International Journal of Mathematical Analysis and Applications
2014; 1(1): 20-26
Published online March 10, 2014 (http://www.aascit.org/journal/ijmaa)


**Keywords**
Solitons,
Wavelet Transform,
Normal Probability Distribution,
Random Processes

Received: February 16, 2014
Revised: March 01, 2014
Accepted: March 02, 2014

# Wavelet analysis of randomized solitary wave solutions

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## Citation

Bharat Bhosale. Wavelet Analysis of Randomized Solitary Wave Solutions. *International Journal of Mathematical Analysis and Applications.* Vol. 1, No. 1, 2014, pp. 20-26

## Abstract

Many physical phenomena are modeled with nonlinear partial differential equations that possess solitary wave solutions called solitons. Solitons exhibit Gaussian forms and in turn engender normal probability distributions. Moreover, solitons fluctuate randomly during evolution. These features intricately relate the solitons to wavelets, statistical distributions and random processes. In the present work, solitons arising as the solutions of Sine-Gordon equation, in particular, are studied from different perspectives treating the soliton evolution as random process. Moreover, useful statistical quantities are computed. In the end, potential applications in spectral data processing involving soliton evolution are explored.

## 1. Introduction

It is apparently seen that the wavelet analysis has entered the realm of applied science via its acoustic and geophysical applications [8, 13]. Now a days, wavelet transform is emerged as the most effective technique for signal processing and image analysis as an alternative to Fourier analysis especially when the signals are random, comprised of fluctuations of different scales and where the very short and very long waves are present in the same signal. Recently, the increasing interest of statistical community to wavelet transform and related techniques is motivated by a number of reasons; the foremost being the one that wavelet transform does not appeal to differentiability, and is thus suitable for the study of continuous but non-differentiable processes exhibiting self similar patterns like fractional Brownian motion which is the most fundamental and interesting random processes [8]. The random processes can be modeled as Gaussian process, whose importance is due to the vital statistical properties inherited from normal probability distribution. In science and engineering, numerous problems deal with time waveforms or signals. The signals are either deterministic or random. The deterministic signals are usually described by mathematical functions. However, a random signal is not purely given by the value of the time because the signal is often accompanied by undesired random waveforms, for example the noise. In signal/image processing, the main task is to separate signals from noisy background, that is, to separate all higher peaks shed mixed intricately with the small ones from the signal/image to be processed. Wavelet decomposition works efficiently for analyzing the signals. In earlier works, the wavelet interaction with solitons arising as the solutions of non-linear partial differential equations viz. Non-linear Schrodinger Equation (NLS), Sine-Gordon equation (SG), Korteweg-de Vries equation (KdV) [2, 4, 5] have been studied. Also, a strong relationships existing between wavelets, solitons and probability distributions [3] has been extensively studied. These studies provided a theoretical framework for soliton analysis in wavelet setup. Most of the solitons have a Gaussian representation [12] and Gaussians themselves are the normal probability distributions [6]. Since the amplitude of the soliton fluctuates randomly during evolution, we can treat soliton as a random process characterized by self similar patterns. Self-similarity is a synonym of scale-invariance, meaning thereby having same properties at different scales; the classical example of which are the fractals. These features of the soliton motivated us to undertake present study involving wavelet analysis of soliton-like randomized processes. In this work, theoretical ideas related to continuous wavelet decomposition of random processes are presented. Besides the random processes, the theory so developed has potential applications in spectroscopy, high energy particle detectors, electron density clouds, nuclear data processing; wavelet based neural networks, image recognition, time series prediction etc.

## 2. Mathematical Pre-Requisites

Random variable or stochastic variate: Let $S$ is a sample space, that is, the set of all possible outcomes of random experiment $E$ and $B$ is a σ-field of subsets of $S$.

A function $X: S \to R$ is called a random variable if the inverse image under $X$ of all semi-closed intervals of the form$(-\infty, x]$, where $x \in R$ are events in $B$, that is,

$$X^{-1}(-\infty, x] = \{\omega \in S: X(\omega) \leq x\} \in B.$$

Distribution function of a random variable: Let $(S, B, P)$ be a probability space and $X$ be a random variable defined on $S$ . The function $F: R \to R$ defined by $F(x) = P(-\infty, x) = P\{\omega \in S: X(\omega) \leq x\}, \forall x \in R$ is called distribution function of the random variable.

The characteristic function of a random variable $X$ for continuous probability function $f(x)$ is defined as $\varphi_X(t) = E(e^{itX}) = \int_{-\infty}^{\infty} e^{itX} f(x)dx = \int_{-\infty}^{\infty} e^{itx} dF(x)$

Random processes: A random variable $X$ assigns a real number $X(s)$ to each outcome $s$ of the experiment, whereas, a random process $X: X(t, s)$ assigns a real function of time $t$ to each outcome $s$ of the experiment whose statistical properties such as mean value, moments, and variance can be described. Moreover, random process is a Gaussian process whose realization consists of random values associated with every point in a range of time such that each such random variable has a normal distribution and every finite collection of those random variables has a multivariate normal distribution.

Normal distribution: The normal distribution, also called Gaussian distribution is a continuous probability distribution.

A continuous random variable $X$ with parameters $\mu$ and $\sigma$, where $-\infty \leq \mu \leq \infty$ and $\sigma > 0$, is said to have a normal distribution or Gaussian distribution if its probability density function is given by

$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ , where $-\infty < x < \infty$

If $X$ is a normal random variable with parameters $\mu$ and $\sigma$, which is denoted by $X \sim N(\mu, \sigma^2)$, then the standard normal variable is defined by transformation $Z = \frac{X-\mu}{\sigma}$. This implies that the standard normal variable $Z$ has mean $\mu_Z = 0$ and variance $\sigma^2 = 1$. Hence, the probability density function of $Z$, called standard normal density, is given by

$\varphi(Z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$ , where $-\infty < x < \infty$.

This implies $f(x) = \frac{1}{\sigma} \varphi\left(\frac{x-\mu}{\sigma}\right)$

The normal probability integral or the area under the normal curve gives the probability $P$ for the interval from the mean to the value $x$ and is given by the definite integral

$P = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{z^2}{2}} dx$

Here, the value of $P$ given by the above integral is also called the error function.

Since the normal distribution is symmetric about mean, the maximum value of $f(x)$ occurs at the point $x = \mu$ and is equal to $f_{max} = [f(x)]_{x=\mu} = \frac{1}{\sigma\sqrt{2\pi}}$

## 3. Continuous Wavelet Transform

The Continuous Wavelet transform (CWT) is a decomposition of a function, $f(x)$, with respect to a basic wavelet $\psi(x)$, given by the convolution of a function with a scaled and translated version of $\psi(x)$

$$W_\psi(a, b)[f] = |a|^{-1/2} \int f(x) \psi^*\left(\frac{x-b}{a}\right) dx \qquad (1)$$

$$= \langle f, \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \rangle = \langle f, \psi_{a,b} \rangle$$

$$= \langle f, U(a, b)\psi \rangle = W_\psi f(a, b)$$

where $\langle ., . \rangle$ is the inner product.

The functions, $f$ and $\psi$ are square integrable functions and $\psi$ satisfies the admissibility condition

$$C_\psi = \int \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty.$$

Subscript '*' denotes complex conjugation, '$a$' is the scale parameter, $a > 0$, '$b$' is the translation parameter. The term $1/\sqrt{|a|}$ is the energy conservative term that keeps energy of the scaled mother wavelet equal to the energy of the original wavelet [14].

The Inverse wavelet transform is

$$f(x) = \frac{1}{C_\psi} \int \int W_\psi f(a, b) \frac{1}{\sqrt{|a|}} \psi\left(\frac{x-b}{a}\right) \frac{dadb}{a^2} \qquad (2)$$

where the admissibility constant, $C_\psi > 0$.

Therefore, any function with compact support which satisfies above requirements can be successfully used as a basic wavelet. For example, the derivatives of Gaussian $exp\,(-x^2)$ can be chosen as basic wavelet as it satisfies the admissibility requirements but not the Gaussian itself.

The spectral representation of CWT is obtained by making substitution for $f(x)$ as the inverse Fourier Transform

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(i\omega x)\,\hat{f}(\omega)d\omega$ in the definition of wavelet transform (1)

$$W_\psi[f(x)](a,b) = \frac{1}{2\pi}|a|^{1/2}\int_{-\infty}^{\infty} \exp(i\omega b)\,\overline{\hat{\psi}(a\omega)}\,\hat{f}(\omega)d\omega. \quad (3)$$

This spectral representation of CWT (3) is useful for evaluating the transform integral in more convenient way.

# 4. Sine-Gordon Equation and its Solitary Wave Solution

The non-linear partial differential equation

$$q_{tt} - q_{xx} + \sin q = 0 \quad (4)$$

is a sine-Gordon equation, where $q = q(x,t)$ represents the phase jump of the wave function in space-time frame $(x,t)$.

The solution $q(x,t)$ of the SGE (4) is

$$q(x,t) = 4\,tan^{-1}\left[exp\left(\pm\frac{x-vt}{\sqrt{1-v^2}}\right)\right] \quad (5)$$

This can also be expressed as

$q(x,t) = q(s)$ with $s = x - vt$.

The expression (5) represents a localized solitary wave called soliton, travelling at a velocity$|v| < 1$.

The SGE with its soliton solution in the form of kinks, breathers, and phonons is well studied in [11]. Note also that the amplitude of the solitary wave randomly fluctuates during the evolution. This feature of the soliton has been particularly used in sequel.

We can write expression (5) as

$$q(x,t) = 4\,tan^{-1}\left[exp\left(\pm\frac{x-vt}{\sqrt{1-v^2}}\right)\right] = \psi(x,t) = \psi\left(\frac{x-b}{a}\right) \quad (6)$$

$\psi(x,t)$ in the above form can be viewed as the function scaled by $a = \sqrt{1-v^2}$ and translated by $b = vt$ in space-time frame $(x,t)$. This we may refer as soliton wavelet considered in later discussion.

The soliton $q(x,t)$ can also be written as the Gaussian integral

$q(x,t) = \int_{-\infty}^{\infty}\hat{q}(k,t)e^{ikx}dk$, where $\hat{q}(k,t)$ is Fourier transform of $q(x,t)$.

Evaluating this integral, we get the Gaussian in the form of normal probability distribution

$$q(x,t) = \frac{1}{2\sqrt{2\pi kt}}e^{-\frac{x^2}{4kt}} \equiv \psi(x,t) = \psi(s) = Ne^{Q(s)} \quad (7)$$

But, the Gaussian function itself is a normal probability density function. Thus, on comparing it with normal probability density function in standard form, we can say that the Gaussian representation (7) corresponds to the normal probability density function of $x$ with mean zero and variance $2kt$.

Thus, we can confirm that the soliton solution (5), $q(x,t) = q(s)$ with $s = x - vt$, has expansion in a Gaussian family of wavelets (7), $\psi(s) = Ne^{Q(s)}$, where $Q(s)$ is a polynomial and $N$ the normalization constant as referred to in [12].

Another important aspect is that, since Gaussian functions and their derivatives belong to space of square-integrable functions or $L^2$ functions, the same can be employed appropriately as the testing or the analyzing functions in wavelet analysis. This gives an insight over the intricate relationship between solitons, wavelets and probability distribution which forms the basis for employing wavelet techniques for analyzing solitons.

## 4.1. Solitons and Probability Distribution

We can now discuss as to how the solitons and probability distributions are interrelated.

Consider the argument of the exponential function, $\frac{x-vt}{\sqrt{1-v^2}}$ from the soliton solution (5) of SGE and obtain the standard normal variate $\eta$ of the random variable $x$, with the transformation

$\eta = \frac{x-\mu}{\sigma}$, where$\mu = vt, \sigma^2 = 1 - v^2$, are the parameters namely mean and the variance of the normal probability distribution.

Indeed, $x$ can be taken to mean as a continuous random variable that has associated with the normal probability density function

$$\psi(x;\mu,\sigma) = \frac{1}{\sqrt{1-v^2}\sqrt{2\pi}}e^{-\frac{(x-vt)^2}{2(1-v^2)}} \quad (8)$$

We can therefore write the normal probability density function for the standard normal variate $\eta$ as

$$\psi(x,t) = \psi(\eta) = \frac{1}{\sqrt{2\pi}}e^{-\eta^2/2} \quad (9)$$

where $-\infty < \eta < \infty$ and $(x,t)$ is space-time frame of wave function of the phase jump $q(x,t)$ in soliton solution.

The expression (9) can be therefore viewed as the normal probability distribution of the standard normal variate $\eta$ with zero mean and unit variance, that is, $\mu = vt = 0$, $\sigma^2 = 1 - v^2 = 1$ . Or in other words, at the initial conditions, $t = 0, v = 0$, the soliton variate $\eta$ becomes the standard normal variate that has the normal probability distribution as given in (8).

We immediately see from (9) that as$\eta \to \infty$,

$\psi(x,t) = \psi(\eta) = \frac{1}{\sqrt{2\pi}}e^{-\eta^2/2} \to 0.$

This confirms that the $\psi(x,t)$ in Gaussian form is indeed a soliton solution of the SGE (4).

We further observe that soliton wavelet $\psi(\eta)$ depicting normal probability distribution, is a non-negative function satisfying the conditions

$$\int_{-\infty}^{\infty}\psi(\eta)d\eta = 1,\ \int_{-\infty}^{\infty}\eta\psi(\eta)d\eta = 0,\ \int_{-\infty}^{\infty}\eta^2\psi(\eta)d\eta = 1.$$

Moreover, $\psi(\eta)$ is at least $n$-times differentiable $(n \geq 1)$ and its $(n-1)^{th}$ derivative satisfies

$lim_{n\to\pm\infty}\,\psi^{(n-1)}(\eta) = 0$

We can therefore derive the wavelets from the derivatives of the soliton $\psi(\eta)$ with $n = 1\ and\ n = 2$

$$\varphi^1(\eta) = \psi'(\eta) = \frac{\eta}{2\pi}e^{-\frac{\eta^2}{2}},\ \varphi^2(\eta) = \psi''(\eta) = \frac{(\eta^2-1)}{2\pi}e^{-\frac{\eta^2}{2}} \quad (10)$$

These wavelets belong to a family of vanishing momenta of the Gaussian obtained as $\frac{d^n}{ds^n}exp\,(-\eta^2/2)$ for $n = 1\ and\ n = 2$ , which are tested analytically as analyzing wavelet.

We know from the theory of wavelets that for the family of vanishing momenta wavelets

$$\varphi^n(\eta) = (-1)^{n+1}\frac{d^n}{ds^n}ex\,p(-\eta^2/2)\,, n > 0,$$

the condition $\int \eta^m\psi(\eta)d\eta = 0$ , $\forall m, 0 \le m < n; n \in Z$ , needs to be satisfied.

Further, the Fourier image of this wavelet family is given by

$\hat{\varphi}^n(\omega) = -\sqrt{2\pi}(-i\omega)^n e^{-\omega^2/2}$ , which has zeroes of order $n$ at $\omega = 0$.

The normalization/admissibility constant $C_{\varphi^n}$ can be evaluated to get

$$C_{\varphi^n} = 2\int_0^\infty \frac{|\hat{\varphi}^n(\omega)|^2}{|\omega|}d\omega = 2\,\pi\Gamma(n) < \infty$$

It is thus evident from the above that the vanishing momenta engendered from the soliton can be judiciously employed as analyzing wavelet for carrying out experimental data/image processing.

### 4.2. Statistical Interpretation

From the statistical theory of normal distribution, the normal probability integral or the area under the normal curve representing the normal probability distribution of $X$, gives the probability $P$ for the interval from the mean $\mu$ to the value $x$.

Thus, the probability that a random value of the normal soliton will fall within the interval $x = \mu - \sigma\ to\ x = \mu + \sigma$ is

$$P[\mu - \sigma < x < \mu + \sigma] = \frac{1}{\sigma\sqrt{2\pi}}\int_{\mu-\sigma}^{\mu+\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx$$

With

$$\mu = vt, \sigma^2 = 1 - v^2 \quad (11)$$

Also, for the standard normal soliton variate $\eta$, the probability that $\eta$ will fall within the interval $[-1, +1]$is

$$P[-1 < \eta < +1] = \frac{1}{\sqrt{2\pi}}\int_{-1}^{+1} e^{-\frac{\eta^2}{2}}dx = \frac{2}{\sqrt{2\pi}}\int_0^1 e^{\frac{-\eta^2}{2}}dx \quad (12)$$

Since the normal distribution is symmetric about mean, the maximum value of the soliton $\psi(x, t)$ occurs at the point $x = \mu$ for all $t$

$$\psi_{max} = [\psi(x,t)]_{x=\mu} = \frac{1}{\sigma\sqrt{2\pi}} = \frac{1}{\sqrt{1-v^2}\sqrt{2\pi}} \quad (13)$$

Also, the points of inflexion of the normal probability curve representing the soliton, $y = \psi(x, t)$ are $x = \mu \pm \sigma = vt \pm \sqrt{1 - v^2}$

Further, for all $\mu = vt,\ \sigma^2 = 1 - v^2$, the area property for the normal probability curve representing soliton can be

$$P(\mu - \sigma) < x < \mu + \sigma = 0.6826$$

$$P(\mu - 2\sigma) < x < \mu + 2\sigma = 0.9544$$

$$P(\mu - 3\sigma) < x < \mu + 3\sigma = 0.9973 \quad (14)$$

Again from the theory of probability distribution, for a continuous function $f(x)$ that defines the probability distribution of the stochastic variate $X$ by the relation that the probability of the value of the variate falling in the infinitesimal interval $x - \frac{1}{2}dx \le X \le x + \frac{1}{2}dx$ is expressible in the form $f(x)dx$ where $f(x)$ is the probability density function.

Thus, for the soliton variate $X$, the probability that the value of the variate will fall in the given interval will be

$$P\left(x - \tfrac{1}{2}dx \le x \le x + \tfrac{1}{2}dx\right) = \psi(x,t)dx \text{ for all } t \quad (15)$$

where $\psi(x, t)$ in this case will be the probability density function.

This together determines the distribution of probabilities under the normal curve representing the soliton and can be of vital importance in computing various statistical quantities useful in statistical inference theory.

### 4.3. Computation of Statistical Quantities

The characteristic function of a random variable $X$ for continuous probability function representing the soliton solution $\psi(x, t)$ for all $t$ can be obtained as

$$\varphi_x(\tau) = E(e^{i\tau X}) = \int_{-\infty}^{\infty} e^{i\tau x}\psi(x,t)dx = \int_{-\infty}^{\infty} e^{i\tau x}\psi(x,t)dx \quad (16)$$

Moreover, the mathematical expectation or mean of soliton random variable $X$ will be

$$E(X) = \mu_x = \int_{-\infty}^{\infty} x\psi(x,t)dx \quad (17)$$

where the infinite integral (17) converges absolutely.

Variance of the soliton variate $X$ : It measures the possible variation of $x$ from its mean value

Mathematically, $Var(X) = E[(X-\mu)^2]$.

By applying the property of mathematical expectation, we can have

$$Var(X) = E[X^2] - \mu^2 = E(X^2) - (vt)^2 \quad (18)$$

For the soliton variable having normal probability distribution, the mean can be computed as

$$E(X) = \mu_x = \int_{-\infty}^{\infty} x\psi(x,t)dx$$

$$= \int_{-\infty}^{\infty} x\left[\frac{1}{\sqrt{1-v^2}\sqrt{2\pi}} e^{-\frac{(x-vt)^2}{2(1-v^2)}}\right] dx$$

By making the substitutions

$\mu = vt, \sigma = \sqrt{1-v^2}$, $u = \frac{x-\mu}{\sqrt{2}\sigma}$, $du = \frac{1}{\sqrt{\sigma}}dx$, we can get

$E(X) = \int_{-\infty}^{\infty}(\sqrt{2}\sigma u + \mu)\left[\frac{1}{\sqrt{2\pi}\sigma}e^{-u^2}\right]dx$

On simplification, the integral above leads to sum of the two integrals; the first being the integrand of odd function and the second being the integrand of even function of $u$.

That eventually leads to

$E(X) = \frac{2\mu}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}du = \mu = vt$

Also, the variance of $X$ can be computed as

$Var(X) = E[(X-\mu)^2]$

$$= \int_{-\infty}^{\infty}(x-vt)^2\left[\frac{1}{\sqrt{1-v^2}\sqrt{2\pi}} e^{-\frac{(x-vt)^2}{2(1-v^2)}}\right] dx$$

Again making the substitutions as above, we eventually lead to

$V(X) = \frac{4\sigma^2}{\sqrt{\pi}}\int_0^{\infty} u^2 e^{-u^2}du = \frac{2\sigma^2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) = \sigma^2 = 1 - v^2$ (19)

Thus, it is justified that the soliton variate $X$ has a normal probability distribution with mean $\mu = vt$, and standard deviation $\sigma = \sqrt{1-v^2}$.

The moment generating function of the soliton solution representing normal distribution can be defined as

$$M_x(\tau) = E(e^{\tau X}) = \int_{-\infty}^{\infty} e^{\tau x}\psi(x,t)dx, \tau \in R$$

$$= \int_{-\infty}^{\infty} e^{\tau x}\left[\frac{1}{\sqrt{1-v^2}\sqrt{2\pi}} e^{-\frac{(x-vt)^2}{2(1-v^2)}}\right] dx$$

Making the substitution $\mu = vt, \sigma = \sqrt{1-v^2},\ u = \frac{x-\mu}{\sqrt{2}\sigma}$, $du = \frac{1}{\sqrt{\sigma}}dx$, we lead to the equation

$M_x(\tau) = \frac{e^{\mu\tau}}{\sqrt{\pi}} e^{\frac{\sigma^2\tau^2}{2}} \int_{-\infty}^{\infty} e^{-\left(u-\frac{\sigma\tau}{\sqrt{2}}\right)^2 du}$

Substituting $v = u - \frac{\sigma\tau}{\sqrt{2}}$, and evaluating the integral, we get

$$M_x(\tau) = e^{(vt)\tau + \frac{(1-v^2)+\tau^2}{2}} \quad (20)$$

For the standard normal soliton variate $\eta$, $\eta = \frac{x-vt}{\sqrt{1-v^2}}$, it follows that the moment generating function is

$$M_\eta(\tau) = e^{\frac{vt\tau}{\sqrt{1-v^2}}} M_\eta\left(\frac{\tau}{\sqrt{1-v^2}}\right) = e^{\frac{\tau^2}{2}}. \quad (21)$$

On expanding $M_\eta(\tau)$ as a Taylor series, we get the series

$M_\eta(\tau) = 1 + \frac{(\tau^2/2)}{1!} + \frac{(\tau^2/2)^2}{2!} + \cdots = 1 + \frac{\tau^2}{2} + \frac{\tau^4}{8} + \cdots$ (22)

Hence, from the above series, the raw moments of soliton variate $\eta$ can be obtained as

$$\mu_1' = E(\eta) = Coefficient\ of\ \frac{\tau}{1!}\ in\ M_\eta(\tau) = 0$$

$$\mu_2' = E(\eta^2) = Coefficient\ of\ \frac{\tau^2}{2}\ in\ M_\eta(\tau) = 1.$$

and the odd order central moments as

$$\mu_{2r+1} = E[(\eta-\mu)^{2r}] = E[(\eta - vt)^{2r}] \text{ for } r = 0,1,2,\ldots$$

Moreover, the even order central moments can be obtained as

$$\mu_{2r} = E[(\eta-\mu)^{2r+1}] = E[(\eta-vt)^{2r+1}]$$

$$= 1.3.5 \ldots (2r-1)(1-v^2)^r \text{ for } r = 0,1,2,\ldots$$

From this, the first four moments are therefore

$$\mu_1 = 0, \mu_2 = (1-v^2), \mu_3 = 0, \mu_4 = 3(1-v^2)^2$$

These computations will help us interpret the soliton solution in statistical framework.

## 5. Wavelet Transform of Random Processes

The foregoing sections described the statistical framework for the wavelet like solitons. Let us now present theoretical idea related to continuous wavelet decomposition of random processes represented by soliton variate.

Considering $(S, B, P)$ as the probability space, where $X = X(t,\omega); t \in R, \omega \in S$ is a second order random process, we can obtain the mathematical expectation

$$E \mid X(t) \mid^2 = \int_S \mid X(t,\omega) \mid^2 dP(\omega) < \infty, \forall t \in R \quad (23)$$

Since the amplitude of the soliton fluctuates randomly during evolution, we can treat soliton as the random or stochastic process $\psi(x,t); t \in R, x \in S$. Moreover, soliton itself is a Gaussian. Hence, we can model the random process as the Gaussian process that possesses vital properties inherited from the normal probability distribution.

In notations, one can write $X \sim GP(\mu, K)$, meaning the random function $X$ is distributed as a Gaussian process with mean $\mu$ and covariance $K$. It thus, makes us possible to obtain derived quantities such as average value of the process over a range of time, the error estimations using sample values at small sets of time.

The mathematical expectation can be worked out as

$$E \mid \psi(t) \mid^2 = \int_S \mid \psi(x,t) \mid^2 dP(x) < \infty, \forall t \in R \quad (24)$$

Further, given process $\psi = \psi(x,t)$, for any function $\varphi: R \to C$ satisfying the admissibility condition for being an analyzing wavelet, the continuous wavelet transform of the random process $\psi$ can be obtained as

$$W_\varphi[\psi](a,b) = \int_R \frac{1}{\sqrt{a}} \bar{\varphi}\left(\frac{x-b}{a}\right) \psi(x,t)dx, b \in R, a > 0, x \in S. \quad (25)$$

This transform is indeed well defined since the random process represented by soliton function $\psi$, and the analyzing wavelet $\varphi$ both belong to the class of square integrable functions as can be seen from expressions (9) and (10).

Moreover, we can define wavelet covariance of the covariance function for all $t$,

$$\psi_x(u,v) = E\overline{\psi(u)}\psi(v), u, v \in R$$

$$\psi_W(a,b,c,s) = EW_a(b)W_b(s)$$

$$= \int_R \frac{1}{\sqrt{ac}} \varphi\left(\frac{u-b}{a}\right) \bar{\varphi}\left(\frac{v-s}{c}\right) \psi_X(u,v)dudv \quad (26)$$

Provided however that the condition

$E\left\{\int_R \mid \bar{\varphi}\left(\frac{u-b}{a}\right) \psi(u,.) \mid du\right\}^2 < \infty$ is satisfied.

It should be noted that in this wavelet exercise, only non-vanishing Fourier image $\tilde{\varphi}(\lambda) \neq 0$ is considered as the testing wavelet [10] rather than employing non-vanishing momenta wavelets as is normally done. With the application of this method, one can extract additional information by studying the correlation of the same signal but at different resolutions.

## 6. Applications in Image/Signal Processing

Wavelets use multi-resolution technique, which is deeply related to signal processing. Wavelets provide a very sparse and efficient representation for piecewise smooth signals. Wavelet transform does not appeal to differentiability, and is thus suitable for the study of continuous but non-differentiable processes like random processes. It is due to this characteristic, wavelet transform can be successfully implemented in the wide range of applications in statistics, engineering and biological sciences. The random processes can be modeled as Gaussian process, whose importance is due to the vital statistical properties inherited from normal probability distribution. In signal processing, the main task is to separate signals from noisy background, that is, to separate all higher peaks shed mixed intricately with the small ones from the signal/image to be processed. This is done first by assuming that the signal is produced by some $M$ number of Gaussian sources, say, $f_{exp}$. Then, all that we need is to find the parameter set $(N, \sigma, x^m)$ by minimizing the difference

$$f(N, \sigma, x^m) = f_{exp} - \sum f_{gauss}, \quad (27)$$

where $f_{gauss}$ is any signal or the random process in the instant case that has Gaussian representation.

Now, employing vanishing momenta wavelets (10) generated from soliton wavelets as the testing function/basic wavelet $\varphi^n$, for $n = 1, 2$, the wavelet images of the Gaussian process are therefore given by

$$W_{\varphi_n}(a,b)[f_{gauss}] = \int \overline{\frac{1}{\sqrt{a}} \varphi_n\left(\frac{x-b}{a}\right)} f_{gauss}(x)dx \quad (28)$$

The integrals in (28) can be evaluated using the Fourier representation

$$W_{\varphi_n}(a,b)[f_{gauss}]$$

$$= \frac{1}{2\pi} |a|^{1/2} \int_{-\infty}^{\infty} \exp(ikb)\, \overline{\widetilde{\varphi_n}(ak)}\, \tilde{f}_{gauss}(k)dk \quad (29)$$

where $\widetilde{\varphi_n}(k) = \sqrt{2\pi}(ik)^n exp(-k^2/2)$.

Finally, proceeding exactly as in [1], we can determine the distribution parameters. The procedure can be adopted in the applications involving any random processes which can be modeled as Gaussian process.

## References


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