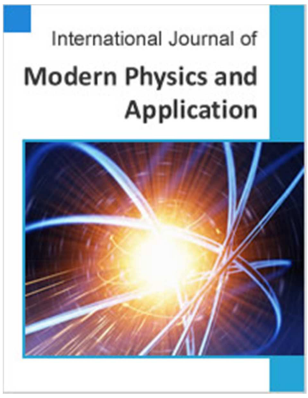
International Journal of Modern Physics and Application
2017; 4(4): 19-23
http://www.aascit.org/journal/ijmpa
ISSN: 2375-3870

American Association for Science and Technology



The Entanglement Entropy of Transverse Field Ising Model

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Citation
Xu-Dong Liu, Yan He. The Entanglement Entropy of Transverse Field Ising Model. *International Journal of Modern Physics and Application*. Vol. 4, No. 4, 2017, pp. 19-23.

Keywords
Entanglement Entropy,
Transverse Field Ising Model,
Jordan-Wigner Transformation,
Fermion Number Parity

Received: March 29, 2017
Accepted: May 3, 2017
Published: August 8, 2017

Abstract
The one dimensional transverse field Ising model is an exact solvable model which is a demonstrating example of quantum critical point. The ground state of the transverse field Ising model allows two types of partitioning, one in real space and one in momentum space, which leads to two different types of entanglement entropies. Although qualitative behavior of the two entanglement entropies away from the quantum critical point agrees, only the real-space entanglement entropy exhibits observable features at the critical point.

1. Introduction

The transverse field Ising model is a well-known model which has been studied for many years [1-5].

Although transverse field Ising model is very simple, it contains a quantum critical point which separates the ferromagnetic and paramagnetic phases, and therefore becomes a demonstrating example of many interesting phenomena.

In one dimension, it is well-known that transverse field Ising model is an exact solvable model. The exact solution of transverse field Ising model is usually based on the Jordan-Wigner transformation, which is a non-local mapping between the spin and fermion operators. Due to the non-local nature of Jordon-Wigner transformation [6], the TFIM with a periodic boundary condition cannot be exactly mapped to a free fermion model. The sign of the hopping term between the last and the first fermions will depends on whether the total fermion number is even or odd. The resulting fermion Hamiltonian is called “a-cycle” problem in [1]. In the a-cycle problem, the excitations are not independent of each other because it depends on the parity of the fermion number parity which is global information. In the previous treatment [2], one usually ignored the sign difference of the hopping term between the last and the first fermions. This approximation makes the a-cycle problem a genuine free fermion problem and is ready to be solved. The error of this approximation is infinitesimally small after the thermodynamic limit has been taken. Since the phase transition only appears after the thermodynamic limit is taken, the above treatment has been justified to discuss the physical properties of the system.

Recent years, there is a lot of experimental progress in small quantum system such as ultra-cold atoms [7, 8], trap ions [9, 10] and superconducting elements [11, 12]. These techniques make it possible to realize model systems in a finite. Also the cold atoms experiments are carried out in relative high temperature comparing to the intrinsic energy scale of the model systems. Thus the finite size effects are experimentally accessible which requires a more careful treatment of the boundary hopping term such that the result will be exact even for finite size system. In order to understand the periodic transverse field Ising model with finite size, one has to impose either a periodic or an anti-periodic boundary condition according to the even or oddness of the fermion number parity, as suggested in [13]. Because of this fermion number parity constraint, the usual Fermi distribution cannot be used to compute the finite temperature thermodynamic quantities. Fortunately, the ground state always contains even number of fermions, therefore it can be easily calculated and allows a detailed study of its entanglement properties.

The entanglement entropy quantifies internal correlations in a bipartite quantum system. It has broad applications ranging from quantum information [14] to black-hole physics [15]. The entanglement entropy of the transverse field Ising model has been analyzed and shown to have interesting scaling behavior [16, 17] by partitioning the system in real space into two parts. Interestingly, by using a Bogoliubov transformation to construct the ground state of the transverse field Ising model, we identify another bipartite structure in momentum space as well. Such a momentum-space entanglement has been discussed in non-interacting fermions [18] and superfluids [19]. We compare the entanglement entropies in real and momentum spaces and found that, although the behavior is qualitatively similar away from the quantum critical point, only the real-space entanglement entropy exhibits features indicating the critical point.

In this paper, we only consider the ferromagnetic transverse field Ising model with even number of lattice sites. We would like to mention that there are more effects due to the boundary hopping term and some of them wil survive even after taking thermodynamic limit. As pointed in [20], for anti-ferromagnetic transverse field Ising model with odd number of lattice sites, there exist the so called boundary frustration which makes the TFIM gapless in the paramagnetic phase. This point is considered with great details in [21], and we will not consider this complicated situation in the current paper.

2. Exact Solution of the Transverse Field Ising Model

The 1D Ising Model with a transverse field is given by

$$\mathrm{H} = -\mathrm{h}\sum_{i=1}^{N} S_i^z - J\sum_{i=1}^{N} S_i^x S_{i+1}^x$$

Here $S^a = \frac{1}{2}\sigma^a$ and σ^a for $a = x, y, z$ are three Pauli matrices. Here we assume the periodic boundary condition such that $S_{N+1}^a = S_1^a$. For convenience, we assume total lattice site number to be even.

To solve this spin problem, we introduce the following Jordan-Wigner transformation as

$$c_j = \exp\left(i\pi\sum_{i=1}^{j-1} S_i^+ S_i^-\right) S_j^- \quad c_j^+ = \exp\left(i\pi\sum_{i=1}^{j-1} S_i^+ S_i^-\right) S_j^+$$

Here $S_i^{\pm} = S_i^x \pm i\, S_i^y$. Then the Hamiltonian becomes

$$\mathrm{H} = \frac{hN}{2} - \mathrm{h}\sum_{i=1}^{N} c_i^+ c_i - \frac{J}{4}\sum_{i=1}^{N}(c_i^+ - c_i)\,(c_{i+1}^+ + c_{i+1})$$
$$+\frac{J}{4}\exp(i\pi N_f)\,(c_N^+ - c_N)(c_1^+ + c_1)$$

Here $N_f = \sum c_i^+ c_i$ is the total fermion number in the system. Due to the spin coupling, the total fermion number is not conserved in the fermion model. The presence of the last term makes the resulting Hamiltonian not a periodic one. In this paper, we only consider the entanglement entropy of the ground state which has even number of fermions. Therefore at zero temperature, we have $\exp(i\pi N_f) = 1$, which corresponds to the anti-periodic boundary condition.

The Hamiltonian can be cast into a form as a BCS superconductor

$$\mathrm{H} = \frac{hN}{2} + \sum_{i=1}^{N}\Big[-\frac{J}{4}(c_i^+ c_{i+1} + c_{i+1}^+ c_i) - h c_i^+ c_i$$
$$+\frac{J}{4}(c_i c_{i+1} + c_{i+1}^+ c_i^+)\Big]$$

with anti-periodic boundary condition $c_{N+1} = c_1$. Making the Fourier transformation by

$$c_j = \sum_k c_k \exp(ikj)$$

where k takes the following values

$$k \in \left\{\pm\frac{\pi}{N}, \pm\frac{3\pi}{N}, \cdots, \pm\frac{(N-1)\pi}{N}\right\}$$

The Hamiltonian becomes

$$\mathrm{H} = \frac{hN}{2} + \sum_{k>0}\Big[\xi_k(c_k^+ c_k + c_{-k}^+ c_{-k})$$
$$+ i\frac{J}{2}\sin k\, c_{-k} c_k - i\sin k\, c_k^+ c_{-k}^+\Big]$$

with $\xi_k = -\frac{J}{2}\cos k - h$.

It can be diagonalized by the Bogoliubov transformation as

$$\begin{pmatrix} c_k \\ c_{-k}^+ \end{pmatrix} = \begin{pmatrix} u_k & v_k \\ -v_k^* & u_k \end{pmatrix}\begin{pmatrix} \eta_k \\ \eta_{-k}^+ \end{pmatrix}$$

Here we assume that u_k is real and v_k is complex. Then we find

$$u_k = \sqrt{\frac{E_k + \xi_k}{2E_k}}\ v_k = i\ \text{sgn}\ k \sqrt{\frac{E_k - \xi_k}{2E_k}}$$

Here we define sgn$k$ to be the sign of k and E_k to be the quasi-particle dispersion

$$E_k = \sqrt{\left(\frac{J}{2}\right)^2 + h^2 + Jh\cos k}$$

One can see that the dispersion becomes gapless when $h = J/2$, which is the quantum critical point of transverse field Ising model.

The ground state is defined by condition $\eta_k|G\rangle = 0$ for all k. It can be rewritten as a standard BCS ground state form as

$$|G\rangle = \prod_{k>0}(u_k + v_k c_k^+ c_{-k}^+)\,|0\rangle$$

3. Entanglement Entropy in Real Space and Momentum Space

By partitioning a system into two parts in real space, the entanglement entropy between the two parts can be defined and it has been studied in the literature [22]. After transforming the transverse field Ising model into a fermion system, the ground state also exhibits interesting features in momentum space and another entanglement entropy can be introduced. In this section, we perform a detailed study of these two types of entanglement entropies.

3.1. Entanglement Entropy in Real Space

By considering two complementary parts of the lattice in real space, the system is bipartite and we follow [16, 17] to derive its real-space entanglement entropy. By calling the two parts l (with L sites) and r (with $N-L$ sites), the reduced density matrix of l is $\rho_l = Tr_r(\rho)$ where ρ is the density matrix of the whole system and the subscript of Tr means only the corresponding degrees of freedom are traced out. The entanglement entropy is then $S_L = -Tr_l(\rho_l \ln \rho_l)$.

The Majorana fermion operators $a_{2j-1} = i(c_j^+ - c_j)$ and $a_{2j} = (c_j^+ + c_j)$ are introduced to simplify the derivation. The entanglement entropy between the two parts can be extracted from the correlation function

$$\langle a_m a_j\rangle = \delta_{mj} + iC_{mj}$$

where

$$C = \begin{pmatrix} \Pi_0 & \Pi_1 & \cdots & \Pi_{L-1} \\ -\Pi_1 & \Pi_0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -\Pi_{L-1} & \cdots & \cdots & \Pi_0 \end{pmatrix}, \Pi_n = \begin{pmatrix} 0 & G(n) \\ -G(-n) & 0 \end{pmatrix}$$

And the correlation function $G(n)$ is given by

$$G(n) = h\,L(n) + \frac{J}{2}L(n+1), L(n) = \frac{1}{N}\sum_k \frac{\cos k}{E_k}$$

The above correlation matrix is anti-symmetric and can be diagonalized as

$$\tilde{C} = VCV^T = \prod_{m=0}^{L-1} \otimes\, v_m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Here $0 \le v_m \le 1$. Then, we can define a new set of $2L$ Majorana operators $\tilde{a}_m = \sum_{n=1}^{2L} V_{mn}\, a_n$ with the correlation matrix given by

$$\langle \tilde{a}_m \tilde{a}_j\rangle = \delta_{mj} + i\tilde{C}_{mj}$$

Here V_{mn} is the matrix element of V. This is equivalent to a set of L decoupled fermions $\tilde{c}_m = (\tilde{a}_{2m-1} + i\tilde{a}_{2m}^+)/2$ with the correlations

$$\langle \tilde{c}_m \tilde{c}_j\rangle = 0, \langle \tilde{c}_m^+ \tilde{c}_j\rangle = \delta_{mj}\frac{1+v_m}{2}$$

The entanglement entropy is the sum of the contributions from each independent fermion and is given by

$$S_L = \sum_{m=0}^{L-1} f\left(\frac{1+v_m}{2}\right)$$

where $f(x) = -x\ln x - (1-x)\ln(1-x)$.

3.2. Entanglement in Momentum Space

On the other hand, the system exhibits a bipartite structure in momentum space as we will explain here. The ground state has a BCS-type structure and can be cast into a Schimdt decomposition [9]:

$$|G\rangle = \prod_{k>0}(u_k + v_k c_k^+ c_{-k}^+)\,|0\rangle$$

$$= \sum_{N=0}^{\infty}\left(\prod_{\sum_k n_k = N} (u_k)^{1-n_k}(v_k)^{n_k}\,|n_k\rangle \otimes |n_{-k}\rangle\right)$$

Here u_k and v_k are given above. We have define

$$|n_k = 0\rangle = |0\rangle, |n_k = 1\rangle = c_k^+|0\rangle$$

One can verify that the coefficients satisfy

$$\sum_{N=0}^{\infty}\left|\prod_{\sum_k n_k = N}(u_k)^{1-n_k}(v_k)^{n_k}\right|^2 = \prod_{k>0}\sum_{n_k=0}^{1}|u_k|^{2(1-n_k)}|v_k|^{2n_k} = 1$$

Due to this structure, the momentum space is bisected into regions with $k > 0$ and $k < 0$, which will be called A and B respectively. The reduced density matrix for A is defined as $\rho_A = Tr_B|G\rangle\langle G|$, which is given by

$$\rho_A = \sum_{N=0}^{\infty}\left(\prod_{\sum_k n_k=N}(u_k)^{2(1-n_k)}(v_k)^{2n_k}\,|n_k\rangle\langle n_k|\right)$$

A direct computation of the entanglement entropy from ρ_A can be quite complicated, but we can circumvent this difficulty by considering

$$\mathrm{Tr}_A(\rho_A^n) = \sum_{N=0}^{\infty}\left|\prod_{\sum_k n_k=N}|u_k|^{2(1-n_k)}|v_k|^{2n_k}\right|^2 = \prod_{k>0}\sum_{n_k=0}^{1}|u_k|^{2(1-n_k)n}|v_k|^{2n_k n}$$

The entanglement entropy in momentum space is then given by

$$S_A = -Tr_A(\rho_A \ln\rho_A) = -\lim_{n\to 1}\frac{d}{dn}Tr_A(\rho_A^n) = \sum_{k>0}(|u_k|^2\ln|u_k|^2 + |v_k|^2\ln|v_k|^2)$$

4. Results and Discussions

In Figure 1 we plot the two types of entanglement entropies of the transverse field Ising model ground state with PBC in real and momentum spaces. The fermion number constraint does not play a significant role in the ground state, so the results from different boundary conditions are essentially the same. For the real-space entanglement entropy, we bisect the system so that the subsystem size is $L = N/2$ (the black curve). This choice is to allow a fair comparison to the momentum-space entanglement entropy of the $k > 0$ states (the red line), which naturally bisects the system in momentum space.

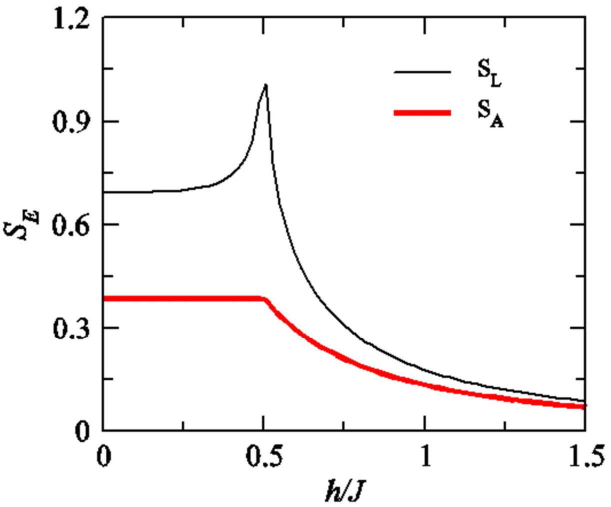

Figure 1. The entanglement entropies of the ground state from real-space partition, SL (black), and from momentum-space partition, SA (red), as functions of h/J. The subsystem size for SL is chosen as N/2, and here N = 20 lattice sites for both cases.

The real-space entanglement entropy has been shown to be proportional to the logarithm of the subsystem size at the critical point [23]. Moreover, it should diverge at the critical point in the thermodynamic limit. In a small system with $N = 20$ lattice sites, a sharp peak in the real-space entanglement entropy is already observable at $h = J/2$. On the other hand, the momentum-space entanglement entropy evolves smoothly across the critical point. Away from the critical point, however, the two entanglement entropies show qualitatively similar behavior. The transverse field Ising model thus provides an interesting example of a quantum system supporting several types internal entanglement. From the results presented here, different entanglement may react to a critical point differently.

Measurements of internal entanglement in many-body systems have advanced significantly with interesting experimental results [24], and the different types of entanglement entropy for the transverse field Ising model may find broader applications in other quantum systems.

Acknowledgements

We thank Peng Li and Jian-Jun Dong for stimulating discussions. Yan He is supported by NSFC under grant No. 11404228.

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