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Equivalent Resistance between Any Two Nodes in an Infinite Globe Network

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Abstract
The study on the equivalent resistance of resistor network is constantly making progress, but there are still some unresolved problems. The equivalent resistance between any two nodes in a finite globe network has been solved by one of us [Zhi-Zhong Tan, J. W. Essam, F. Y. Wu, Physical Review E. 90, 012130 (2014)]. However, the equivalent resistance between any two nodes in an infinite case has not yet been solved. This paper, by taking the limit method, gives three integral formula for the equivalent resistance between any two nodes in the infinite globe network. Six specific examples are given as simple application of this method.

1. Introduction

Ever since the German physicist Kirchhoff (1824-1888) described Kirchhoff's current law and Kirchhoff's voltage law, humans have begun to address a number of abstract and complex scientific problems by establishing the model of resistor networks [1-14]. Yet to establish a universal formula for the equivalent resistance of an arbitrary $m\times n$ resistor network has remained a difficult scientific problem [1-6]. Scholars have explored this field for over 160 years [1-14], and have gained a wealth of research achievements. Studies on the natural graphene network, certain metallic or non-metallic compound crystal structures, the structure of multiferroic magnetic materials, the structures of fullerene C_{60} and carbon nanotube, and other structures, may all need a model for constructing a resistor network. The establishment and study of resistor network models have important theoretical and practical values.

Although resistor networks have been studied for a tremendously long time, due to the inherent difficulty it was not until 2004 that a major breakthrough occurred. In 2004 [4] a theory for calculating the equivalent resistance of an arbitrary $m\times n$ resistor network was established. However, the structure of the formula obtained [4] is a bit complex, and the results are always expressed with double summation, clearly not conducive to the application. Therefore, we still need a simple calculation method.

In 2011 research on the equivalent resistance of the resistor network made a new progress. One of us [1] established various forms of resistor networks, and in particular established a new method for calculating the equivalent resistance – Recursion-transform method [13] (which is a single summation method different from the double summation method established in [4]). However, the Recursion-transform method proposed in 2011 [1] was not recognized by the international academic community until 2013 when we published an article on the cobweb in the IOP science journal [8]. In 2014, one of the researchers in our group (from China) collaborated with researchers from the United States and the United Kingdom, and used the single summation method [1] to study a globe resistor network, and obtained a satisfactory formula for the equivalent resistance [10].

Here we call the network illustrated in Fig.1 an $(m\text{-}1)\times n$ globe network. This network has m-1 latitude lines and n longitude lines. Set the south pole O as the origin of the curvilinear coordinate, and the curvilinear coordinates where node d_1 is located mark the first longitude line, and the coordinates of d_1 are $d_1(0,y_1)$. The coordinates of the other node d_2 are $d_2(x,y_2)$. Let the resistor of all unit resistors on the longitude lines be r_0, and that on the latitude lines be r. With the above definitions the formula for the equivalent resistance of a globe network in the finite condition was provided by a previous study [10]:

$$R_{mn}(d_1,d_2)=\frac{(y_2-y_1)^2}{mn}r_0+\frac{r}{m}\sum_{i=2}^{m}\frac{\cosh(nL_i)(S_{1,i}^2+S_{2,i}^2)-2\cosh[(n-2x)L_i]S_{1,i}S_{2,i}}{\sinh(2L_i)\sinh(nL_i)}. \quad (1)$$

where

$$\lambda_i=e^{2L_i},\ \overline{\lambda}_i=e^{-2L_i},\ S_{k,i}=\sin(y_k\theta_i), \quad (2)$$

and $\theta_i=(i-1)\pi/m$, $h=r/r_0$, $\lambda_i\cdot\overline{\lambda}_i=1$. And

$$\lambda_i=1+h-h\cos\theta_i+\sqrt{(1+h-h\cos\theta_i)^2-1}. \quad (3)$$

Formula (1) is generally applicable, and therefore, applies in cases where m,n are any natural numbers. However, equation (1) considers only the finite condition, and the study [10] did not provide a result for the infinite condition. The present study aims to provide an integral formula for the equivalent resistance between any two nodes in an infinite globe network by taking the limit of equation (1). Three examples are provided to illustrate a simple application of the new formula.

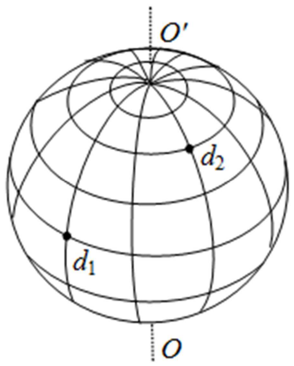

Fig. 1. *A globe resistor network, in which the unit resistors in the longitude lines and latitude lines are r_0 and r, respectively.*

2. The Main Results on an Infinite Globe Network

In the globe network shown in Figure 1, the coordinates of nodes d_1 and d_2 are $d_1(0,y_1)$ and $d_2(x,y_2)$, respectively. Let the resistor element on the longitude lines and latitude lines be r_0 and r, respectively. We have following four new results.

Result 1. When $n\to\infty$, but m is finite, we have

$$R_{m\times\infty}\{(0,y_1),(x,y_2)\}=\frac{r}{m}\sum_{i=2}^{m}\frac{(\sin y_1\theta_i-\sin y_2\theta_i)^2-2(\overline{\lambda}_i^x-1)\sin y_1\theta_i\sin y_2\theta_i}{\sqrt{(1+h-h\cos\theta_i)^2-1}}. \quad (4)$$

Equation (4) gives the equivalent resistance of a semi-infinite network, i.e., infinite on the latitude lines but finite on the longitude lines.

Result 2. When $m,n\to\infty$, but y_1 and y_2 are limited, we have

$$R_{m\times\infty}\{(0,y_1),(x,y_2)\}=\frac{r}{\pi}\int_0^{\pi}\frac{(\sin y_1\theta-\sin y_2\theta)^2-2(\overline{\lambda}_\theta^x-1)\sin(y_1\theta)\sin(y_2\theta)}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta, \quad (5)$$

where $\lambda_\theta\cdot\overline{\lambda}_\theta=1$, and

$$\overline{\lambda}_\theta=1+h-h\cos\theta-\sqrt{(1+h-h\cos\theta)^2-1}. \quad (6)$$

Result 3. When $m,n\to\infty$, and $d_1=(0,0)$ is at the south pole, then

$$R_{\infty\times\infty}(\{0,0\},\{x,y\})=\frac{r}{2\pi}\int_0^{\pi}\frac{1-\cos(2y\theta)}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta. \quad (7)$$

Result 4. When $m,n\to\infty$, and $y_1,y_2\to\infty$, then

$$R_{\infty\times\infty}\{(0,y_1),(x,y_2)\}=\frac{r}{\pi}\int_0^{\pi}\frac{1-\overline{\lambda}_\theta^x\cos(y_1-y_2)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta. \quad (8)$$

Equation (8) describes the equivalent resistance for a bi-infinite network, i.e., both the latitude lines and the longitude lines are infinite. It represents the characteristics of the equivalent resistance over a large-scale range.

3. Derivation of the Main Results

When $n\to\infty$ but m is finite, by using equation (3) to obtain $\lambda_i>1>\overline{\lambda}_i>0$, then after taking the limit:

$$\lim_{n\to\infty}\frac{\cosh(nL_i)}{\sinh(nL_i)}=1,\quad \lim_{n\to\infty}\frac{\cosh[(n-2x)L_i]}{\sinh(nL_i)}=\overline{\lambda}^x. \quad (9)$$

Because $\sinh(2L_i)=\frac{1}{2}(e^{2L_i}-e^{-2L_i})=\sqrt{(1+h-h\cos\theta_i)^2-1}$, then by taking the limit of equation (1) and substituting (9) into this limit we obtain (4).

Before proving equation (5) we need to give an integral definition: Let $\theta_k=2k\pi/m$, then $\Delta\theta_k=\theta_{k+1}-\theta_k=2\pi/m$. To take the limit as $m\to\infty$, we get

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m}f(\theta_k)=\frac{1}{2\pi}\int_0^{\pi}f(\theta)\,d\theta. \tag{10}$$

This is an identity equation that applies to any non-singular function $f(\theta_k)$.

Using equation (10) and taking the limit of equation (4) as $m\to\infty$, equation (5) can instantly be derived.

To prove equation (7), we must simplify the numerator of equation (5) as d_1 is at the south pole and the coordinate of d_2 is $(0,y)$,

$$(\sin 0-\sin y\theta)^2=\frac{1}{2}(1-\cos y\theta)$$

From this and Eq. (5), thus Eq. (7) can easily be obtained.

Next equation (8) will be proved. According to standard trigonometric functions, it is easy to obtain the following:

$$\sin(y_1\theta_i)-\sin(y_2\theta_i)=2\sin(\frac{y_1-y_2}{2}\theta_i)\cos(\frac{y_1+y_2}{2}\theta_i),$$

$$\sin(y_1\theta_i)\sin(y_2\theta_i)=\frac{1}{2}[\cos(y_1-y_2)\theta_i-\cos(y_1+y_2)\pi]. \tag{11}$$

When $m\to\infty$ and $y_1,y_2\to\infty$, and using the following conversion

$$(y_1,y_2)\Rightarrow\left(\frac{m}{2}+p,\frac{m}{2}+q\right),$$

where $p,q\ll m$ are integers, and $(p-q)=(y_1-y_2)$ is finite, we obtain

$$[\sin(y_1\theta_i)-\sin(y_2\theta_i)]^2=\begin{cases}0, & t=\text{even}\\ 2[1-\cos(y_1-y_2)\theta_i], & t=\text{odd}\end{cases}$$

$$2\sin(y_1\theta_i)\sin(y_2\theta_i)=\begin{cases}\cos(y_1-y_2)\theta_i+1, & t=\text{even}\\ \cos(y_1-y_2)\theta_i-1, & t=\text{odd}\end{cases}. \tag{12}$$

Thus, when $m,n\to\infty$, from Eq.(4) and (12) we obtain

$$\begin{aligned}&R_{\infty\times\infty}\{(0,y_1),(x,y_2)\}\\ &=\lim_{m\to\infty}\frac{r}{m}\sum_{i=2}^{m}\frac{(1-\overline{\lambda}_{2i}^{x})[1+\cos(y_1-y_2)\theta_{2i}]}{\sqrt{(1+h-h\cos\theta_{2i})^2-1}}\\ &+\lim_{m\to\infty}\frac{r}{m}\sum_{i=2}^{m}\frac{(1+\overline{\lambda}_{2i-1}^{x})[1-\cos(y_1-y_2)\theta_{2i-1}]}{\sqrt{(1+h-h\cos\theta_{2i-1})^2-1}}.\end{aligned} \tag{13}$$

Substituting equation (10) into equation (13), we have

$$\begin{aligned}&R_{\infty\times\infty}\{(0,y_1),(x,y_2)\}\\ &=\frac{r}{2\pi}\int_0^{\pi}\frac{(1-\overline{\lambda}_{\theta}^{x})[1+\cos(y_1-y_2)\theta]}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta\\ &+\frac{r}{2\pi}\int_0^{\pi}\frac{(1+\overline{\lambda}_{\theta}^{x})[1-\cos(y_1-y_2)\theta]}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta\\ &=\frac{r}{\pi}\int_0^{\pi}\frac{1-\overline{\lambda}_{\theta}^{x}\cos(y_1-y_2)\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta\end{aligned} \tag{14}$$

Equation (14) proves equation (8).

4. Six Simple Applications

Application 1. To compute the equivalent resistance between two adjacent nodes $d_1=(0,0)$ and $d_2=(x,1)$ in an infinite network, from equation (7), we get

$$\begin{aligned}&R_{\infty\times\infty}\{(0,0),(x,1)\}\\ &=\frac{r}{2\pi}\int_0^{\pi}\frac{1-\cos 2\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta\\ &=\frac{r}{2\pi}\int_0^{\pi}\frac{4\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}{\sqrt{h(1+h\sin^2\frac{\theta}{2})}}d\theta\\ &=\frac{4r_0}{\pi}\int_0^{1}\frac{x^2}{\sqrt{1+h^{-1}-x^2}}dx\\ &=\frac{2r_0}{\pi}\left((\frac{1+h}{h})\arcsin\sqrt{\frac{h}{1+h}}-\frac{1}{\sqrt{h}}\right).\end{aligned} \tag{15}$$

In particular, when $h=r/r_0=1$, equation (15) gives

$$R_{\infty\times\infty}\{(0,0),(x,1)\}=r_0(1-\frac{2}{\pi}). \tag{16}$$

Application 2. To calculate the equivalent resistance between two adjacent nodes on the same longitude line in an infinite network, where $y\to\infty$. We use equation (8), to get

$$\begin{aligned}&R_{\infty\times\infty}\{(0,y),(0,y+1)\}\\ &=\frac{r}{\pi}\int_0^{\pi}\frac{1-\cos\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta\\ &=\frac{r}{\pi}\int_0^{\pi}\frac{\sin\frac{\theta}{2}}{\sqrt{h(1+h\sin^2\frac{\theta}{2})}}d\theta=\frac{r_0}{\pi}\int_0^{1}\frac{2}{\sqrt{1+h^{-1}-x^2}}dx\\ &=\frac{2r_0}{\pi}\arcsin\left(x\sqrt{\frac{h}{1+h}}\right)_0^1=\frac{2r_0}{\pi}\arcsin\sqrt{\frac{h}{1+h}}.\end{aligned} \tag{17}$$

In particular, when $h=r/r_0=1$, equation (17) gives

$$R_1=\frac{1}{2}r_0\ . \tag{18}$$

Application 3. We shall calculate the equivalent resistance between two nodes spaced two grids apart on the same longitude line in an infinite network, where $y \to \infty$. According to equation (8), we have

$$\begin{aligned}
&R_{\infty\times\infty}\{(0,y),(0,y+2)\}\\
&=\frac{r}{\pi}\int_0^\pi \frac{1-\cos 2\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta\\
&=\frac{r}{\pi}\int_0^\pi \frac{4\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}}{\sqrt{h(1+h\sin^2\frac{\theta}{2})}}d\theta=\frac{r_0}{\pi}\int_0^\pi \frac{8x^2}{\sqrt{1+h^{-1}-x^2}}dx\\
&=\frac{4r_0}{\pi}\left(\frac{(1+h)}{h}\arcsin\sqrt{\frac{h}{1+h}}-\frac{1}{\sqrt{h}}\right).
\end{aligned} \tag{19}$$

In particular, when $h=r/r_0=1$, equation (19) gives

$$R_{\infty\times\infty}\{(0,y),(0,y+2)\}=2(1-\frac{2}{\pi})r_0. \tag{20}$$

Noticing that, when we compare equation (15) with equation (19), we find

$$R_{\infty\times\infty}\{(0,y),(0,y+2)\}=2R_{\infty\times\infty}\{(0,0),(x,1)\}. \tag{21}$$

This is an interesting identity as it is find for the first time.

Application 4. To calculate the equivalent resistance between two adjacent nodes on the same latitude line in an infinite network we use equation (8) to get the following

$$\begin{aligned}
&R_{\infty\times\infty}\{(0,y),(1,y)\}\\
&=\frac{r}{\pi}\int_0^\pi \frac{\sqrt{(1+h-h\cos\theta)^2-1}-h(1-\cos\theta)}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta.
\end{aligned}$$

By transforming equation and applying equation (17), we get

$$\begin{aligned}
&R_{\infty\times\infty}\{(0,y),(1,y)\}\\
&=\frac{r}{\pi}\int_0^\pi \frac{\sqrt{(1+h-h\cos\theta)^2-1}-h(1-\cos\theta)}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta\\
&=\frac{r}{\pi}\int_0^\pi d\theta-\frac{rh}{\pi}\int_0^\pi\left(\frac{1-\cos\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\right)d\theta\\
&=r-\frac{2r}{\pi}\arcsin\sqrt{\frac{h}{1+h}}.
\end{aligned} \tag{22}$$

In particular, when $h=r/r_0=1$, equation (22) gives

$$R_{\infty\times\infty}\{(0,y),(1,y)\}=\frac{1}{2}r. \tag{23}$$

Application 5. To compute the equivalent resistance between two nodes spaced two grids apart $d_1=(0,0)$ and $d_2=(x,2)$ in an infinite network, from equation (7), we get

$$R_{\infty\times\infty}(\{0,0\},\{x,2\})=\frac{r}{\pi}\int_0^\pi \frac{\sin^2(2\theta)}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta. \tag{24}$$

Next we calculate the integral value of Eq. (24) as follows,

$$\begin{aligned}
&R_{\infty\times\infty}(\{0,0\},\{x,2\})\\
&=\frac{r}{\pi}\int_0^\pi \frac{4\sin^2\theta\cos^2\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta\\
&=-\frac{4r}{\pi}\int_0^\pi \frac{\sin\theta\cos^2\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d(\cos\theta)\\
&=-\frac{4r}{\pi h}\int_o^\pi \cos^2\theta\sqrt{\frac{1-\cos^2\theta}{(1-\cos\theta)(2h^{-1}+1-\cos\theta)}}d(\cos\theta)\\
&=-\frac{4r}{\pi h}\int_0^\pi \cos^2\theta\sqrt{\frac{1+\cos\theta}{2h^{-1}+1-\cos\theta}}d(1+\cos\theta)\\
&=\frac{4r}{\pi h}\int_0^2 (x-1)^2\sqrt{\frac{x}{2h^{-1}+2-x}}dx\\
&=\frac{4r}{\pi h}\int_0^2 \frac{(x-1)^2x}{\sqrt{2ax-x^2}}dx\\
&=\frac{4r}{\pi h}\int_{-a}^{2-a} \frac{(t+a-1)^2(t+a)}{\sqrt{a^2-t^2}}dt \quad (a=h^{-1}+1)\\
&=\frac{4r}{\pi h}\int_{-a}^{2-a} \frac{[t^2-a^2+2t(a-1)+(a-1)^2+a^2](t+a)}{\sqrt{a^2-t^2}}dt\\
&=\frac{-4r}{\pi h}\int_{-a}^{2-a}(t+a)\sqrt{a^2-t^2}dt\\
&+\frac{4r}{\pi h}\int_{-a}^{2-a}\frac{[2(a-1)(t^2-a^2)+(2a-1)^2(t+a)]}{\sqrt{a^2-t^2}}dt\\
&=\frac{-4r}{\pi h}\int_{-a}^{2-a}t\sqrt{a^2-t^2}dt-\frac{4r}{\pi h}(3a-2)\int_{-a}^{2-a}\sqrt{a^2-t^2}dt\\
&+\frac{4r}{\pi h}(2a-1)^2\int_{-a}^{2-a}\frac{t+a}{\sqrt{a^2-t^2}}dt\\
&=\frac{4r}{3\pi h}\sqrt{(a^2-t^2)^3}\Big|_{-a}^{2-a}-\frac{4r}{\pi h}(3a-2)\\
&\times\left(\frac{t}{2}\sqrt{a^2-t^2}+\frac{a^2}{2}\arcsin\frac{t}{a}\right)\Big|_{-a}^{2-a}\\
&+\frac{4r}{\pi h}(2a-1)^2\left(-\sqrt{a^2-t^2}+a\arcsin\frac{t}{a}\right)\Big|_{-a}^{2-a}\\
&=\frac{32r}{3\pi h}\sqrt{(a-1)^3}-\frac{4r}{\pi h}(3a-2)\\
&\times\left[(2-a)\sqrt{a-1}+\frac{a^2}{2}\arcsin\frac{2-a}{a}+\frac{a^2\pi}{4}\right]\\
&+\frac{4r}{\pi h}(2a-1)^2\left[-2\sqrt{a-1}+a(\arcsin\frac{2-a}{a}+\frac{\pi}{2})\right]
\end{aligned}$$

$$=\frac{2ra}{\pi h}(5a^2-6a+2)\left(\arcsin\frac{2-a}{a}+\frac{\pi}{2}\right)$$
$$+\frac{4r}{\pi h}[\frac{8}{3}(a-1)+2-5a^2]\sqrt{a-1}$$

Since $a=h^{-1}+1$, then

$$R_{\infty\times\infty}(\{0,0\},\{x,2\})$$
$$=\frac{2r(h^{-1}+1)}{\pi h}(5h^{-2}+4h^{-1}+1)\left(\arcsin\frac{h-1}{h+1}+\frac{\pi}{2}\right)$$
$$+\frac{4r}{\pi h}\left(\frac{8h^{-1}}{3}+2-5(h^{-1}+1)^2\right)\sqrt{h^{-1}}$$
$$=\frac{2r(h+1)}{\pi h^4}(5+4h+h^2)\left(\arcsin\frac{h-1}{h+1}+\frac{\pi}{2}\right) -\frac{4r}{\pi h^4}\left(3h^2+\frac{22}{3}h+5\right)\sqrt{h}. \tag{25}$$

In particular, when $h=r/r_0=1$, Eq.(25) gives

$$R_{\infty\times\infty}(\{0,0\},\{x,2\})=4\left(5-\frac{46}{3\pi}\right)r. \tag{26}$$

*Application 6***.** To calculate the equivalent resistance between two diagonal nodes in an infinite network, where $y\to\infty$. From equation (8) we have

$$R_{\infty\times\infty}\{(0,y),(1,y+1)\}$$
$$=\frac{r}{\pi}\int_0^{\pi}\frac{1-\overline{\lambda}_\theta\cos\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta$$
$$=\frac{r}{\pi}\int_0^{\pi}\frac{1-[1+h-h\cos\theta-\sqrt{(1+h-h\cos\theta)^2-1}]\cos\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta$$
$$=\frac{r}{\pi}\int_0^{\pi}\left(\cos\theta+\frac{1-[1+h-h\cos\theta]\cos\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}\right)d\theta$$
$$=\frac{r}{\pi}(1+h)\int_0^{\pi}\frac{1-\cos\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta -\frac{rh}{\pi}\int_0^{\pi}\frac{\sin^2\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta$$
$$=\frac{r}{\pi}(1+h)\int_0^{\pi}\frac{1-\cos\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\theta -\frac{rh}{\pi}\int_{-1}^{1}\frac{\sin\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\cos\theta. \tag{27}$$

Since

$$\int_{-1}^{1}\frac{\sin\theta}{\sqrt{(1+h-h\cos\theta)^2-1}}d\cos\theta$$
$$=\int_{-1}^{1}\sqrt{\frac{1-\cos^2\theta}{(1+h-h\cos\theta)^2-1}}d\cos\theta$$
$$=\int_{-1}^{1}\sqrt{\frac{1-x^2}{h(1-x)(2+h-hx)}}dx$$
$$=\frac{1}{h}\int_{-1}^{1}\sqrt{\frac{1+x}{(2h^{-1}+1-x)}}dx$$
$$=-\frac{1}{h}\sqrt{(1+x)(2h^{-1}+1-x)}\Big|_{-1}^{1} +\frac{(h+1)}{h^2}\int_{-1}^{1}\sqrt{\frac{1}{(1+x)(2h^{-1}+1-x)}}dx$$
$$=-\frac{2}{h\sqrt{h}}+2\frac{(h+1)}{h^2}\arctan\sqrt{h}. \tag{28}$$

Substituting (28) into (27) and using (17) yields

$$R_{\infty\times\infty}\{(0,y),(1,y+1)\}$$
$$=(1+h)\frac{2r_0}{\pi}\arcsin\sqrt{\frac{h}{1+h}}$$
$$+\frac{2r}{\pi}\left(-\frac{1}{\sqrt{h}}+\frac{(h+1)}{h}\arctan\sqrt{h}\right)$$
$$+\frac{2r}{\pi}\left(-\frac{1}{\sqrt{h}}+\frac{(h+1)}{h}\arctan\sqrt{h}\right). \tag{29}$$

Since $\arcsin\sqrt{\frac{h}{1+h}}=\arctan\sqrt{h}$, from (29) we have

$$R_{\infty\times\infty}\{(0,y),(1,y+1)\}=\frac{2}{\pi\sqrt{h}}r=\frac{2}{\pi}\sqrt{r_0 r}. \tag{30}$$

By comparing equation (18) and equation (23) we see that, when $h=r/r_0=1$, the equivalent resistances between two adjacent nodes on a longitude line that are on a latitude line are the same. This reflects the basic isotropic property of equivalent resistance in an infinite globe network when $r=r_0$.

5. Conclusion

Scientists have solved many abstract and complex scientific problems by establishing resistor network models. Over the past 160 years studies in the field of resistor networks have made continuous progress, but due to the inherent difficulties some problems still remain unsolved. In addition to a method proposed earlier [1] for calculating the equivalent resistance – the single summation method provides a new theory and method for researching resistor networks, and has been applied in multiple studies [1,7,8,10-14]. In the present paper, by taking limit of the equivalent resistance in a finite globe network, three integral formulas for the equivalent resistance between any two nodes in an infinite globe network are derived. These formulae are demonstrated using three specific examples.

It should be noted in particular that, in the infinite network condition, because of the network is independent of the boundaries, equation (8) also applies to resistor networks of any rectangular configuration. In other words, the equivalent resistance between any two nodes in an infinite rectangular network (either planar or three-dimensional) can also be expressed using equation (8). Hence, the resistance formula (8) is significant for research and valuable for applications.

We emphasize that our results in this paper are original research, because our results including four main results and six examples are given for the first time, which has never been given before. You know that the original results (1) cannot represent all because many properties will not automatically run out, such as formulae (4), (5), (7) and (8) are different from formula (1). Thus our study is useful and important, which can provide convenience for future applications.

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