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Dual Phase Lag, Isotropic Thermoelastic, Laplace Transform, Hankel Transform, Plane Axisymmetric, Diffusion

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# Axi-Symmetric Propagation in a Thermoelastic Diffusion with Phase Lags

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#### Abstract

The purpose of this paper is to depict the effect of thermal and diffusion phase lags due to axisymmetric heat supply for a disc. The problem is discussed within the context of DPLT and DPLD models. The upper and lower surfaces of the disc are traction free and subjected to an axisymmetric heat supply. The solution is found by using Laplace and Hankel transform technique and a direct approach without the use of potential functions. The analytical expressions of displacements, stresses and chemical potential, temperature and mass concentration are computed in transformed domain. Numerical inversion technique has been applied to obtain the results in the physical domain. Numerically simulated results are depicted graphically. The effect of diffusion and thermal phase-lags are shown on the various components. Some particular cases of result are also deduced from the present investigation.

# **1. Introduction**

Classical Fourier heat conduction law implies an infinitely fast propagation of a thermal signal which is violated in ultra-fast heat conduction system due to its very small dimensions and short timescales. Catteno [1] and Vernotte [2] proposed a thermal wave with a single phase lag in which the temperature gradient after a certain elapsed time was given by  $q + \tau_q \frac{\partial q}{\partial t} = -k\nabla T$ , where  $\tau_q$  denotes the relaxation time required for thermal physics to take account of hyperbolic effect within the medium. Here when  $\tau_q > 0$ , the thermal wave propagates through the medium with a finite speed of  $\sqrt{\alpha/\tau_q}$ , where  $\alpha$  is thermal diffusivity. when  $\tau_q$  approaches zero, the thermal wave has an infinite speed and thus the single phase lag model reduces to traditional Fourier model. The dual phase lag model of heat conduction was proposed by Tzou [3]  $q + \tau_q \frac{\partial q}{\partial t} = -k(\nabla T + \tau_t \frac{\partial}{\partial t}\nabla T)$ , where the temperature gradient  $\nabla T$  at a point P of the material at time  $t + \tau_t$  corresponds to the heat flux vector q at the same time at the time  $t + \tau_q$ . Here K is thermal conductivity of the material. The delay time  $\tau_t$  is interpreted as that caused by the microstructural interactions and is called the phase lag of temperature gradient. The other

delay time  $\tau_a$  interpreted as the relaxation time due to the fast transient effects of thermal inertia and is called the phase lag of heat flux. This universal model is claimed to be able to bridge the gap between microscopic and macroscopic approaches, covering a wide range of heat transfer models. If  $\tau_t = 0$ , Tzou [3] refers to the model as single phase model. Numerous efforts have been invested in the development of an explicit mathematical solution to the heat conduction equation under dual phase lag model. Quintanilla [4] compared two different mathematical hyperbolic models proposed by Tzou. Kumar and Mukhopadhyay [5] investigated the propagation of harmonic waves of assigned frequency by employing the thermoelasticity theory with three phase lags. Chou and Yang [6] discussed two dimensional dual phase lag thermal behaviour in single-/multi-layer structures using CESE method. Zhou et. al [7] proposed an axisymmetric dual-phase-lag bio heat model for laser heating of living tissues. Kumar et. al [8] discussed effect of viscosity on wave propagation in anisotropic thermoelastic medium with three phase lag model. Abbas et. al [9] investigated two-dimensional problem in a transversely isotropic thermoelastic medium without energy dissipation and with two temperatures due to a thermal source. Liu et. al [10] analysed thermal damage in a laser-Irradiated based on non-Fourier model. Ying and Yun [11] built a fractional dualphase-lag model and the corresponding bio-heat transfer equation. Mondal et. al [12] investigated fractional order two-temperature dual-phase-lag thermoelasticity with variable thermal conductivity. Abdallah [13] used uncoupled thermoelastic model based on dual phase lag to investigate the thermoelastic properties of a semi-infinite medium. Bhattacharya and Kanoria [14] investigated the problem of elasto-thermo-diffusion inside a spherical shell. Kaushal et. al [15] analysed wave propagation in temperature rate dependent thermoelasticity with two temperatures. Kaushal et. al [16] discussed propagation of waves in generalized thermoelastic continua with two temperature. Rukolaine [17] investigated unphysical effects of the dual-phase-lag model of heat conduction. Tripathi et. al [18] analysed generalized thermoelastic diffusion problem in a thick circular plate with axisymmetric heat supply.

Diffusion is defined as the spontaneous movement of the particles from high concentration region to the low

concentration region, and it occurs in response to a concentration gradient expressed as the change in concentration due to change in position. Thermal diffusion utilizes the transfer of heat across a thin liquid or gas to accomplish isotope separation. The thermodiffusion in elastic solids is due to coupling of fields of temperature, mass diffusion and that of strain in addition to heat and mass exchange with the environment. Dual phase lag diffusion model was considered by Kumar and Gupta [19, 20], Chiriță, et. al [21] study the propagation of plane time harmonic waves in the infinite space filled by a time differential dualphase-lag thermoelastic material. Sherief and Hamza [22] considered the two-dimensional problem of a thick plate whose lower and upper surfaces are traction free and subjected to a given axisymmetric temperature distribution is considered within the context of the theory of generalized thermoelasticity with one relaxation time. Kumar et. al [23] investigated thermomechanical interactions for dual -phaselag in a homogeneous isotropic thick circular plate in the light of two-temperature thermoelasticity theory.

Here in this investigation, a generalized form of mass diffusion equation is introduced instead of classical Fick's diffusion theory by using two diffusion phase-lags in axisymmetric form. One phase-lag of diffusing mass flux vector, represents the delayed time required for the diffusion of the mass flux and the other phase-lag of chemical potential, represents the delayed time required for the establishment of the potential gradient. The basic equations for the isotropic thermoelastic diffusion medium in the context of dual-phase-lag heat transfer (DPLT) and dualphase-lag diffusion (DPLD) models in axisymmetric form are presented. The components of displacements, stresses and chemical potential, Temperature and mass concentration are computed numerically. Numerically computed results are depicted graphically. The effect of diffusion and thermal phase-lags are shown on the various components.

#### 2. Basic Equations

The basic equations of motion, heat conduction and mass diffusion in a homogeneous isotropic thermoelastic solid with DPLT and DPLD models in the absence of body forces, heat sources and mass diffusion sources are

$$(\lambda + \mu)\nabla(\nabla . u) + \mu\nabla^2 u - \beta_1\nabla T - \beta_2\nabla C = \rho\ddot{u}$$
(1)

$$\left(1+\tau_t\frac{\partial}{\partial t}\right)KT_{,ii} = \left(1+\tau_q\frac{\partial}{\partial t}+\tau_q^2\frac{\partial^2}{\partial t^2}\right)\left[\rho C_E\dot{T}+\beta_1 T_0\dot{e}_{kk}+aT_0\dot{C}\right]$$
(2)

$$\left(1+\tau_p\frac{\partial}{\partial t}\right)\left(D\beta_2\nabla^2(\nabla .\,u)+Da\nabla^2 T-Db\nabla^2 C\right)+\frac{\partial}{\partial t}\left(1+\tau_\eta\frac{\partial}{\partial t}+\tau_\eta^2\frac{\partial^2}{\partial t^2}\right)C=0$$
(3)

and the constitutive relations are

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij} (\lambda e_{kk} - \beta_1 T - \beta_2 C) \tag{4}$$

$$\rho T_0 S = \left(1 + \tau_q \frac{\partial}{\partial t} + \tau_q^2 \frac{\partial^2}{\partial t^2}\right) \left(\rho C_E T + \beta_1 T_0 e_{kk} + a T_0 C\right)$$
(5)

$$P = -\beta_2 e_{kk} - aT - bC \tag{6}$$

Where  $\lambda$ ,  $\mu$  are Lame's,  $\rho$  is the density assumed to be independent of time, D is the diffusivity, P is the chemical potential per unit mass, C is the concentration,  $u_i$  are components of displacement vector u, K is the coefficient of thermal conductivity,  $C_E$  is the specific heat at constant strain,  $T = \vartheta - T_0$  is small temperature increment,  $\vartheta$  is the absolute temperature of the medium,  $T_0$  is the reference temperature of the body such that  $\left|\frac{T}{T_0}\right| \ll 1$ , a and b are respectively, the coefficients describing the measure of thermodiffusion and mass diffusion effect respectively,  $\sigma_{ii}$ and  $e_{ii}$  are the components of stress and strain respectively,  $e_{kk}$  is dilatation, S is the entropy per unit mass,  $\beta_1 = (3\lambda +$  $2\mu)\alpha_t$ ,  $\beta_2 = (3\lambda + 2\mu)\alpha_c$ ,  $\alpha_c$  is the coefficient of linear diffusion expansion and  $\alpha_t$  is the coefficient of thermal linear expansion.  $au_t$ ,  $au_q$ ,  $au_p$ ,  $au_\eta$  are respectively phase lag of temperature gradient, the phase lag of heat flux, the phase lag of chemical potential, and phase lag of diffusing mass flux vector. In above equations, a comma followed by suffix denotes spatial derivative and a superposed dot denotes derivative with respect to time.

# 3. Formulation and Solution of the Problem

Consider a disc of diameter 2b occupying the space D defined by  $0 \le r \le \infty, -b \le z \le b$ , Let the ring be subjected to an axisymmetric heat supply depending on the radial and axial directions of the cylindrical co-ordinate system. The initial temperature of the disc is given by a constant temperature  $T_0$ , and the heat flux  $g_0F(r,z)$  is prescribed on the upper and lower boundary surfaces. Under these conditions, the thermoelastic quantities for the disc are required to be determined. We take a cylindrical polar co-ordinate system  $(r, \theta, z)$  with symmetry about z – axis. As the problem considered is plane axisymmetric, the field component  $u_{\theta} = 0$ , and  $u_r, u_z, T$  and C are independent of  $\theta$  and restrict our analysis to the two dimensional problem with

$$u = (u_r, 0, u_z) \tag{7}$$

Equations (1)-(6) with the aid of (7) take the form

$$(\lambda+\mu)\frac{\partial e}{\partial r}+\mu\left(\nabla^2-\frac{1}{r^2}\right)u_r-\beta_1\frac{\partial T}{\partial r}-\beta_2\frac{\partial C}{\partial r}=\rho\frac{\partial^2 u_r}{\partial t^2} \quad (8)$$

$$(\lambda + \mu)\frac{\partial e}{\partial z} + \mu \nabla^2 u_z - \beta_1 \frac{\partial T}{\partial z} - \beta_2 \frac{\partial C}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2} \qquad (9)$$

$$(1+\tau_t\frac{\partial}{\partial t}) \operatorname{K} \nabla^2 T = \left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q^2}{2}\frac{\partial^2}{\partial t^2}\right) \left[\rho C_E \dot{T} + \beta_1 T_0 \frac{\partial}{\partial t} divu + a T_0 \frac{\partial C}{\partial t}\right]$$
(10)

$$(1+\tau_p\frac{\partial}{\partial t})(D\beta_2\nabla^2 di vu + Da\nabla^2 T - Db\nabla^2 C) + \frac{\partial}{\partial t}\left(1 + \tau_\eta\frac{\partial}{\partial t} + \frac{\tau_\eta^2}{2}\frac{\partial^2}{\partial t^2}\right)C = 0$$
(11)

and Constitutive relations

$$\sigma_{rr} = 2\mu e_{rr} + \lambda e - \beta_1 T - \beta_2 C \tag{12}$$

$$\sigma_{\theta\theta} = 2\mu e_{\theta\theta} + \lambda e - \beta_1 T - \beta_2 \tag{13}$$

$$\sigma_{zz} = 2\mu e_{zz} + \lambda e - \beta_1 T - \beta_2 C \tag{14}$$

$$\sigma_{rz} = \mu e_{rz}, \, \sigma_{r\theta} = 0, \, \sigma_{z\theta} = 0 \tag{15}$$

$$P = -\beta_2 e - aT + bC \tag{16}$$

where

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}, e_{rr} = \frac{\partial u_r}{\partial r}, e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta}, e_{zz} = \frac{\partial u_z}{\partial z},$$
$$e_{rz} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)$$
(17)

To facilitate the solution, the following dimensionless quantities are introduced

$$r' = \frac{\omega_{1}^{*}}{c_{1}}r, z' = \frac{\omega_{1}^{*}}{c_{1}}z, (u'_{r}, u'_{z}) = \frac{\omega_{1}^{*}}{c_{1}}(u_{r}, u_{z}), t' = \omega_{1}^{*}t, (\sigma'_{rr}, \sigma'_{\theta\theta}, \sigma'_{zz}, \sigma'_{rz}) = \frac{1}{\beta_{1}T_{0}}(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz})$$

$$(T', C') = \frac{\beta_1}{\rho c_1^2} (T, C), (\tau'_q, \tau'_t, \tau'_p, \tau'_\eta) = \omega_1^* (\tau_q, \tau_t, \tau_p, \tau_\eta), \omega_1^* = \frac{\rho c_E c_1^2}{\kappa}, c_1^2 = \frac{\lambda + 2\mu}{\rho}$$
(18)

Using (18) in equations (8)-(11) and after that suppressing the primes and then applying the Laplace transform defined by (19)

$$\bar{f}(r,z,s) = \int_0^\infty f(r,z,t)e^{-st}dt$$
(19)

$$\tilde{f}(\xi, z, s) = \int_0^\infty \hat{f}(r, z, s) r J_n(r\xi) dr$$
(20)

on the resulting quantities and simplifying we obtain

$$\nabla^2 \overline{T} + \nabla^2 \overline{C} - (\nabla^2 - s^2)\overline{e} = 0$$
<sup>(21)</sup>

$$(\nabla^2 - \frac{\tau'_q}{\tau'_t} ks)\bar{T} - \frac{\kappa a \tau_0 \tau'_q s}{\rho c_E \beta_2 \tau'_t} \bar{C} - \frac{\kappa \beta_1^2 \tau_0}{\rho^2 c_E c_1^2} \frac{\tau'_q}{\tau'_t} s \bar{e} = 0$$
(22)

$$Da\nabla^2 \frac{\rho c_1^2}{\beta_1} \overline{T} - \left( Db \frac{\rho c_1^2}{\beta_2} \nabla^2 - \frac{s \tau_\eta' k c_1^2}{\tau_p' \beta_2 c_E} \right) \overline{C} + D\beta_2 \nabla^2 \overline{e} = 0 \quad (23)$$

where  $\tau'_q = 1 + s\tau_q + \frac{s^2\tau_q^2}{2}$ ,  $\tau'_\eta = 1 + s\tau_\eta + \frac{s^2\tau_\eta^2}{2}$ ,  $\tau'_p = 1 + s\tau_p$ ,  $\tau'_t = 1 + s\tau_t$ 

Eliminating  $\overline{T}$ ,  $\overline{C}$ ,  $\overline{e}$  from equations (21)-(23), we obtain

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)(\bar{T}, \bar{C}, \bar{e}) = 0$$
(24)

The solutions of the equation (24) can be written in the form

$$\bar{T} = \sum_{i=1}^{3} \bar{T}_i, \, \bar{e} = \sum_{i=1}^{3} \bar{e}_i, \, \bar{C} = \sum_{i=1}^{3} \bar{C}_i \tag{25}$$

where  $\bar{T}_i$ ,  $\bar{e}_i$ , and  $\bar{C}_i$  are solutions of the following equation

$$(\nabla^2 - k_i^2)(\bar{T}_i, \bar{e}_i, \bar{C}_i) = 0, i = 1, 2, 3$$
(26)

On taking Hankel transform defined by (20) on (26), we obtain

$$(D^{2} - \xi^{2} - k_{i}^{2}) \left( \bar{T}_{i}^{*}, \bar{e}_{i}^{*}, \bar{C}_{i}^{*} \right) = 0$$
 (27)

$$\overline{T}^* = \sum_{i=1}^3 A_i(\xi, s) \cosh(q_i z)$$
(28)

$$\bar{C}^* = \sum_{i=1}^3 d_i A_i(\xi, s) \cosh(q_i z)$$
(29)

$$\bar{e}^* = \sum_{i=1}^3 f_i A_i(\xi, s) \cosh(q_i z)$$
(30)

$$d_{i} = \frac{\zeta_{33}q_{i}^{*} - 6)\zeta_{21}\zeta_{33}q_{i}^{*} + \zeta_{23}\zeta_{31}}{-q_{i}^{2}(\zeta_{22}\zeta_{33} + \zeta_{23}\zeta_{32}) + \zeta_{23}\zeta'_{32}}$$

$$f_i = \frac{-\zeta_{32}q_i^4 + (\zeta'_{32} + \zeta_{21}\zeta_{32})q_i^2 - \zeta_{21}\zeta'_{32}}{-q_i^2(\zeta_{22}\zeta_{33} + \zeta_{23}\zeta_{32}) + \zeta_{23}\zeta'_{32}}$$

$$\zeta_{31} = \frac{Da\rho c_1^2}{\beta_1}, \zeta_{32} = Db \frac{\rho c_1^2}{\beta_2}, \zeta'_{32} = \frac{s \tau'_{\eta} k c_1^2}{\tau'_p \beta_2 c_E}, \zeta_{33} = D\beta_2$$

Applying inversion of Hankel transform on (28), (29) and (30), we get

$$\overline{T} = \int_0^\infty \{\sum_{i=1}^3 A_i(\xi, s) \cosh(q_i z)\} \xi J_0(\xi r) d\xi$$
(31)

$$\bar{C} = \int_0^{\omega} \{ \sum_{i=1}^3 d_i A_i(\xi, s) \cosh(q_i z) \} \xi J_0(\xi r) d\xi$$
(32)

$$\bar{e} = \int_0^\infty \{\sum_{i=1}^3 f_i A_i(\xi, s) \cosh(q_i z)\} \xi J_0(\xi r) d\xi$$
(33)

Using (8)-(11), (18) and (31)-(33), we obtain the displacement components in the transformed domain as

$$q_{i} = \sqrt{\xi^{2} + k_{i}^{2}}, \zeta_{21} = \frac{\tau_{q}'}{\tau_{t}'} ks, \zeta_{22} = \frac{\kappa a \tau_{0} \tau_{q}'s}{\rho C_{E} \beta_{2} \tau_{t}'}, \zeta_{23} = \frac{\kappa \beta_{1}^{2} \tau_{0}}{\rho^{2} C_{E} c_{1}^{2}} \frac{\tau_{q}'}{\tau_{t}'} s,$$
$$\bar{u}_{r}(r, z, s) = \int_{0}^{\infty} \xi^{2} J_{1}(\xi r) \left[ E(\xi, s) \cosh(qz) + \sum_{i=1}^{3} \frac{\lambda_{i}}{\left(\frac{\mu q_{i}^{2}}{\rho c_{1}^{2}} - s^{2}\right)} \cosh(q_{i}z) \right] d\xi$$
(34)

$$\bar{u}_{z}(r,z,s) = \int_{0}^{\infty} \xi J_{0}(\xi r) \left[ F(\xi,s) \sinh(qz) - \sum_{i=1}^{3} \frac{\lambda_{i}}{\left(\frac{\mu q_{i}^{2}}{\rho c_{1}^{2}} s^{2}\right)} \sinh(q_{i}z) \right] d\xi$$

$$(35)$$

where

as

$$\lambda_i = \left(\frac{\lambda + \mu}{\rho c_1^2} f_i - 1 - d_i\right) A_i, F(\xi, s) = \frac{\xi^2 E(\xi, s)}{q}, q = \sqrt{\xi^2 + \frac{\rho c_1^2}{\mu} s^2}$$

Substituting the values of (31)-(35) in (12)-(15) and with the aid of (18) yield

$$\overline{\sigma_{\theta\theta}} = \frac{2\mu}{\beta_1 T_0} \int_0^\infty \xi^2 J_1(\xi r) \left[ E(\xi, s) \cosh(qz) + \sum_{i=1}^3 \frac{\lambda_i}{\left(\frac{\mu q_i^2}{\rho c_1^2} s^2\right)} \cosh(q_i z) \right] d\xi + \int_0^\infty \sum_{i=1}^3 \eta_i \cosh(q_i z) \,\xi J_0(\xi r) \,d\xi \tag{36}$$

$$\overline{\sigma_{rr}} = \frac{2\mu}{\beta_1 T_0} \int_0^\infty \xi^3 \left(\frac{1}{\xi_r} J_1(\xi_r) - J_0(\xi_r)\right) \left[ E(\xi, s) \cosh(qz) + \sum_{i=1}^3 \frac{\lambda_i}{\left(\frac{\mu q_i^2}{\rho c_1^2} - s^2\right)} \cosh(q_i z) \right] d\xi + \int_0^\infty \sum_{i=1}^3 \eta_i \cosh(q_i z) \,\xi J_0(\xi_r) \,d\xi \tag{37}$$

$$\overline{\sigma_{zz}} = \frac{2\mu}{\beta_1 T_0} \int_0^\infty \xi J_0(\xi r) \left[ F(\xi, s)q \cosh(qz) + \sum_{i=1}^3 \frac{\lambda_i q_i^2}{\left(\frac{\mu q_i^2}{\rho c_1^2} - s^2\right)} \cosh(q_i z) \right] d\xi + \int_0^\infty \sum_{i=1}^3 \eta_i \cosh(q_i z) \,\xi J_0(\xi r) \,d\xi \tag{38}$$

$$\overline{\sigma_{rz}} = \frac{\mu}{2\beta_1 T_0} \int_0^\infty \xi^2 J_1(\xi r) \left[ \left( \frac{q^2 - \xi^2}{q} \right) E(\xi, s) \operatorname{qsinh}(qz) + 2\sum_{i=1}^3 \frac{\lambda_i}{\left( \frac{\mu q_i^2}{\rho c_1^2} - s^2 \right)} q_i \operatorname{sinh}(q_i z) \right] d\xi$$
(39)

$$\overline{P}(r,z,s) = \int_0^\infty \sum_{i=1}^3 \zeta_i \cosh(q_i z) \,\xi J_0(\xi r) \,d\xi \tag{40}$$

where 
$$\eta_i = \left(\frac{f_i - \rho c_1^2 - \rho c_1^2 d_i}{\beta_1 T_0}\right) A_i, F(\xi, s) = \frac{\xi^2 (E(\xi, s))}{q}, \zeta_i = \left(-\beta_2 f_i - \frac{\rho c_1^2}{\beta_1} + \frac{b\rho c_1^2}{\beta_2} d_i\right) A_i(\xi, s)$$

#### $\frac{\partial T}{\partial z} = \pm g_0 F(r, z),$ (41)

$$\sigma_{zz} = 0, \tag{42}$$

$$\sigma_{rz} = 0, \tag{43}$$

$$P = f(r, t) = \frac{H(t-r)}{\pi(ct)^2}$$
(44)

Applying Laplace transform and Hankel transform on both

where 
$$\eta_i = \left(\frac{f_i - \rho c_1^2 - \rho c_1^2 d_i}{\beta_1 T_0}\right) A_i$$
,  $F(\xi, s) = \frac{\xi^2 (E(\xi, s))}{q}$ ,  $\zeta_i = \left(-\beta_2 f_i - \frac{\rho c_1^2}{\beta_1} + \frac{b \rho c_1^2}{\beta_2} d_i\right) A_i$   
**4. Boundary Conditions**

We consider a thermal source and chemical potential source (disc load). The disc load which emanates from origin of the coordinates and expands radically at constant rate 'c' over along with vanishing of stress components at the stress free surface at  $z = \pm b$ . Mathematically, these can be written

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sides of the boundary conditions (41)-(44), we obtain

$$\frac{dT}{dz} = g_0 \bar{F}(\xi, z) \tag{45}$$

$$\overline{\sigma_{zz}} = 0$$
 (46)

$$\overline{\sigma_{rz}} = 0 \tag{47}$$

$$\bar{P} = \bar{f}(\xi, s) = \frac{1}{\pi c \xi \sqrt{\xi^2 + \frac{s^2}{c^2} - \frac{s}{c}}}$$
(48)

Substitute the values of  $\overline{T}$ ,  $\overline{\sigma_{zz}}$ ,  $\overline{\sigma_{rz}}$ ,  $\overline{P}$  in (45)-(48), we obtain the values of unknown parameters as

$$A_{1} = \frac{\Delta_{1}}{\Delta}, A_{2} = \frac{\Delta_{2}}{\Delta}, A_{3} = \frac{\Delta_{3}}{\Delta}, E(\xi, s) = \frac{\Delta_{4}}{\Delta}$$
$$D = \begin{vmatrix} D_{11} & D_{12} & D_{13} & 0 \\ D_{21} & D_{22} & D_{23} & \frac{-2m}{b_{1}T_{0}}x^{2}q\cos(qb) \\ D_{31} & D_{32} & D_{33} & \frac{q^{2} - x^{2}}{q}\sinh(qb) \\ D_{41} & D_{42} & D_{43} & 0 \end{vmatrix}$$

 $\Delta_{1i} = q_i \sinh(q_i b), \ \Delta_{2i} = (\mu_i q_i^2 + \eta_i) \cosh(q_i b), \ \Delta_{3i} =$ 

$$D(j,K) = (-1)^{j+M} \sum_{n=m}^{\min(j,M)} \frac{n^M (2n)!}{(M-n)! \, n! \, (n-1)! \, (j-n)! \, (2n-j)!}$$

Where K is an even integer, whose value depends on the word length of computer used. M=K/2, and m is an integer part of (j + 1)/2. The optimal value of K was chosen as described in Gaver-Stehfast algorithm, for the fast convergence of results with desired accuracy. The Romberg numerical integration technique [27] with variable step size was used to evaluate the results involved.

#### **6. Particular Cases**

- (1) If we neglect the diffusion effect (i.e.,  $\beta_2 = a = b = 0$ ), we obtain the expressions for components of displacement, stress and temperature distribution in thermoelastic isotropic half space.
- (2) If  $\tau_q = \tau_t = 0$ , we obtain traditional Fourier model from dual phase lag model.
- (3) If  $\tau_p = \tau_n = 0$ , then it reduces to DPLT model.
- (4) If  $\tau_q = 0$  and  $\tau_p = 0$ , then DPLT and DPLD models reduce to single phase heat model (SPLT) and single phase diffusion model (SPLD)

#### 7. Numerical Results and Discussion

The mathematical model is prepared with copper material for purposes of numerical computation. The material constants for the problem are taken from Dhaliwal and Singh [28]

$$\begin{array}{l} \lambda=7.76\times 10^{10}Nm^{-2}, \mu=3.86\times 10^{10}Nm^{-2}, K=386JK^{-1}m^{-1}s^{-1}, \beta_1=5.518\times 10^6Nm^{-2}deg^{-1}, \rho= \end{array}$$

$$\mu_i q_i \sinh(q_i b), \Delta_{4i} = \zeta_i \cosh(q_i b), i=1, 2, 3 \text{ and } \Delta_i$$
  
s obtained from  $\Delta$ , by interchanging i<sup>th</sup> column with  
 $g_0 F(\xi, b) = 0 \frac{1}{\pi c \xi \sqrt{\xi^2 + \frac{s^2}{c^2} - \frac{s}{c}}} \int_{x}^{t}$ , where t denotes transpose.

#### 5. Inversion of Double Transform

Due to the complexity of the solution in the Laplace transform domain, the inverse of the Laplace transform is obtained by using the Gaver-Stehfast algorithm. Graver [24] and Stehfast [25, 26] derived the formula given below. By this method, the inverse f(t) of Laplace transform  $\overline{f}(s)$  is approximated by

$$f(t) = \frac{\log 2}{t} \sum_{j=1}^{k} D(j, K) F\left(j \frac{\log 2}{t}\right)$$

with

$$\begin{array}{l} 8954 \; Kgm^{-3}, \, a = 1.2 \; \times 10^4 m^2/s^2 kb = 0.9 \; \times \\ 10^6 m^5/kgs^2 D = 0.88 \; \times 10^{-8} kgs/m^3, \beta_2 = 61.38 \; \times \\ 10^6 Nm^{-2} deg^{-1}, T_0 = 293 \mathrm{K}, \, C_E = 383.1 \; Jkg^{-1}K^{-1} \end{array}$$

An investigation has been conducted to compare the effect of phase lags of heat transfer and diffusion on normal displacement  $u_3$ , Chemical potential function P, Temperature change T and mass concentration C by keeping one phase lag fixed and varying the values of other phase lag and vice versa in both the cases. The graphs have been plotted in the range  $0 \le r \le 10$ .

We consider the following cases

- a)  $\tau_t = 0.08$ ,  $\tau_q = 0$ , 0.02 and 0.04
- b)  $\tau_q = 0.08$ ,  $\tau_t = 0$ , 0.02 and 0.04
- c)  $\tau_{\eta} = 0.08$ ,  $\tau_{p} = 0$ , 0.02 and 0.04
- d)  $\tau_p$ =0.08,  $\tau_\eta$  = 0,0.02 and 0.04

In all figures solid line corresponds to the value of phase lag =0, small dashed line corresponds to the value of phase lag= 0.02, long dashed line with dots corresponds to value of phase lag= 0.04

Figures (1)-(4), exhibits variations of axial displacement  $u_z$  with distance r corresponding to the cases (a), (b), (c), (d) respectively.

In Fig. 1. near the loading surface, there is a sharp decrease in range  $0 \le r \le 2$  corresponding to  $\tau_q = .02$  and variations are in wave form for the rest two curves.

In Fig. 2. Opposite oscillatory behaviour away from the loading surface is observed for  $\tau_q = 0$  and 0.02 whereas trends are similar for  $\tau_q = .04$  and  $\tau_q = 0$ .

In Fig. 3. for  $\tau_p = 0$ , variations are in wave form whereas for  $\tau_p = 0.02$  and  $\tau_p = 0.04$ , there is a sharp increase near the loading surface and away from it, trend is oscillatory.

In Fig. 4. illustrates the variations of  $u_z$  corresponding to case (d). Here variations are similar as discussed in figure 3 with interchanging  $\tau_p$  and  $\tau_\eta$ .

Figures (5)-(8) represent the variations in chemical potential function P with distance r corresponding to the four cases. In these figures, we observe that ther is a sharp decrease in the range  $0 \le r \le 2$  and variations are oscillatory in the rest with different amplitudes and trends.

Variations of temperature change T with distance r have been shown in figures (9)-(12) corresponding to the cases (a),(b),(c) and (d) respectively.

In Fig. 9. we observe that in the range  $2 \le r \le 4$  and  $7 \le r \le 9$  values corresponding to  $\tau_q = 0$  and  $\tau_q = 0.04$  are smaller than for  $\tau_q = 0.02$  whereas behaviour is opposite in the rest.

In Figures (10)-(12), there is a sharp increase in the range  $0 \le r \le 2$  and afterwards behaviour is oscillatory with different magnitudes and patterns. Small variations are observed near zero in Fig. 12 in the range  $4 \le r \le 9$ .

In Figures (13)-(16) show variations in mass concentration C with distance r corresponding to the four cases respectively. Here, we observe a descending behaviour in the cases. As r increases, values of mass concentration are decreasing.

In Fig. 13. we observe that, as  $\tau_t$  increases, there is a deficiency in the wave.

In Fig. 14. For  $\tau_q = .02$  and  $\tau_q = .04$  trends are similar with difference in amplitude whereas for  $\tau_q = 0$ , trends are different. Small variations near zero are observed in Fig. 15 and Fig. 16. in the range  $4 \le r \le 8$  and trends are descending oscillatory in the rest.



Fig. 1. Variations of axial displacement  $u_z$  with distance r (case a).



Fig. 2. Variations of axial displacement  $u_z$  with distance r (case b).



Fig. 3. Variations of axial displacement  $u_z$  with distance r (case c).







Fig. 5. Variations of chemical potential function P with distance r (case a).





Fig. 7. Variations of chemical potential function P with distance r(case c).







Fig. 9. Variations of temperature change T with distance r(case a).







Fig. 11. Variations of temperature change T with distance r (case c).



Fig. 13. Variations of mass concentration C with distance r (case a).



Fig. 15. Variations of mass concentration C with distance r (case c).



Fig. 16. Variations of mass concentration C with distance r (case d).

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## 8. Conclusion

From the graphs, effects of phase lags are computed and comparison of variations is made. It is observed that change in phase lags changes the behaviour of deformations of the various components of stresses, displacements, chemical mass potential function, temperature change and concentration. Small difference in phase lags results in big difference of thermal waves. A sound impact of diffusion and thermal phase-lags on the various quantities is observed. The use of diffusion phase-lags in the equation of mass diffusion gives a more realistic model of thermoelastic diffusion media as it allows a delayed response between the relative mass flux vector and the potential gradient. The result of the problem is useful in the two dimensional problem of dynamic response due to various sources of thermodiffusion which has various geophysical and industrial applications.

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