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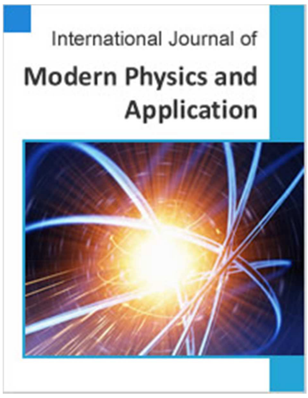

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General Field Equation for Electromagnetism and Gravitation

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Abstract

Einstein showed that, the effect of gravitational field on a space-time is explained mathematically using Ricci tensor. Also, it is clear that the effect of electromagnetic filed on a space-time is explained with electromagnetic tensor which satisfies Maxwell's equations. In real physics world, both electromagnetic and gravitational fields exist in a space-time simultaneously. So, the space-time should be considered, simultaneously using two second rank tensors. In this manuscript, a new approach for writing a general field equation for both gravitation and electromagnetism in a four dimensional space-time is proposed. As a result, a relationship between electromagnetism and gravitation is obtained.

1. Introduction

Since 1914, that Albert Einstein proposed a new and complete definition of gravitation, there have been several attempts to unify electromagnetism and gravitation, included Kaluza theory with the fifth dimension and some attempts in the domain of Quantum mechanics. But all of them were unsuccessful in unifying electromagnetism with gravitation. For example, some years after the suggestion of Kaluza theory, at 1939 Einstein refused to accept his theory in a letter mentioning that, his research is frustrated: As no arbitrary constants occur in the equations, the theory would lead to electromagnetic and gravitation fields of the same order of magnitude. Therefore one would be unable to explain the empirical fact that the electrostatic force between two particles is so much stronger than the gravitational force. This means that a consistent theory of matter could not be based on these equations, [1].

Einstein, himself, tried to do this unification, as he said: It would be a great step forward to unify in a single picture the gravitational and electromagnetic fields, then there would be a worthy completion of the epoch of theoretical physics, [2].

Einstein was trying to unify these two fields in a theory based on fields and not particles, as it has been mentioned in his paper in 1935: A complete field theory knows only fields and not the concepts of particles and motions, [3]. Actually, Einstein wanted the fields to be absorbed in geometry and to formulate electromagnetism in geometry, like gravitation. But electromagnetism has not been absorbed in geometry in none of the previous theories.

Einstein showed that, the effect of gravitational field on a space time is explained with a symmetric rank 2 tensor, namely Ricci tensor. In addition, it is known that, the effect of electromagnetic filed on a space-time is explained with a rank 2 antisymmetric tensor, which satisfies Maxwell's equations.

In real physics world, both electromagnetic and gravitational fields exist in a space-time simultaneously. Therefore, the space-time should be considered, simultaneously by two second rank tensors which one of them is symmetric and the other one is antisymmetric. But what is the relationship between these tensors and how can they be introduced in a single equation? In the following, a new way is proposed for writing a general field equation for gravitation and electromagnetism in a four dimensional space-time while considering the above mentioned tensors.

2. Mathematical Background

The class of totally antisymmetric tensors is an important class of tensors of type (0, s). This class contains covariant tensors with antisymmetric property in every pair of their arguments, i.e.

$$T(X_1,...,X_\mu,...,X_\upsilon,...,X_s)=-T(X_1,...,X_\upsilon,...,X_\mu,...,X_s) \quad (1)$$

for all pairs of indices μ and υ and for all X's [4]. By applying the alternating operator A to a general tensor T of type (0,s), this kind of tensor can be formed. Applying operator A to T give the linear combination defined by

$$AT(X_1,...,X_s)=\frac{1}{s!}\sum_{\upsilon_1,...,\upsilon_s}\mathrm{sgn}(\upsilon_1,...,\upsilon_s)T(X_{\upsilon_1},...,X_{\upsilon_s}) \quad (2)$$

where in this summation, $(\upsilon_1,...,\upsilon_s)$ are an even or an odd permutation of (1,…,s) integer numbers, and based on that, $\mathrm{sgn}(\upsilon_1,...,\upsilon_s)=\pm 1$, and equation (2) is to be valid for every $(X_1,...,X_s)$.

When T is totally antisymmetric, applying the A operator to it simply reproduces T. However, when $s\rangle n$ (the dimension of the vector space), applying the A operator to T reduces T to zero; simply put, there is no totally antisymmetric tensor of type (0, s) for $s\rangle n$.

Antisymmetric tensors of type (0,s) are called s-forms. Since they must vanish when any two of their arguments coincide, it follows that the dimension of the vector space for s-forms can be obtained using: $\frac{n!}{s!(n-s)!}$.

This space is denoted by $\Lambda^s T_p^*$.

By applying the A operator to the basis elements of the following tensor product, a basis for $\Lambda^s T_p^*$ can be obtained:

$$A(e^{\upsilon_1}\otimes...\otimes e^{\upsilon_s}) \quad (3)$$

The resulting basis elements can be written as the exterior or the wedge product of the e^{υ}, s as the following:

$$e^{\upsilon_1}\wedge e^{\upsilon_2}\wedge....\wedge e^{\upsilon_s} \quad (\upsilon_1\rangle\upsilon_2\rangle....\rangle\upsilon_s) \quad (4)$$

By extending the summation only over strictly descending sequences, a general s-form can be written as:

$$\Omega=\Omega_{\upsilon_1...\upsilon_s}e^{\upsilon_1}\wedge e^{\upsilon_2}\wedge...e^{\upsilon_s} \quad (\upsilon_1\rangle\upsilon_2\rangle....\rangle\upsilon_s) \quad (5)$$

Considering that interchanging a pair of indices is equal to interchanging the corresponding elements in the wedge product, it can be deduced that interchanging the elements in a wedge product must be accompanied by a change of sign:

$$e^\upsilon\wedge e^\tau=-e^\tau\wedge e^\upsilon \quad (6)$$

The expression for an s-form in a local coordinate basis is:

$$\Omega=\Omega_{\upsilon_1...\upsilon_s}dx^{\upsilon_1}\wedge...\wedge dx^{\upsilon_s} \quad (7)$$

To obtain a (p+q) form, the wedge product of any p-form Ω^1 and a q-form Ω^2 can be formed by the rule

$$\Omega^1\wedge\Omega^2=A(\Omega^1\otimes\Omega^2) \quad (8)$$

which must accordingly vanish identically if $(p+q)\rangle n$.

By definition,

$$\Omega^1\wedge\Omega^2=(\Omega^1_{\upsilon_1...\upsilon_p}e^{\upsilon_1}\wedge...\wedge e^{\upsilon_p})\wedge(\Omega^2_{\tau_1...\tau_q}e^{\tau_1}\wedge...\wedge e^{\tau_q}) \quad (9)$$

where $(\upsilon_1,...,\upsilon_p)$ and $(\tau_1,...,\tau_q)$ are strictly descending sequences. Since each of the q basis elements $e^{\tau_1},...,e^{\tau_q}$ must go through p interchanges before $\Omega^1\wedge\Omega^2$ can be brought to the form required of $\Omega^2\wedge\Omega^1$, consequently:

$$\begin{aligned}\Omega^1\wedge\Omega^2&=(-1)^{pq}(\Omega^2_{\tau_1...\tau_q}e^{\tau_1}\wedge...\wedge e^{\tau_q})\wedge(\Omega^1_{\upsilon_1...\upsilon_p}e^{\upsilon_1}\wedge...\wedge e^{\upsilon_p})\\&=(-1)^{pq}\Omega^2\wedge\Omega^1\end{aligned} \quad (10)$$

If we choose a suitable coordinate, it will be seen that our electrogravity (EG) equations, will be appropriate for infinitely small four dimensional regions. Let x_1, x_2 and x_3 be the space coordinates and x_4 be the time coordinate in appropriate unit. Here the appropriate unit is the coordinate in which the time unit is chosen so that the light speed is equal unit (c=1) in the local coordinate. If a unit measure is chosen, the coordinates with a given orientation of the coordinates have direct physical meaning in the sense of the theory of relativity. In theory of relativity the following expression, has a value which is independent of the orientation of the local system of coordinates:

$$ds^2=-dx_1{}^2-dx_2{}^2-dx_3{}^2+dx_4{}^2 \quad (11)$$

Let *ds* be the magnitude of linear element pertaining to points of the four-dimensional continuum in infinite proximity. To the mentioned linear element or to the two infinitely proximate point events, there are correspond definite differentials dx_1, dx_2, dx_3, and dx_4. In this system the dx_υ is represented here by definite linear homogeneous expression of the dx_σ:

$$dx_{\upsilon} = \sum_{\sigma} \alpha_{\upsilon\sigma} dx_{\sigma} \tag{12}$$

Inserting these expressions in above equation, we obtain:

$$ds^2 = \sum_{\tau\sigma} g_{\sigma\tau} dx_{\sigma} dx_{\tau} \tag{13}$$

where $g_{\sigma\tau}$ are functions of x_{σ}. These are independent from the orientation and the state of motion of the local system of the coordinates. *ds* is independent of any particular choice of coordinates.

If it is possible to choose the system of coordinate in the finite region in such a way that the $g_{\mu\nu}$ has constant values:

$$g_{\mu\upsilon} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{bmatrix} \tag{14}$$

It will be seen that a free material point moves, relatively to this system, with uniform motion on a straight line. But if a new space-time coordinates x_1, x_2, x_3 and x_4, is chosen, the $g_{\mu\nu}$ in the new system will not be constant, but functions of space and time, and the motion of free material point will be a curvilinear motion. This motion must be interpreted as a motion under the influence of the EG field. So it can be found the occurrence of an EG field connected with the space-time variables of $g_{\mu\upsilon}$. So the $g_{\mu\upsilon}$ representing the EG field at the same time define the metrical properties of the space-time.

Now a four-vector can be defined using $g_{\mu\upsilon}$. Let's use the following metric for this purpose:

$$ds^2 = g_{tt}dt^2 - g_{rr}dr^2 - g_{\theta\theta}d\theta^2 - g_{\varphi\varphi}d\varphi^2, \tag{15}$$

A four vector, h_{μ}, can be defined using $g_{\mu\upsilon}$ as following:

$$h_{\mu} = (h_1, h_2, h_3, h_4) \text{ where, } h_1 = \sqrt{g_{tt}}, h_2 = \sqrt{g_{rr}},$$

$$h_3 = \sqrt{g_{\theta\theta}}, h_4 = \sqrt{g_{\varphi\varphi}} \text{ or } h_{\mu} = \left(\sqrt{g_{\mu\mu}}\right) \tag{16}$$

As an example, for Schwarzschild metric:

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{(1 - \frac{2m}{r})}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2 \tag{17}$$

This four-vector will be:

$$h_{\mu} = (\sqrt{1 - \frac{2m}{r}}, \frac{1}{\sqrt{1 - \frac{2m}{r}}}, r, r\sin\theta) \tag{18}$$

And for any arbitrary metric like

$$\begin{aligned} ds^2 = {} & g_{11}dx_1{}^2 + g_{12}dx_1dx_2 + g_{13}dx_1dx_3 + g_{14}dx_1dx_4 + \\ & g_{21}dx_2dx_1 + g_{22}dx_2{}^2 + g_{23}dx_2dx_3 + g_{24}dx_2dx_4 + \\ & g_{31}dx_3dx_1 + g_{32}dx_3dx_2 + g_{33}dx_3{}^2 + g_{34}dx_3dx_4 + \\ & g_{41}dx_4dx_1 + g_{42}dx_4dx_2 + g_{43}dx_4dx_3 + g_{44}dx_4{}^2 \end{aligned} \tag{19}$$

The four-vector will be:

$$h_{\mu} = (\sqrt{g_{11}}, \sqrt{g_{22}}, \sqrt{g_{33}}, \sqrt{g_{44}}) \tag{20}$$

As $g^{\alpha\tau}R^{\mu}_{\upsilon\sigma\tau} = R^{\mu\alpha}_{\upsilon\sigma}$ where $R^{\mu}_{\upsilon\sigma\tau}$ is the Riemann tensor, one can write:

$$h_{\alpha}R^{\mu\alpha}_{\upsilon\sigma} = R^{\mu\alpha}_{\upsilon\sigma\alpha} \tag{21}$$

By contracting (21), twice, it is obtained: $R^{\mu\alpha}_{\upsilon\mu\alpha} = R_{\upsilon}$, where we call curvature four vector to R_{υ}. Now to find the geometrical form of the antisymmetric tensor, which was mentioned in introduction, and its relationship with gravitational geometry, we define $S_{\mu\upsilon}$ using the following wedge product:

$$S_{\mu\upsilon} = R_{\mu} \wedge h_{\upsilon} \tag{22}$$

where $S_{\mu\upsilon}$ is an antisymmetric tensor. Now the EG tensor can be defined as following:

$$\frac{\partial S_{\mu\upsilon}}{\partial x_{\sigma}} - \Gamma^{\tau}_{\mu\sigma}S_{\tau\upsilon} - \Gamma^{\tau}_{\sigma\upsilon}S_{\mu\tau} + h_{\mu}R_{\sigma\upsilon} \tag{23}$$

where $R_{\sigma\upsilon}$ is the Ricci tensor.

3. The EG Field Equations in the Absence of Matter

The mathematical importance of the above mentioned EG tensor is that, if there is a coordinate system with reference to which the $g_{\mu\upsilon}$ are constant (K_0 coordinate), then all components of the EG tensor will vanish. If any new system of coordinates (K_1), is chosen, in place of the original one (K_0), the $g_{\mu\upsilon}$ will not be constant, but in consequence of its tensor nature, the transformed components of the EG tensor will still vanish in the new system (K_1). Relatively to this system, all components of the EG tensor vanish in any other system of coordinates. Thus the required equations of the matter-free EG field in any case must be satisfied if all components of the EG tensor vanish.

So using eq. 23, the equations of matter-free field are:

$$\frac{\partial S_{\mu\upsilon}}{\partial x_{\sigma}} - \Gamma^{\tau}_{\mu\sigma}S_{\tau\upsilon} - \Gamma^{\tau}_{\sigma\upsilon}S_{\mu\tau} + h_{\mu}R_{\sigma\upsilon} = 0 \tag{24}$$

In approximation, for considering the electromagnetic field behaviors, it is better to focus on microscopic scales. If some electric charges are placed in the media, it is clear that, in this scale the effects of gravitational field are so much smaller than electromagnetic field, so the gravitational field can be ignored here. As the existing mass (the mass of electric charges m') is very small ($m' \to 0$), then $T_{\upsilon\sigma} \to 0$ where $T_{\upsilon\sigma}$ is the energy-momentum tensor, thus according to the Einstein field equation: $R_{\sigma\upsilon} \to 0$ and expression (23) turns into:

$$\frac{\partial S_{\mu\upsilon}}{\partial x_\sigma} - \Gamma^\tau_{\mu\sigma} S_{\tau\upsilon} - \Gamma^\tau_{\sigma\upsilon} S_{\mu\tau} \qquad (25)$$

Using 25, in addition, the two following tensors can be written:

$$\frac{\partial S_{\upsilon\sigma}}{\partial x_\mu} - \Gamma^\tau{}_{\mu\upsilon} S_{\tau\sigma} - \Gamma^\tau{}_{\mu\sigma} S_{\upsilon\tau} \qquad (26)$$

$$\frac{\partial S_{\sigma\mu}}{\partial x_\nu} - \Gamma^\tau{}_{\upsilon\sigma} S_{\tau\mu} - \Gamma^\tau{}_{\upsilon\mu} S_{\sigma\tau} \qquad (27)$$

From eq. 24, it is seen that all components of the EG tensor vanished, so the summation of the above 3 tensors (25, 26 and 27) must be vanished also. Now adding 25, 26 and 27, it is obtained:

$$\begin{aligned}&\frac{\partial S_{\mu\upsilon}}{\partial x_\sigma} + \frac{\partial S_{\upsilon\sigma}}{\partial x_\mu} + \frac{\partial S_{\sigma\mu}}{\partial x_\upsilon} - \Gamma^\tau{}_{\mu\sigma} S_{\tau\upsilon} - \Gamma^\tau{}_{\sigma\upsilon} S_{\mu\tau} - \\ &\Gamma^\tau{}_{\mu\upsilon} S_{\tau\sigma} - \Gamma^\tau{}_{\mu\sigma} S_{\upsilon\tau} - \Gamma^\tau{}_{\upsilon\sigma} S_{\tau\mu} - \Gamma^\tau{}_{\upsilon\mu} S_{\sigma\tau} = 0\end{aligned} \qquad (28)$$

and using $S_{\mu\upsilon} = -S_{\upsilon\mu}$ in equation (28):

$$\frac{\partial S_{\mu\upsilon}}{\partial x_\sigma} + \frac{\partial S_{\upsilon\sigma}}{\partial x_\mu} + \frac{\partial S_{\sigma\mu}}{\partial x_\upsilon} = 0 \qquad (29)$$

4. The General Form of the EG Field Equations

The field equations (eq. 24), which are obtained for matter free space-time, are to be compared with the field equation $\nabla^2\varphi = 0$ of Newton's theory or $R_{\mu\upsilon} = 0$ of Einstein gravity field equations in vacuum. We require the equation corresponding to Poisson's equation: $\nabla^2\varphi = 4k\pi\rho$ or

$$R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} R = -kT_{\mu\upsilon} \qquad (30)$$

of Einstein general form of the gravitational field equation. For this purpose $T_{\mu\upsilon\sigma}$ is defined as following:

$$T_{\mu\upsilon\sigma} = g_{\alpha\upsilon}(kT^\alpha_\sigma + k'T'^\alpha_\sigma)h_\mu \qquad (31)$$

Where k and k' are two constants related to the gravity and electromagnetism respectively, and

$$T_{\upsilon\sigma} = g_{\alpha\upsilon} T^\alpha_\sigma \qquad (32)$$

is the energy-momentum tensor and T'^α_σ is the electromagnetic energy tensor. Thus, instead of eq. 24 we write:

$$\frac{\partial S_{\mu\upsilon}}{\partial x_\sigma} - \Gamma^\tau_{\mu\sigma} S_{\tau\upsilon} - \Gamma^\tau_{\sigma\upsilon} S_{\mu\tau} + h_\mu R_{\sigma\upsilon} - \frac{1}{2} g_{\sigma\upsilon} h_\mu R = -T_{\mu\upsilon\sigma} \qquad (33)$$

Therefore equation (33) is the required general form of the EG field equations.

Equation (33) in its special forms easily gives us the general form of the Einstein field equations in general relativity and all Maxwell's equations as following.

As Einstein has mentioned in his paper, (The Foundation of the General Theory of Relativity), The Christoffel symbols are used as the field components of gravitation in general relativity. So, in the absence of matter, by approximation, if the gravitational field is very small, then: $\Gamma^\tau_{\sigma\upsilon} \to 0$, and according to the Einstein vacuum equation: $R_{\sigma\upsilon} \to 0$. For example, it can be reached to this approximation in a region without any matter unless some electrical charges, like electron, that have a very small mass. So in the mentioned approximation, eq. 33 turns into:

$$\frac{\partial S_{\mu\upsilon}}{\partial x_\sigma} = -T_{\mu\upsilon\sigma} \qquad (34)$$

Let's suppose that, we are working in a region that the total electric charge density of the area is ρ_q and the current density is j_i . In one special case, we suppose that all components of $T_{\mu\nu\sigma}$ are zero except the four components of $T_{4\upsilon 4}$. Now a 4_vector $T_{4\upsilon 4}$ can be defined as:

$$T_{4\upsilon 4} = k(\rho_q, j_1, j_2, j_3) = j_\upsilon \qquad (35)$$

Now from eqs. 34 and 35:

$$\frac{\partial S_{\mu\upsilon}}{\partial x_\mu} = -k(\rho_q, j_1, j_2, j_3) = j_\upsilon \Rightarrow \frac{\partial S_{\mu\upsilon}}{\partial x_\mu} = j_\upsilon \qquad (36)$$

where $S_{\mu\nu}$ is an antisymmetric tensor.

Therefore, from eqs. 29 and 36, on identifying $S_{\mu\upsilon}$ with the electromagnetic tensor $F_{\mu\upsilon}$, one will recognize eqs. 29 and 36 as the Maxwell's equations. Therefore, $S_{\mu\upsilon}$ can be identified as the electromagnetic tensor ($F_{\mu\upsilon}$).

Moreover, the equation 33 in its special form easily gives us the general form of the Einstein field equations for gravitation.

5. Conclusion

In this manuscript, a unified field equation (eq. 33), for both electromagnetism and gravity (Electrogravity field equation) is presented. From general relativity it is well-known that, the Christoffel symbols are considered as the field components of gravitation. So, in this research, it has been found a relationship between the gravitation and electromagnetism. We have found $F_{\alpha\beta}$ in terms of $\Gamma^{\tau}_{\mu\upsilon}$ as following:

$$F_{\alpha\beta} = g^{\mu\upsilon} h_{\alpha} \left(\frac{\partial \Gamma^{\tau}_{\mu\upsilon}}{\partial x_{\tau}} - \frac{\partial \Gamma^{\tau}_{\mu\tau}}{\partial x_{\upsilon}} - \Gamma^{\tau}_{k\tau}\Gamma^{k}_{\mu\upsilon} + \Gamma^{\tau}_{k\upsilon}\Gamma^{k}_{\mu\tau} \right) \wedge h_{\beta} \quad (37)$$

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