, y)$$

Proof: We have

$$f_G\{x, f_G(x, y)\} = f^{-1}[f(x).ff^{-1}\{f(x).f(y)\}^{1/2}]^{1/2}$$

$$= f^{-1}[f(x).\{f(x).f(y)\}^{1/2}]^{1/2}$$

$$\& f_G\{y, f_G(x, y)\} = f^{-1}[f(x).ff^{-1}\{f(x).f(y)\}^{1/2}]^{1/2}$$

$$= f^{-1}[f(y).\{f(x).f(y)\}^{1/2}]^{1/2}$$

Therefore,

$$f_G[f_G\{x, f_G(x, y)\}, f_G\{(y, f_G(x, y)\}]$$

$$= f^{-1}[ff_G\{x, f_G(x, y)\}.ff_G\{y, f_G(x, y)\}]^{1/2}$$

$$= f^{-1}[f^{-1}\{f(x).f(y)\}^{1/2}\}^{1/2}]$$

$$= f_G(x, y)$$

Remark:

This definition / formulation can be applied in searching for / constructing of a number of definitions / formulations for average.

The technique can be summarized in the following steps:

Select a suitable function which is invertible.

Then find out the inverse function of the function selected.

Then apply the function selected in the definition of Generalized f_G - Mean defined by (15).

4. Derivation of Various Average from Generalized f_G -Mean

Arithmetic Mean:

Let the invertible function $f(.)$ be such that

$$f: x \to e^{x}$$

i.e. f maps x to e^{x} i.e. $f(x) = e^{x}$

Here, $f^{-1}(e^{x}) = x$ i.e. $f^{-1}(y) = \log_e y$, putting $y = e^{x}$

This implies that $f^{-1}(.)$ is a function with

$$f^{-1}: x \to \log_e x$$

i.e. f^{-1} maps x to $\log_e x$ i.e. $f^{-1}(x) = \log_e x$

Therefore, if $f(x) = e^{x}$

then the Generalized f_G - Mean becomes

$$\frac{1}{n}(x_1 + x_2 + \ldots\ldots\ldots\ldots\ldots + x_n)$$

which is nothing but Pythagorean arithmetic mean defined by (1).

Geometric Mean:

Let the invertible function $f(.)$ be such that

$$f: x \to x$$

i.e. f maps x to x i.e. $f(x) = x$

Then $f^{-1}(x) = x$

i.e. f^{-1} also maps x to x

Therefore, if $f(x) = x$

then the Generalized f_G - Mean becomes

$$(x_1 x_2 x_3 \ldots\ldots x_n)^{1/n}$$

which is nothing but Pythagorean geometric mean defined by (2).

Harmonic Mean:

Let the invertible function $f(.)$ be such that

$$f: x \to e^{1/x}$$

i.e. f maps x to e^{x} i.e. $f(x) = e^{1/x}$

Here, $f^{-1}(e^{1/x}) = x$

i.e. $f^{-1}(y) = (\log_e y)^{-1}$, putting $y = e^{1/x}$

Thus, f^{-1} maps x to $(\log_e x)^{-1}$

Therefore, if $f(x) = e^{1/x}$

then the Generalized f_G - Mean becomes

$$\{\frac{1}{n}(x_1^{-1} + x_2^{-1} + \ldots\ldots + x_n^{-1})\}^{-1}$$

which is nothing but Pythagorean harmonic mean defined by (3).

Quadratic Mean:

Let $f(x) = exp(x^2)$

Then, $f^{-1}\{exp(x^2)\} = x$

i.e. $f^{-1}(y) = (\log_e y)^{1/2}$, putting $y = exp(x^2)$

This means, f^{-1} maps x to $(\log_e x)^{1/2}$

Therefore, if $f(x) = exp(x^2)$

then the Generalized f_G - Mean becomes

$$\{\frac{1}{n}(x_1^2 + x_2^2 + \ldots\ldots\ldots\ldots\ldots + x_n^2)\}^{1/2}$$

which is nothing but the quadratic mean defined by (4).

Square Root Mean:

Let $f(x) = exp(x^{1/2})$

Then $f^{-1}\{exp(x^{1/2})\} = x$

i.e. $f^{-1}(y) = (\log_e y)^2$, putting $y = exp(x^{1/2})$

This means, f^{-1} maps x to $(\log_e x)^2$

Therefore, if $f(x) = exp(x^{1/2})$

then the Generalized f_G - Mean becomes

$$\{\frac{1}{n}(x_1^{1/2} + x_2^{1/2} + \ldots\ldots\ldots\ldots\ldots + x_n^{1/2})\}^2$$

which is nothing but the square root mean defined by (5).

Cubic Mean:

Let $f(x) = exp(x^3)$

Then, $f^{-1}\{exp(x^3)\} = x$

i.e. $f^{-1}(y) = (\log_e y)^{1/3}$, putting $y = exp(x^3)$

This implies, f^{-1} maps x to $(\log_e x)^{1/3}$

Therefore, if $f(x) = exp(x^3)$

then the Generalized f_G - Mean becomes

$$\{\frac{1}{n}(x_1^3 + x_2^3 + \ldots\ldots\ldots\ldots\ldots + x_n^3)\}^{\frac{1}{3}}$$

which is nothing but the cubic mean defined by (6).

Cube Root Mean:

Let the invertible function $f(.)$ be such that

$$f: x \to exp(x^{1/3})$$

i.e. $f(x) = exp(x^{1/3})$

Then $f^{-1}\{exp(x^{1/3})\} = x$

i.e. $f^{-1}(y) = (\log_e y)^3$, putting $y = exp(x^{1/3})$

Thus $f^{-1}(.)$ is a function with

$$f^{-1}: x \to (\log_e x)^3$$

Thus if in the Generalized f_G -Mean, the function $f(.)$ is selected as

$$f(x) = exp(x^{1/3})$$

then the Generalized f_G -Mean becomes

$$\frac{1}{n}(x_1^{1/3} + x_2^{1/3} + \ldots\ldots\ldots\ldots\ldots + x_n^{1/3})\}^3$$

This is nothing but the cube root mean defined by (7).

Generalized p-Mean:

Let the invertible function $f(.)$ be such that

$$f: x \to exp(x^{1/p})$$

i.e. $f(x) = exp(x^{1/p})$

Then $f^{-1}\{exp(x^{1/p})\} = x$

i.e. $f^{-1}(y) = (\log_e y)^p$, putting $y = exp(x^{1/p})$

Thus $f^{-1}(.)$ is a function with

$$f^{-1}: x \to (\log_e x)^p$$

Therefore, if $f(x) = exp(x^{1/p})$

then the Generalized f_G - Mean becomes

$$\{\frac{1}{n}(x_1^p + x_2^p + \ldots\ldots\ldots\ldots\ldots + x_n^p\}^{\frac{1}{p}}$$

This is nothing but the Generalized p-Mean defined by (8).

Generalized p^{th} Root Mean:

Let the invertible function $f(.)$ be such that

$$f: x \rightarrow exp\,(x^{1/p})$$

i.e. $f(x) = exp\,(x^{1/p})$

Then $f^{-1}\{exp\,(x^{1/p})\} = x$

i.e. $f^{-1}(y) = (log_e\, y)^p$, putting $y = exp\,(x^{1/p})$

Thus $f^{-1}(.)$ is a function with

$$f^{-1}: x \rightarrow (log_e\, x)^p$$

Thus if in the Generalized f_G -Mean, $f(.)$ is selected as

$$f(x) = exp\,(x^{1/p})$$

then the Generalized f_G -Mean becomes

$$\{\tfrac{1}{n}(\,x_1{}^{1/p} + \,x_2{}^{1/p} \,+\, \ldots\ldots\ldots\ldots . + x_n{}^{1/p})\}^p$$

This is nothing but the Generalized p^{th} Root Mean defined by (9).

5. Generalized f_G -Mean of a Function

From the Generalized f_G -Mean, one can define the Generalized f_G -Mean of a function

$$g = g(.) = g(x)$$

of x by

$$f_G\,\{\,g(x_1), g(x_2), \ldots\ldots\ldots, g(x_n)\}$$

$= f^{-1}[\{f(g_1)f(g_2)\ldots\ldots.\ f(g_n)\}^{1/n}]$ (4.1)

where

$$g_1 = g(x_1),\ g_2 = g(x_2), \ldots\ldots\ldots ,\ g_n = g(x_n)$$

6. Some Definitions/Formulations of Average of Function

From this definition, one can obtain the definitions / formulations of various means as mentioned above, for a function of variable, as follows:

Arithmetic Mean:

$$\text{Substituting } f(x) = exp\,\{g(x)\}$$

in (4.1), Arithmetic Mean of $g(x)$ can be obtained as

$$\tfrac{1}{n}\,\{g(x_1) + g(x_2) + \ \ldots\ldots\ldots\ldots . + g(x_n)\}$$

In particular

Arithmetic Mean of $g(x) = x^2$ is

$$\frac{1}{n}(\,x_1{}^2 + \,x_2{}^2 \,+\, \ldots\ldots\ldots\ldots . + x_n{}^2)$$

Arithmetic Mean of $g(x) = |x|$ is

$$\frac{1}{n}(|x_1| \,+\, |x_2| + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . + |x_n|)$$

Arithmetic Mean of $g(x) = x^3$ is

$$\frac{1}{n}(\,x_1{}^3 + \,x_2{}^3 \,+\, \ldots\ldots\ldots\ldots . + x_n{}^3)$$

Arithmetic Mean of $g(x) = x^p$ is

$$\frac{1}{n}(\,x_1{}^p + \,x_2{}^p \,+\, \ldots\ldots\ldots\ldots . + x_n{}^p)$$

Arithmetic Mean of $g(x) = x^{1/p}$ is

$$\frac{1}{n}(\,x_1{}^{1/p} + \,x_2{}^{1/p} \,+\, \ldots\ldots\ldots\ldots . + x_n{}^{1/p})$$

Arithmetic Mean of $g(x) = e^x$ is

$$\frac{1}{n}(e^{\,x_1} + e^{\,x_2} \,+\, \ldots\ldots\ldots\ldots . + e^{x_n})$$

Arithmetic Mean of $g(x) = log\,x$ is

$$\frac{1}{n}(log\,x_1 + \,log\,x_2 + \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . + log\,x_n)$$

Geometric Mean:

Geometric Mean of $g(x)$ can be obtained as

$$\{\,g(x_1).\,g(x_2).\,g(x_3)\ \ \ldots\ldots\ldots\ldots\ \ g(x_n)\}^{1/n}$$

Harmonic Mean:

Harmonic Mean of $g(x)$ can be obtained as

$$\frac{1}{\frac{1}{n}\{\frac{1}{g(x_1)} + \frac{1}{g(x_2)} + \ldots\ldots\ldots\ldots . + \frac{1}{g(x_n)}\}}$$

Quadratic Mean:

Quadratic Mean of $g(x)$ can be obtained as

$$\{\tfrac{1}{n}(\,g_1{}^2 + \,g_2{}^2 \,+\, \ldots\ldots\ldots\ldots . + g_n{}^2)\}^{1/2}$$

where $g_1 = g(x_1),\ g_2 = g(x_2), \ldots\ldots\ldots ,\ g_n = g(x_n)$

Cubic Mean:

Cubic Mean of $g(x)$ can be obtained as

$$\{\tfrac{1}{n}(\,g_1{}^3 + \,g_2{}^3 \,+\, \ldots\ldots\ldots\ldots . + g_n{}^3)\}^{1/3}$$

Generalized p Mean:

Generalized p Mean of $g(x)$ can be obtained as

$$\{\tfrac{1}{n}(\,g_1{}^p + \,g_2{}^p \,+\, \ldots\ldots\ldots\ldots . + g_n{}^p)\}^{1/p}$$

Generalized p^{th} Root Mean:

Generalized p^{th} Root Mean of $g(x)$ can be obtained as

$$\{\tfrac{1}{n}(\,g_1{}^{1/p} + \,g_2{}^{1/p} \,+\, \ldots\ldots\ldots\ldots . + g_n{}^{1/p})\}^p$$

e Mean:

e Mean of $g(x)$ can be obtained as

$$log_e\{\frac{1}{n}(e^{g_1} + e^{g_2} + \dots\dots\dots\dots\dots. + e^{g_n}\}$$

Scale s -Mean:

Scale s Mean or simply s Mean of $g(x)$ can be obtained as

$$\frac{1}{s}\{\frac{1}{n}(s.g_1 + s.g_2 + \dots\dots\dots\dots\dots + s.g_n)\}$$

Shift a -Mean:

Shift a - Mean of $g(x)$ can be obtained as

$$\frac{1}{n}\{(g_1 - a) + (g_2 - a) + \dots\dots\dots\dots + (g_n - a)\} + a$$

Shift a – Inverse Scale s – Mean:

Shift (a) – Inverse Scale s - Mean of $g(x)$ can be obtained as

$$s\{\frac{1}{n}(\frac{g_1 - a}{s} + \frac{g_2 - a}{s} + \dots\dots\dots + \frac{g_n - a}{s})\} + a$$

7. Conclusion

One can conclude that the Generalized f_G - Mean constructed here can be regarded as a source from where lots of definitions/formulations can be derived for various types of averages.

Different types of formulations of average are necessary for handling different types of data. That is why we need more and more formulations of average.

The types of average, formulated here, have been derived from the Generalized f_G - Mean constructed here. However, this Generalized f_G - Mean, for generating means, may not be sufficient to yield many types of averages to deal with many types of data. Thus, there is necessity of further study on searching for more and more techniques of defining / formulating of more types of averages.

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