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On the generalization of the discretized continuous algorithm for optimal proportional control problem

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Abstract
This paper seeks to find solution to the generalization of a class of continuous-time optimal control model with special preference to those whose control efforts are proportional to the state of the dynamical system with and without delay in the state variables. The adoption of the Augmented Lagrangian method in the transformation of the constrained problem into an unconstrained sequential nonlinear quadratic problem allows for the use of the well-posed Broydon-Fletcher-Goldberg-Shanno (BFGS) embedded Quasi-Newton algorithm. The symmetric and positive definite properties of the constructed control operator guarantees the invertibility of the BFGS embedded in the algorithm and as well induces faster convergence. Numerical results were considered; result tested and responded favourably to the analytical solution with linear convergence.

1. Introduction

1.1. Motivation

Similar works on optimal proportional control had been carried out for both delay and non-delay in the state trajectories of the constraint using Quasi-Newton Augmented Lagrangian method by Olotu and Dawodu [14, 15] but for real coefficients However, most real life problems are formulations of higher order differential equations which are reduced to systems of differential equations of Vector-Matrix coefficients. Therefore, direct application of the existing method for the real coefficients may constitute an ill-posed problem, hence the novelty of this Algorithm.

1.2. Literature Review

In the development of this paper, a review of the earlier work on optimal control in the area of the function space algorithm for solving both continuous and discrete linear quadratic optimal control problem given by Polak [17] was carried out. Later work was by Poljak [18] on the rate of convergence of the quadratic penalty function method which has substantially influenced the present day developments in multiplier method, an extension of the quadratic method of multipliers that was first proposed independently by Hestenes [8] and Powell [19]. The outstanding publication of Ibiejugba and Onumanyi [9] on the construction of control operator to

circumvent the numerical set-back in function space algorithm was also reviewed to help under-study the operational properties of the penalized matrix operator so as to avoid the problem of ill-conditioning. Many line search direct and indirect methods were as well reviewed but superior to most of these methods are the direct methods which first discretize and later optimize the optimal control problems using well proven iterative methods. The choice of the discretization schemes is determined by its ability to generate sparse matrices that will prompt the convergence of the developed scheme and reduce computational errors. The newton-type iterative methods with finite sets of variables and constraints earlier proposed by Bett [3] had
over years been discovered to be very appropriate in solving nonlinear programming models formulated from optimal control problems when compared to the Conjugate Gradient Methods. Most acceptable of the newton type methods is the BFGS embedded Quasi--Newton algorithm which, in practice, had been proven to have faster rate of convergence with minimal error. The BFGS update formula inculcated in the algorithm according to Bertsekas [2] is an improvement over that of Davidon-Fletcher–Powell (DFP) because of its ease in estimating the inverse Hessian.

Many authors such as Olotu & Olorunshola [16], Olotu & Adekunle [11,12], Olotu & Akeremale [13], Olotu and Dawodu[14,15] and Gollman et al [7] have adopted this direct method of first discretizing the continuous-time optimal control problem with different order of numerical scheme and later optimizing with well-known iterative schemes after transforming the constrained problem into unconstrained nonlinear programming problem by any of the methods of penalty or multiplier, referred to as the "Discretized Continuous Algorithm". Extensive work was done by Olotu and Adekunle on the discretized optimal control with vector and matrix coefficients for both delay and non-delay in the state variable but much more recent are those carried out by Olotu and Dawodu on the above referenced delayed and nondelayed proportional control problems with their feedback laws having control efforts proportional to the states with feedback gains, constant estimates of those by the Riccati law. In the course of discretization, the rational proportionality law analyzed by Gollman et al [7], which assumes that the ratio of the time delays in state and control is a rational factor, was adopted. However, this approach is an improvement over the unified approach by Colonius and Hinrichsen [5] and Soliman et al [20] to control problems with delays in the state variable using the theory of necessary conditions for optimization problems in function spaces.

2. General Formulation of the Problem

Consider the generalized optimal proportional control problem

$$\text{Min } J(\mathbf{X},\mathbf{W}) = \tfrac{1}{2}\int_0^Z [\mathbf{X}^T(t)P(t)\mathbf{X}(t)+\mathbf{W}^T(t)Q(t)\mathbf{W}(t)]dt \quad (1)$$

Subject to:

$$\dot{\mathbf{X}}(t) = A\mathbf{X}(t) + B\mathbf{W}(t) + C\mathbf{X}(t-d), \quad 0 \le t \le Z \quad (2)$$

$$\mathbf{X}(t) = \bar{\boldsymbol{\alpha}}(t), \quad t \in [-d,0], \ d \ge 0 \quad (3)$$

$$\mathbf{W}(t) = M\mathbf{X}(t) \quad (4)$$

$$\mathbf{X}(0) = \mathbf{X}_0 \quad (5)$$

Where $\mathbf{X}(\mathbf{t}) \in C^2[0,Z]$ *and* $\mathbf{W}(\mathbf{t}) \in C^2[0,Z]$ are the respective m and n dimensional state and control variables that are twice differentiable in the closed interval $[0,Z]$, $M_{m\times n}$ a time-invariant coefficient matrix, $P_{m\times m}$ and $Q_{n\times n}$ are real, symmetric and positive definite constant coefficient matrices while $A_{n\times m}$, $B_{n\times n}$ and $C_{n\times m}$ are constant matrices that are not necessarily positive definite. $\bar{\boldsymbol{\alpha}}(t)$ is the pre-shaped function or initial value profile defined on the delay interval $[-d,0]$ for which the values of the state trajectory $\mathbf{X}(\mathrm{t})$ are known.

3. Materials and Methods of Solution

3.1. Overview of the Methodology

The conceptual framework for the formulation of the algorithm in solving the model above is based purely on the direct numerical approach to solving unconstrained numerical problems for which the functions are defined in the given interval. The discretization of the continuous-time functions into discrete time functions using the third order composite Simpson's rule was used both for the constraint and performance index to obtain the constrained sequential nonlinear quadratic formulation of the control problem. The nonlinear programming problem was converted to an unconstrained quadratic problem using the Augmented Lagrangian function amenable to the well-posed 2-rank Broydon-Fletcher-Goldberg-Shanno (BFGS) embedded Quasi-Newton iterative scheme. The choice of our iterative method with higher numerical order was to generate a highly sparse quadratic operator in the course of discretization. This will help induce rate of convergence, reduce computational error and consequently increase the level of accuracy. However, in the course of discretization the control variables are appropriately discretized by partitioning the interval $[0,Z]$ into N equal intervals with knots $0 = \mathrm{t}_0 < \mathrm{t}_1 < \mathrm{t}_2 < \cdots < \mathrm{t}_N = \mathrm{Z}$ where $\Delta t_i = \frac{Z-0}{N} = a\times 10^{-q}$ $a,q \in \mathbb{Z}^+$ and $\mathbf{X}_{-k} = \bar{\boldsymbol{\alpha}}(-kh)$, $k = 0,1,2,\ldots,r$ is defined by the pre-shaped function defined on the delay interval $[-d,0]$.

3.2. Discretization of the Performance Index

Suppose the generalized m-dimensional state and n-dimensional control vector variables of the optimal control problem are $\mathbf{X}^{\mathrm{T}} = (\mathrm{X}^{(1)}, \mathrm{X}^{(2)}, \ldots, \mathrm{X}^{(m)})^{\mathrm{T}} \in \mathbb{R}^m$ and $\mathbf{W}^{\mathrm{T}} = (\mathrm{W}^{(1)}, \mathrm{W}^{(2)}, \ldots, \mathrm{W}^{(n)})^{\mathrm{T}} \in \mathbb{R}^n$ respectively. The discretization of the state vector of the performance index (objective criterion) defined in equation (1) above into N partitions, with a uniform steplength $h = \Delta t_i$, by Simpsons rule stated in Burden et al [4] gives

$$Min\, J(\mathbf{X}) = \frac{1}{2}\int_{\mathbf{0}}^{Z}[\mathbf{X}^{\mathrm{T}}(\mathrm{t})P(t)\mathbf{X}(\mathrm{t})]dt \approx \frac{1}{2}\left\{X_0^T\bar{P}(t_0)X_0 + 4\mathbf{X}_1^{\mathrm{T}}\bar{P}(t_1)\mathbf{X}_1 + 2\mathbf{X}_2^{\mathrm{T}}\bar{P}(t_2)\mathbf{X}_2 + \ldots + 2\mathbf{X}_{N-2}^{\mathrm{T}}\bar{P}(t_{N-2})\mathbf{X}_{N-2} + 4\mathbf{X}_{N-1}^{\mathrm{T}}\bar{P}(t_{N-1})\mathbf{X}_{N-1} + \mathbf{X}_N^{\mathrm{T}}(\mathbf{t}_N)\bar{P}(t_N)\mathbf{X}_N\right\}$$

Where $\bar{P}(t_i) = {}^{h}\!/_{3}\,P(t_i)$, $\bar{C} = {}^{1}\!/_{2}\,X_0^T\bar{P}(t_0)X_0$ and $\mathbf{X}^{\mathrm{T}} = \bigcup_{k=1}^{m}(\mathrm{X}_1^{(k)}, \mathrm{X}_2^{(k)}, \ldots, \mathrm{X}_N^{(k)})^{\mathrm{T}} \in \mathbb{R}^{mN}$ for every $X^{(k)} \subset \mathbf{X} \in \mathbb{R}^m$. This generates a block diagonal coefficient matrix $\bar{A}$ of dimension $mN \times mN$ (number of entries) partitioned into N mesh points which entries $[\bar{a}_{ii}]$ are $4\bar{P}(t_i)$ for $i = 1, 3, \ldots, (N-1)$, , $2\bar{P}(t_i)$ for $i = 2, 4, \ldots, (N-2)$, $\bar{P}(t_i)$ for $i = N$ and 0 elsewhere. Similarly, the discretization of the control variable of the performance index partitioned into $N+1$ mesh points generates a block diagonal coefficient matrix $\bar{D}$ of dimension $n(N+1) \times n(N+1)$ whose entries $[\bar{d}_{ii}]$ are $\bar{Q}(t_{i-1})$ for $i = 1$,N, $4\bar{Q}(t_{i-1})$ for $i = 2, 4, \ldots, N-2$, $2\bar{Q}(t_{i-1})$ for $i = 3, 5, \ldots, N-1$ and 0 elsewhere; given that $\mathbf{W}^{\mathrm{T}} = \bigcup_{l=1}^{n}(\mathrm{W}_0^{(l)}, \mathrm{W}_1^{(l)}, \ldots, \mathrm{W}_N^{(l)})^{\mathrm{T}} \in \mathbb{R}^{n(N+1)}$ $\forall W^{(l)} \subset \mathbf{W} \in \mathbb{R}^n$, $\bar{Q}(t_i) = {}^{h}\!/_{3}\,Q(t_i)$ and the i^{th} element corresponds to the i^{th} block. $\mathbf{Z}^{\mathrm{T}} = \{\bigcup_{k=1,l=1}^{k=m,l=n}(\mathrm{X}_1^{(k)}, \mathrm{X}_2^{(k)}, \ldots, \mathrm{X}_N^{(k)}, \mathrm{W}_0^{(l)}, \mathrm{W}_1^{(l)}, \ldots, \mathrm{W}_N^{(l)})^{\mathrm{T}}\}$ $\in \mathbb{R}^{(mN+nN+n)}$ is the derived augmented variable of the performance index $J(\mathbf{X},\mathbf{W})$ in equation (1) and $\bar{V} = \left[\begin{array}{c|c} \bar{A} & (0) \\ \hline (0) & \bar{D} \end{array}\right]$ is the generated augmented symmetric and positive definite block matrix of order $(mN + nN + n)$ with entries given below by,

$$\bar{V} = [\bar{v}_{ii}] = \begin{cases} 4\bar{P}(t_i)\ , & i = 1, 3, \ldots, (N-1) \\ 2\bar{P}t_i)\ , & i = 2, 4, \ldots, (N-2) \\ \bar{P}(t_i)\ , & i = N \\ \bar{Q}(t_i)\ , & i = N+1 \\ 4\bar{Q}(t_i)\ , & i = N+2, N+4, \ldots, 2N-2, 2N \\ 2\bar{Q}(t_i)\ , & i = N+3, N+5, \ldots, 2N-1 \\ \bar{Q}(t_i)\ , & i = 2N+1 \\ 0\ , & elsewhere \end{cases} \tag{6}$$

Considering then the generalized proportional control case with $\mathbf{W}(\mathbf{t}) = M\mathbf{X}(\mathbf{t})$, the continuous time objective function can be expressed as,

$$Min\, J(\mathbf{X}, \mathbf{W}) = \frac{1}{2}\int_0^Z[\mathbf{X}^{\mathrm{T}}(\mathrm{t})P\mathbf{X}(\mathrm{t}) + \mathbf{W}^{\mathrm{T}}(\mathrm{t})Q\mathbf{W}(\mathrm{t})]dt$$

$$\approx \frac{1}{2}\int_0^Z\left\{\mathbf{X}^{\mathrm{T}}(\mathrm{t})[P + M^TQM]\mathbf{X}(\mathrm{t})\right\}dt$$

$$ {}^{1}\!/_{2}\left\{\mathbf{X}_0^T\bar{S}(t_0)\mathbf{X}_0 + 2\sum_{i=1}^{N/2-1}\mathbf{X}_{2i}^{\mathbf{T}}\bar{S}(t_{2i})\mathbf{X}_{2i} + 4\sum_{i=1}^{N/2}\mathbf{X}_{2i\text{-}1}^{\mathbf{T}}\bar{S}(t_{2i\text{-}1})\mathbf{X}_{2\mathrm{i}} + \mathbf{X}_N^{\mathbf{T}}\bar{S}(t_N)\mathbf{X}_N\right\} \tag{7}$$

Where $\mathbf{W}(t) = M\mathbf{X}(t)$ and $\bar{S} = {}^{h}\!/_{3}\,(P + M^TQM)$ (8)

This generates a block diagonal coefficient matrix $[\bar{v}_{ii}] \in \mathbb{R}^{mN \times mN}$ of dimension $mN \times mN$ (number of block entries) partitioned into N mesh points and whose entries are defined below by

$$\bar{V} = [\bar{v}_{ii}] = \begin{cases} 4\bar{S}(t_i)\ , & i = 1, 3, \ldots, (N-1) \\ 2\bar{S}(t_i)\ , & i = 2, 4, \ldots, (N-2) \\ \bar{S}(t_i)\ , & i = N \\ 0\ , & elsewhere \end{cases} \tag{9}$$

where $\bar{C} = {}^{1}\!/_{2}\,X_0^T\bar{S}(t_i)X_0$ (10)

Hence, the discretized objective value is compactly expressed as

$$F(\mathbf{Z}) = \frac{1}{2}\mathbf{Z}^{\mathrm{T}}\bar{V}\mathbf{Z} + \bar{C} \tag{11}$$

However, since the coefficient matrices $P(t_i)$ and $Q(t_i)$ are constant, then $P(t_i) = P$ and $Q(t_i) = Q\ \ \forall i.$

3.3. Discretization of the Constraint

The generalized constraint with the delays on the state vector variable of the optimal control problem is represented below as

$$\dot{\mathbf{X}}(t) = A\mathbf{X}(t) + B\mathbf{W}(t) + C\mathbf{X}(t-d),\ t \in [0, Z] \tag{12}$$

$$\mathbf{X}(t) = \bar{\boldsymbol{\alpha}}(t),\ t \in [-d, 0] \tag{13}$$

$$\mathbf{X}(0) = \mathbf{0},$$

Where,

$$\left.\begin{array}{l} A \in \mathbb{R}^{n\times m}, B \in \mathbb{R}^{n\times n}\ C \in \mathbb{R}^{n\times m} \\ \mathbf{X} \in \mathbb{R}^m\ \ and\ \ \mathbf{W} \in \mathbb{R}^n \end{array}\right\} \tag{14}$$

To discretize the constraint, we employ the *third order two steps implicit Simpson's rule* according to Burden et al [4], generated through Newton-Gregory forward interpolated formula as stated below

$$\mathbf{X}_{k+2} - \mathbf{X}_k = \frac{h}{3}\left[\mathbf{f}(\mathbf{X}_k) + 4\mathbf{f}(\mathbf{X}_{k+1}) + \mathbf{f}(\mathbf{X}_{k+2})\right] + \mathbf{O}(h^4) \quad (15)$$

Assuming that $h = \frac{d}{r}$ *for* $r \neq 0 \in \mathbb{N}$ to give $\mathbf{X}_{-k} \approx \bar{\boldsymbol{\alpha}}(-kh)$, . *for* $k = 1, 2, ..., r$, then the discretized constraint in equation (12) over the control interval $[0, Z]$ using the given initial value profile $\bar{\boldsymbol{\alpha}}(t) \approx \bar{\boldsymbol{\alpha}}(-kh)$ in equation (13) as fixed values over the delay intervals $[-d, 0]$ becomes#

$$\begin{aligned} &T\mathbf{X}_k + U\mathbf{X}_{k+1} + \mathbf{X}_{k+2} + V[\mathbf{W}_k + 4\mathbf{W}_{k+1} + \mathbf{W}_{k+2}] + \\ &S[\mathbf{X}_{k-r} + 4\mathbf{X}_{k+1-r} + \mathbf{X}_{k+2-r}] = 0 \end{aligned} \quad (16)$$

for,

$$T = (3I_{n\times m} + hA) * inv(hA - 3I_{n\times m}),$$
$$U = 4hA * inv(hA - 3I_{n\times m}),$$
$$V = hB * inv(hA - 3I_{n\times n}) \text{ and}$$
$$S = hC * inv(hA - 3I_{n\times m}).$$

While,

$$\mathbf{X}_{k-r} = (X_{k-r}^{(1)}, X_{k-r}^{(2)}, ..., X_{k-r}^{(m)}) \in \mathbb{R}^{mn}$$

Setting $k = 0, 1, 2, ..., (N-1)$, equation (16) then gives,

for $k = 0$

$$U\mathbf{X}_1 + \mathbf{X}_2 + V(\mathbf{W}_0 + 4\mathbf{W}_1 + \mathbf{W}_2) = -T\mathbf{X}_0 - S(\mathbf{X}_{-r} + 4\mathbf{X}_{1-r} + \mathbf{X}_{2-r})$$

for $k = 1$,

$$T\mathbf{X}_1 + U\mathbf{X}_2 + \mathbf{X}_3 + V(\mathbf{W}_1 + 4\mathbf{W}_2 + \mathbf{W}_3) = -S(\mathbf{X}_{1-r} + 4\mathbf{X}_{2-r} + \mathbf{X}_{3-r})$$

for $k = 2$,

$$T\mathbf{X}_2 + U\mathbf{X}_3 + \mathbf{X}_4 + V(\mathbf{W}_2 + 4\mathbf{W}_3 + \mathbf{W}_4) = -S(\mathbf{X}_{2-r} + 4\mathbf{X}_{3-r} + \mathbf{X}_{4-r})$$

for $k = r - 2$,

$$T\mathbf{X}_{r-2} + U\mathbf{X}_{r-1} + \mathbf{X}_r + V(\mathbf{W}_{r-2} + 4\mathbf{W}_{r-1} + \mathbf{W}_r) = -S(\mathbf{X}_{-2} + 4\mathbf{X}_{-1} + \mathbf{X}_0)$$

for $k = r - 1$,

$$S\mathbf{X}_1 + T\mathbf{X}_{r-1} + U\mathbf{X}_r + \mathbf{X}_{r+1} + V(\mathbf{W}_{r-1} + 4\mathbf{W}_r + \mathbf{W}_{r+1}) = -S(\mathbf{X}_{-1} + 4\mathbf{X}_0)$$

for $k = r$,

$$4S\mathbf{X}_1 + S\mathbf{X}_2 + T\mathbf{X}_r + U\mathbf{X}_{r+1} + \mathbf{X}_{r+2} + V(\mathbf{W}_r + 4\mathbf{W}_{r+1} + \mathbf{W}_{r+2}) = -S\mathbf{X}_0$$

for $k = r + l$ for $l = 1, 2, 3......., (N - 2)$,

$$S(\mathbf{X}_l + 4\mathbf{X}_{l+1} + \mathbf{X}_{l+2}) + T\mathbf{X}_{r+l} + U\mathbf{X}_{r+l+1} + \mathbf{X}_{r+l+2} + V(\mathbf{W}_{r+l} + 4\mathbf{W}_{r+l+1} + \mathbf{W}_{r+l+2}) = \mathbf{0}$$

The above expressions give a linear system of the form below

$$[J_1 | J_2] Z = J\mathbf{Z} = H \quad (17)$$

Where $[J_1 | J_2] = J = [j_{ij}]$ is a block matrix of dimension $m(N-1) \times (mN + nN + n)$ developed from the augmentation of the discretized block matrices J_1 and J_2 of the state and control variables with dimensions $m(N-1) \times mN$ and $m(N-1) \times m(N+1)$ respectively. J is then a sparse augmented coefficient matrix with principal block diagonal elements equal to U for every i, j such that $i = j = 1, 2, ...N-1$, lower diagonal block elements T for every i, j such that for $i = 1, 2, ..., N-1$, then $j = i - 1$, upper diagonal block elements $I_{n\times m}$ for every i, j such that for $i = 1, 2, ..., N-1$, then $j = i + 1$, $4S$ such that for every $i = r, r+1, r+2, ..., N-1$, then $j = i + 1 - r$, $\bar{S}$ such that for every $i = r-1, r, ..., N-1$, then $j = i + 2 - r$ and for every $i = r+1, r+2, ..., N-1$, then $j = i - r$. Other block entries are such that for every $i = 1, 2, ..., N-1$, then $J_{ij} = 4V$ *for* $j = N + 1 + i$, $J_{ij} = V$ for every $j = N + i$ *and* $N + 2 + i$ and 0 elsewhere.

Similarly, for the $(mN + nN + n) \times 1$ dimensional row vector $H = [h_{ij}]$, the elements are defined as $[h_{i1}] = -T\mathbf{X}_0 - S(\mathbf{X}_{-r} + 4\mathbf{X}_{1-r} + \mathbf{X}_{2-r})$ for $i = 1$, $-S(\mathbf{X}_{i-1-r} + 4\mathbf{X}_{i-r} + \mathbf{X}_{i+1-r})$ for $i = 2, 3, ..., r-1$ $-S(\mathbf{X}_{-1} + 4\mathbf{X}_0)$ for $i = r$, $-S\mathbf{X}_0$ for $i = r+1$ and 0 for $r+1, r+3, ..., N-1$ such that all the subscripts of the variable $\mathbf{X}$ are negative $[\mathbf{X}_i : i < 0]$. The coefficient block matrix J and vector H can respectively be represented below as follows:

$$J = [j_{ij}] = \begin{cases} T & 1 \le i \le N-1 & j = i-1 \\ U & 1 \le i \le N-1 & j = i \\ I_{n\times m} & 2 \le i \le N-1 & j = i-1 \\ S & \begin{bmatrix} r-1 \le i \le N-1 \\ r+1 \le i \le N-1 \end{bmatrix} & \begin{matrix} j = i+2-r \\ j = i-r \end{matrix} \\ 4S & r \le i \le N-1 & j = i+1-r \quad \Big\} r \ge 3 \\ V & \begin{bmatrix} 1 \le i \le N-1 \\ 1 \le i \le N-1 \end{bmatrix} & \begin{matrix} j = N+i \\ j = N+2+i \end{matrix} \\ 4V & 1 \le i \le N-1 & j = N+1+i \\ 0 & \textit{elsewhere} & \end{cases} \quad (18)$$

and

$$H = [h_{i1}] = \begin{cases} -T\mathbf{X}_0 - S(\mathbf{X}_{-r} + 4\mathbf{X}_{1-r} + \mathbf{X}_{2-r}) & i = 1 \\ -S(\mathbf{X}_{i-1-r} + 4\mathbf{X}_{i-r} + \mathbf{X}_{i+1-r}) & 2 \le i \le r-1 \\ -S(\mathbf{X}_{-1} + 4\mathbf{X}_0) & i = r \\ -S\mathbf{X}_0 & i = r+1 \\ 0 & i = r+2, r+3, ..., N-1 \end{cases} \quad (19)$$

3.4. Generalized Proportional Control with Nonzero Delay

Imposing the proportional feedback law $\mathbf{W}(t) = M\mathbf{X}(t)$ with nonzero delay coefficient ($d \neq 0$), then the control problem in equation (12) above with M a $n \times m$ dimensional matrix will be expressed as,

$$\dot{\mathbf{X}}(t) = A\mathbf{X}(t) + BM\mathbf{X}(t) + C\mathbf{X}(t-d) = [A+BM]\mathbf{X}(t) + C\mathbf{X}(t-d) = \bar{A}\mathbf{X}(t) + C\mathbf{X}(t-d) \quad (20)$$

This then gives the discretized constraint equation defined by,

$$\bar{T}\mathbf{X}_k + \bar{U}\mathbf{X}_{k+1} + \mathbf{X}_{k+2} + \bar{S}[\mathbf{X}_{k-r} + 4\mathbf{X}_{k+1-r} + \mathbf{X}_{k+2-r}] = \mathbf{0} \quad (21)$$

$$\left.\begin{array}{l} for\ \bar{T} = (3I_{m\times m} + h\bar{A}) * inv(h\bar{A} - 3I_{m\times m}), \\ \bar{U} = 4h\bar{A} * inv(h\bar{A} - 3I_{m\times m})\ and \\ \bar{S} = hC * inv(hA - 3I_{m\times m}). \end{array}\right\} \quad (22)$$

Slotting $k = 0,1,2,...,(N-1)$ into equation (21) generates the matrix $\bar{J}$ of dimension $m(N-1)\times mN$ defined below by,

$$\bar{J} = [\bar{j}_{ij}] = \begin{cases} \bar{S}, & 2 \le i \le N-1,\ j = i-1 \\ \bar{T}, & 1 \le i \le N-1,\ \ j = i \\ I_{m\times m}, & 1 \le i \le N-1,\ \ j = i+1 \\ 0, & elsewhere \end{cases} \quad (23)$$

With entries given by $[h_{11}] = -\bar{F}\mathbf{X}_0$ (for known initial value $\mathbf{X(0)} = \mathbf{X}_0$), $[h_{i1}] = 0$ for $i = 2,3,......,(N-1)$.

The column vector $\bar{H}$ of order $m(N-1)\times 1$ with block entries given below as ,

$$H = [h_{i1}] = \begin{cases} -\bar{T}\mathbf{X}_0 + \bar{S}(\mathbf{X}_{-r} + 4\mathbf{X}_{1-r} + \mathbf{X}_{2-r}) & i = 1 \\ -\bar{S}(\mathbf{X}_{i-1-r} + 4\mathbf{X}_{i-r} + \mathbf{X}_{i+1-r}), & 2 \le i \le r-1 \\ -\bar{S}(\mathbf{X}_{-1} + 4\mathbf{X}_0), & i = r \\ -\bar{S}\mathbf{X}_0 \quad , & i = r+1 \\ 0 \quad , & i = r+2, r+3, ..., N-1 \end{cases}$$

where $\mathbf{Z}^{T} = \bigcup_{k=1}^{m} (X_1^{(k)}, X_2^{(k)}, ...X_N^{(k)})^{T} \in \mathbb{R}^{mN}$

3.5. Generalized Proportional Control with Zero Delay

A proportional control model without delay is that for which the feedback law is $\mathbf{W}(t) = M\mathbf{X}(t)$ and the delay coefficient is zero ($d = 0$) to reduce the general model in equation (12) above to the equation below.

$$\dot{\mathbf{X}}(t) = (A + C + BM)\mathbf{X}(t) = \bar{C}\mathbf{X}(t), \quad 0 \le t \le Z \quad (25)$$

This then gives the discretized constraint equation defined by,

$$\bar{F}\mathbf{X}_k + \bar{G}\mathbf{X}_{k+1} + \mathbf{X}_{k+2} = \mathbf{0} \quad (26)$$

$$\left.\begin{array}{l} for\ \bar{C} = (A + C + MB), \\ \bar{F} = (3I_{m\times m} + h\bar{C}) * inv(h\bar{C} - 3I_{m\times m}) \\ and\ \bar{G} = 4h\bar{C} * inv(h\bar{C} - 3I_{m\times m}). \end{array}\right\} \quad (27)$$

Slotting $k = 0,1,2,.....,(N-1)$ into equation (26), generates the matrix $\bar{J}$ of dimension $m(N-1)\times mN$ defined below by,

$$\bar{J} = [\bar{j}_{ij}] = \begin{cases} \bar{F}, & 2 \le i \le N-1,\ j = i-1 \\ \bar{G}, & 1 \le i \le N-1,\ j = i \\ I_{m\times m}, & 1 \le i \le N-1,\ j = i+1 \\ 0, & elsewhere \end{cases} \quad (28)$$

The column vector $\bar{H}$ is of order $m(N-1)\times 1$ given by $[h_{11}] = -\bar{F}\mathbf{X}_0$ (for known initial value $\mathbf{X(0)} = \mathbf{X}_0$), $[h_{i1}] = 0$ for $i = 2,3,......,(N-1)$.

3.6. The Augmented Lagrangian formulation

The above discretized optimal control problem becomes a large sparse nonlinear constrained quadratic programming problem represented in matrix algebra as, Minimize $F(\mathbf{Z}) = \frac{1}{2}\mathbf{Z}^{\mathbf{T}}\bar{V}\mathbf{Z} + \bar{C}$ subject to $\bar{\mathbf{J}}\mathbf{Z} = \bar{\mathrm{H}}$ (29)

Theorem 3.1

Considering a problem of the form Min $f(x)$ subject to $x \in X, h(x) = 0$ where $f: R^n \to R$; $h: R^n \to R^m$ are twice continuously differentiable functions $X \subset R^n$, then for a given scalar c, there exists an augmented lagrangian function $L_{\mu_k}: R^n \times R^m \to R$ defined by $L_{\mu_k}(X_k, \lambda_k, \mu_k) = f(x_k) + \lambda_k^T h(X_k) + \frac{1}{2}\mu_k \|h(X_k)\|^2$, where the multiplier vector λ_k in R^n is updated by $\lambda_{k+1} = \lambda_k + \mu_k h(X_k)$ and the penalty parameter μ_k is chosen such that $\mu_{k+1} > \mu_k > 0\ \forall k = 0,1,2,...$ so as to minimize $L_{\mu_k}(.)$ over R^n. See Bertsekas [1, 2].

Substituting the developed discretized variables in equation (29) into the augmented lagrangian function defined in theorem 3.1, as earlier reviewed by Fiacco [6], with μ as the penalty parameter and λ as the lagrangian multiplier, gives the unconstrained minimization problem below:

$$Min\, L_\mu(\mathbf{Z}, \lambda, \mu) = \frac{1}{2}\mathbf{Z}^T\bar{V}\mathbf{Z} + \bar{C} + \lambda^T|\bar{J}\mathbf{Z} - \bar{H}| + \frac{\mu}{2}\|\bar{J}\mathbf{Z} - \bar{H}\|^2 \quad (30)$$

Upon expansion, it gives

$$Min\ L_{\mu}(\mathbf{Z},\lambda,\mu) = {}^{1}\!/_{2}\,\mathbf{Z}^{T}\bar{V}_{\mu k}\mathbf{Z} + \bar{M}^{T}_{\mu k}\mathbf{Z} + \bar{N}_{\mu k} \quad (31)$$

Where L_{μ} is the penalized lagrangian,

$\bar{V}_{\mu K} = \left[\bar{V} + \left(\mu_j \bar{J}^T \bar{J}\right)\right] \in \mathbb{R}^{(mN+nN+n\times)(mN+nN+n))}$ is the quadratic control operator,

$\bar{M}_{\mu k} = \left(\lambda_j{}^T \bar{J} - \mu_j \bar{H}^T \bar{J}\right) \in \mathbb{R}^{(mN+nN+n)\times 1}$ *and*

$\bar{N}_{\mu k} = \left(\frac{\mu_j}{2}\bar{H}^T\bar{H} - \lambda_j{}^T\bar{H} + \bar{C}\right) \in \mathbb{R}$

The matrices dimensions for the delayed and non-delayed proportional cases are specified below with the control operator $\bar{V}_{\mu k}$ symmetric and positive definite, Olotu et al [13].

$\bar{V}_{\mu k} \in \mathbb{R}^{mN\times mN}$, $\bar{M}_{\mu k} \in \mathbb{R}^{mN\times 1}$, $\bar{N}_{\mu k} \in \mathbb{R}$

and $\mathbf{Z} \in \mathbb{R}^{mN\times 1}$ (32)

Considering the next iterative process, then the variable changes from $\mathbf{Z}_k$ to $\mathbf{Z}_{k+1}$ by a steplength α_k in the direction of S_k defined by

$$\mathbf{Z}_{k+1} = \mathbf{Z}_k + \alpha_k S_k \quad (33)$$

and

$$S_k = -B_k g_k \quad (34)$$

where $B_k \approx [\nabla^2 F(\mathbf{Z}_k)]^{-1}$ (approximate inverse Hessian) and $g_k = [\nabla F(\mathbf{Z}_k)]$

Therefore, slotting equation (33) into equation (31) gives $k = 1, 2, \ldots\ldots$ (35)

At the optimum (stationary) point, the optimal steplength $\alpha_k = \alpha_k^*$ is obtained thus:

$$\frac{\partial L(\mathbf{Z}_{k+1},\lambda,\mu)}{\partial \alpha_k} = [(\frac{\partial L(\mathbf{Z}_{k+1},\lambda,\mu)}{\partial \mathbf{Z}_{k+1}}) \times (\frac{\partial \mathbf{Z}_{k+1}}{\partial \alpha_k})]$$
$$= (\mathbf{Z}_{k+1}^T \bar{V}_{\mu k} + \bar{M}_{\mu k}) \times (S_k) = [(\mathbf{Z}_k + \alpha_k S_k)^T \bar{V}_{\mu k} + \bar{M}_{\mu k}][S_k] = 0$$

For $S_k \neq 0$, the optimal steplength is therefore given as,

$$\alpha_k^* = -\frac{(\bar{M}_{\mu k} S_k + \mathbf{Z}_k^T \bar{V}_{\mu k} S_k)}{(S_k^T \bar{V}_{\mu k} S_k)} \quad (36)$$

Moreover, instead of using the "exact minimizer" at the point Z_{k+1} satisfying the equation $\left|\nabla_Z L(\mathbf{Z}_{k+1},\lambda,\mu)\right| = 0$, we employ the inexact minimizer $\left|\nabla_Z L(\mathbf{Z}_{k+1},\lambda,\mu)\right| = \left|\frac{\partial L}{\partial \mathbf{Z}}(\mathbf{Z}_{k+1},\lambda,\mu)\right| \le \varepsilon_k = T$ with $\varepsilon_k \ge 0$ being the current approximation errors. The above exogenous sequence of scalars converges to zero but, in theory, is usually truncated at a tolerance value $\varepsilon_T \approx T^*$, a truncation condition (termination criteria) that guarantees convergence after a number of iterations

3.7. The Algorithm formulation

According to Nocedal et al [10], adopting the Augmented Lagrangian Function (outer loop) for the formulation of the unconstrained programming problem helps to reduce the possibility of ill conditioning, largely preserves smoothness, induce convergence at a faster rate and makes algorithm amenable to standard software for unconstrained or bound-constrained optimization. However, by numerical experience, the BFGS embedded Quasi-Newton method (inner loop) exhibits either a linear or superliner convergence near the optimal value (state variable for the optimal control problem) since it is less influenced by errors in the computation of the optimal steplength according to Olotu and Dawodu [14,15]. Stated below is the algorithm clearly demonstrated by the flowchart (see appendix) for both the delay and non-delay generalized proportional control problems; putting into consideration the key parameters such as the optimal steplength α_i^* in the gradient direction S_i and the Lagrange update formula for λ_{j+1}.

Quasi-Newton Algorithm for Generalized Proportional Control Problem

1. INPUT given variables $\bar{V} \in \mathbb{R}^{mN\times mN}$, $\bar{J} \in \mathbb{R}^{m(N-1)\times 1}$, $\bar{H} \in \mathbb{R}^{m(N-1)\times 1}$ and $\bar{C} \in \mathbb{R}$
2. CHOOSE $\mathbf{Z}_0 \in \mathbb{R}^{mN}$, $B_0 = I$ (identity), T^* (Tol.), initialize $\mu_j > 0 \in \mathbb{R}, \lambda_j^T > 0 \in \mathbb{R}^{m(N-1)\times 1}$ by setting $j = 0$
3. (3a) Set $i = 0$ and $g_0 = \nabla_Z L_{\mu}(\mathbf{Z}_{0,0}) = \nabla L_{\mu 0}$

   (3b) Compute $\bar{V}_{\mu i}, \bar{M}_{\mu i}$ and $\bar{N}_{\mu i}$

   (3c) Set $S_i = -[B_i]g_i$ (search direction)

   (3d) Compute $\alpha_i^* = (\bar{M}_{\mu} S_i + Z_{j,i}^T \bar{V}_{\mu} S_i)(S_i^T \bar{V}_{\mu} S_i)^{-1}$ (steplength) in the direction of S_i)

   (3e) Set $\mathbf{Z}_{j,i+1} = \mathbf{Z}_{j,i} + \alpha_i^* p_i$

   (3f) Compute $g_{i+1} = \nabla_Z L_{\mu}(\mathbf{Z}_{j,i+1}, \lambda_j, \mu_j)$

   (3g) if $\left\|\nabla_Z L_{\mu}(\mathbf{Z}_{j,i+1}, \lambda_j, \mu_j\right\| \le T^*$ go to step 4 else go to 3h

   (3h) Set $q_i = g_{i+1} - g_i$ and $p_i = \mathbf{Z}_{j,i+1} - \mathbf{Z}_{j,i}$

   (3i) Compute BFGS

   $$B_i^u = [1 + \frac{q_i^T B_i q_i}{p_i^T q_i}][\frac{p_i p_i^T}{p_i^T q_i}] - [\frac{(p_i^T q_i B_i) - (B_i q_i p_i^T)}{p_i^T q_i}]$$

   (3j) Set $B_{i+1} = B_i + B_i^u$ and repeat process (3a-3f) in the inner loop for next $i = i + 1$
4. IF $\left\|\bar{J}\mathbf{Z}_{j,i+1} - \bar{H}\right\| \le T^*$ stop! then compute $\mathbf{W}_{j,i+1}^* = M\mathbf{Z}_{j,i+1}^*$ else go to step 5
5. UPDATE $\mu_{j+1} = \mu_0 \times 2^{j+1}$ (penalty) and $\lambda_{j+1} = \lambda_j + \mu_j\left(\bar{J}\mathbf{Z}_{j,i} - \bar{H}\right)$ (multiplier)
6. GO T0 step 3 for next $j = j + 1$ (outer loop)

4. The Analytical Approach

Theorem 3.2

Considering the one-dimensional optimal proportional control problem given as:

$\text{Min}\, J(x,w) = \frac{1}{2}\int_0^T [px(t)^2 + qw(t)^2]dt$ subject to $\dot{x}(t) = ax(t) + bw(t),\ t \in [0,T]$, $x(0)=x_0$, $w(t) = mx(t)\ \ p,q,a,b,m \in R \text{ and } p,q > 0$; where the optimal control $w^*(t)$ proportional to the solution $x^*(t)$ of the state system, at a constant rate m for $m \in \mathbb{R}$, minimizes the performance index $J(x,w)$ over T . Then there exist a unique solution that satisfies the condition $a+bm<0$ with the proportional control constant and optimal objective values defined below as

$$m = -\frac{1}{b}\left[a + \sqrt{\frac{(pb^2+qa^2)}{q}}\right]$$

and

$$J^*(m) = \frac{x_0^2(p+qm^2)}{4(a+bm)}\left[e^{2(a+bm)T} - 1\right]$$

respectively.

(Olotu & Dawodu [14]).

Considering the generalized proportional control problem in section 2.0 with non-delay by setting $C = 0$, then as a corollary to theorem 3.2 above, the optimal objective value and feedback law are stated below:

Proportional control constant:

$$M = -inv(B)\left\{A + Sqrt[(PB^2 + QA^2)*inv(Q)]\right\}$$

State trajectory:

$$\mathbf{X}(t) = \mathbf{X}(0) * e\,\text{xp}(A+BM)t,\ t \in [0,Z]$$

Optimal control law:

$$\mathbf{W}(t) = M\mathbf{X}(t) = -\left\{inv(B)*[A + Sqrt[(PB^2+QA^2)*inv(Q)]\right\}\mathbf{X}(t)$$

Optimal objective value:

$$J^*(M) = X_0^2(0)[P+QM^2]*inv[4(A+BM)][\exp[2(A+BM)t)-1]$$

Provided $A+BM \prec 0$, (negative-definite), $P \succ 0$, $Q \succ 0$ (positive-definite) and $B \neq 0$ (non-singular)

5. Results and Analysis

5.1. Hypothetical Example

In this section, we demonstrate the reliability of our approach to the discretized optimal proportional control problem and result compared with the solution obtained by the analytical method using Euler-Lagrange. All computations in the following example were performed in the MATLAB environment, running on a Microsoft Window 7 operating system with DELL processor of 1.67 GHz Intel® Atom (TM) CPU.

Example: Consider the non-delayed generalized optimal proportional control problem

$$Min\, J(.) = \tfrac{1}{2}\int_0^{10}[2x_1^2 + x_1x_2 + x_2^2 + w_1^2 + \tfrac{1}{2}w_1w_2 + w_2^2]dt \quad (37)$$

Subject to:

$$\left.\begin{array}{l}\dot{x}_1 = x_1 - w_1,\ x_2 = x_2 + w_1 + w_2 \\ \mathbf{X}(0) = (1\ \ 1), \mathbf{W}(t) = M\mathbf{X}(t)\end{array}\right\} \quad (38)$$

From above, the vector-matrix coefficients of the 2-dimensional state and control variables $\mathbf{X}(t) = (x_1(t), x_2(t))$ *and* $\mathbf{W}(t) = (w_1(t), w_2(t))$ respectively are represented below as:

$$P = \begin{pmatrix} 2 & \tfrac{1}{2} \\ \tfrac{1}{2} & 1 \end{pmatrix},\ Q = \begin{pmatrix} 1 & \tfrac{1}{4} \\ \tfrac{1}{4} & 1 \end{pmatrix},\ A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and}$$

$$\mathbf{X}^T(0) = (1\ \ 1)$$

Where P and Q are real, symmetric and positive definite and the optimal proportional control constant M obtained from the matrix representation of the derived formula in subsection (4.0) above.

$$M = -inv(B)*\left\{A + \left((PB^2 + QA^2)*inv(Q)\right)^{\tfrac{1}{2}}\right\} \quad (39)$$

$$M = \begin{pmatrix} 2.0000 & 0 \\ -2.5164 & -1.7127 \end{pmatrix} \text{ and } A+BM = \begin{pmatrix} -1.0000 & 0 \\ -0.5164 & -0.7127 \end{pmatrix} \quad (40)$$

The matrix $A+BM$ in equation (40) above is negative-semi definite (i.e. $A+BM \prec 0$) to guarantee convergence of the problem to a unique solution, even for increasingly large values of the final time Z. The analytical objective value from the proportional control problem with the parameters given above is J_A = 4.519348, for Z=10. The numerical objective value from the Quasi-Newton based Augmented Lagrangian Method, using MATLAB subroutine, is $J_N = 4.519137$. Here, we take $\mu = 10^4$, $\varepsilon = 10^{-3}$, $h = 0.5$ *for* large $Z = 10$ to obtain the numerical results for selected values of the state (X_N) and control (W_N) variables as shown in the table below. The graphical representation of the response of the state and control variables to time within the specified interval is displayed below in figures 1 and 2.

5.2. Discussions

The numerical objective JN =4.519311 value improves and converges, though slower, closer to the analytical objective value by reducing the mesh interval from $h = 0.5$ to $h = 0.1$ for an increasing value of the penalty estimate from $\mu = 10^4$ to $\mu = 10^5$ for fixed value of the tolerance $\varepsilon = 10^{-3}$. The graph of the result for the changed parameters is represented in figures 3 and 4 below to demonstrate the linear convergence of the scheme for increasing values of μ . Analyses on similar results for real coefficients were earlier given by Olotu and Dawodu [14, 15].

Table 1. *Numerical Results of State and Control Variables for the given example using h = 0.5, Z=10.*

TIME	XN(1)	XN(2)	WN(1)	WN(2)
0.0	1.0000	1.0000	2.0000	-4.2291
0.5	0.4876	0.5517	0.9752	-2.1718
1.0	0.2928	0.3255	0.5856	-1.2944
1.5	0.1810	0.1548	0.3619	-0.7204
2.0	0.1057	0.1043	0.2115	-0.4448
2.5	0.0688	0.0258	0.1377	-0.2175
3.0	0.0362	0.0391	0.0724	-0.1580
3.5	0.0285	-0.0149	0.0570	-0.0463
4.0	0.0096	0.0285	0.0191	-0.0665
4.5	0.0149	-0.0265	0.0298	0.0113
5.0	-0.0017	0.0265	-0.0034	-0.0410
5.5	0.0116	-0.0352	0.0232	0.0311
6.0	-0.0079	0.0330	-0.1570	-0.0367
6.5	0.0128	-0.0412	0.0258	0.0384
7.0	-0.0129	0.0411	-0.0258	-0.3790
7.5	0.0165	-0.0482	0.0330	0.0409
8.0	-0.0187	0.0502	-0.0373	-0.0390
8.5	0.0225	-0.0565	0.0449	0.0403
9.0	-0.0262	0.0601	-0.0524	-0.0370
9.5	0.0310	-0.0661	0.0620	0.0352
10.0	-0.0364	0.0707	-0.0728	-0.0294

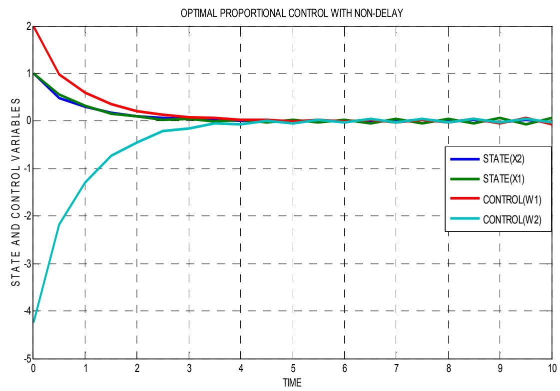


Fig 1. *State and Control trajectories with h=0.5, Tol (ε) =10-3*

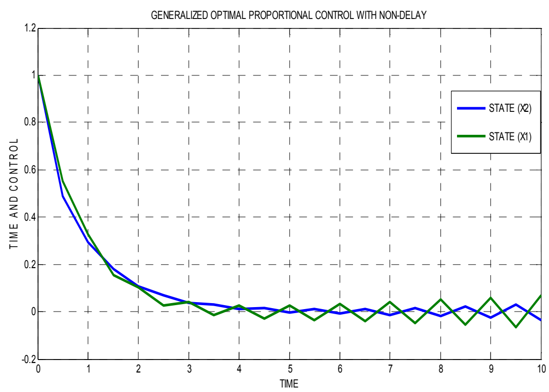


Fig 2. *State trajectory with h=0.5, Tol (ε) = 10-3*

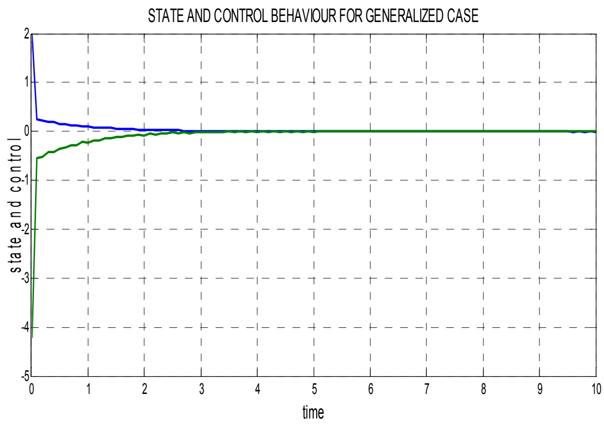


Fig 3. *State and Control trajectories with h=0.1, Tol (ε) = 10-3*

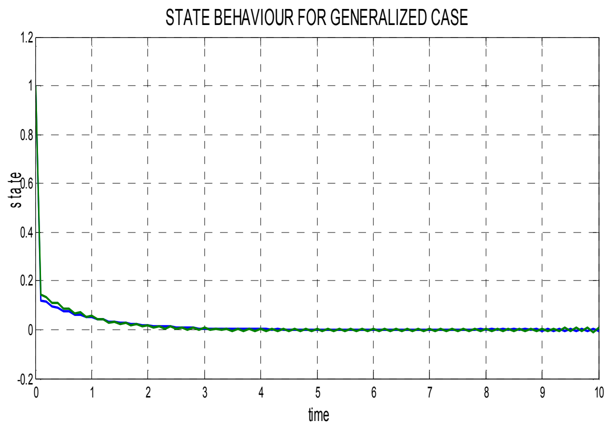


Fig 4. *State trajectory with h = 0.1, Tol (ε) = 10-3*

5.3. Error and Convergence Analyses

Given a sequence $\{\mathbf{Z}(t_k)\} = \{\mathbf{Z}_k\} = \{e_k\} \subset R^{mN\times nN}$ with e_k converging to the optimal solution $\mathbf{Z}^*$ (i.e. $e_k \to \mathbf{Z}^*$) with the rate of convergence measured in terms of the error function $e_k : R^{mN\times mN} \to R$ such that $e_k \geq 0$, $\forall \mathbf{Z}_k \in R^{mN+mN}$ and $e(\mathbf{Z}^*) = 0$. Assuming $e(\mathbf{Z}^*) \neq 0$, $\forall \mathbf{Z}_k$ and

$$\beta = \underset{t\to\infty}{Lim}\frac{e_{k+1}}{e_k^p} = \underset{t\to\infty}{Lim}\frac{\left\|\mathbf{Z}_{k+1} - \mathbf{Z}^*\right\|}{\left(\left\|\mathbf{Z}_k - \mathbf{Z}^*\right\|\right)^p} \in R^+ \tag{41}$$

Then for $p = 1$, $\mathbf{Z}(t_k) = \mathbf{Z}_k$ is said to converge *linearly, superlinearly* or *sub-linearly* if $0 < \beta < 1$, $\beta = 0$ or $\beta = 1$ respectively with the convergence ratio β. If $p = 2$, $\mathbf{Z}_k$ is said to converge *quadratically* if $0 < \beta < 1$ with the convergence ratio β. The convergence ratio profile of the given hypothetical example using the Quasi-newton embedded Augmented Lagrangian algorithm for increasing values of the penalty parameter (μ) is shown in table 2 below.

Table 2. *Convergence ratio profile.*

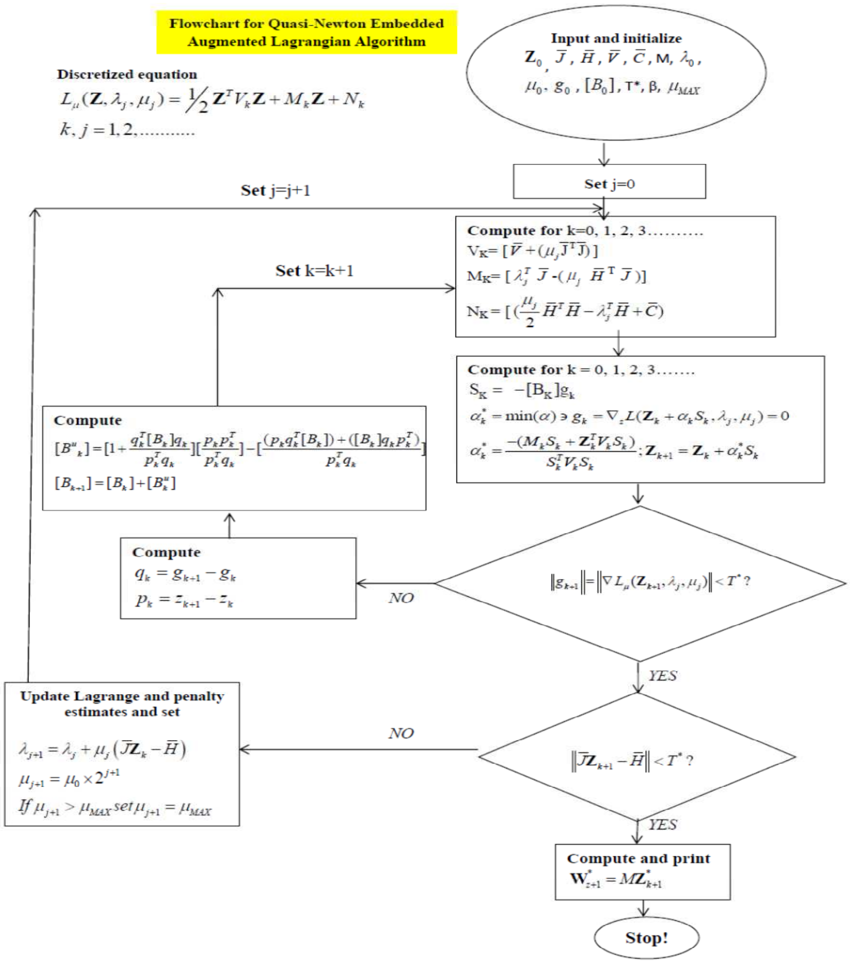
penalty parameter (μ)	Objective value (r)	convergence ratio (β)
1×10^{1}	4.514611	-
1×10^{2}	4.516107	0.614534
1×10^{3}	4.518343	0.310042
1×10^{4}	4.519137	0.210040
1×10^{5}	4.519311	0.174002

The result on the table shows that the convergence ratio hovers round the average figure of $\beta = \mathbf{0.32715}$ for increasing values of the penalty parameter with longer processing time which makes the convergence linear. This convergence is satisfactory for optimization algorithms since the convergence is not close to one.

6. Conclusion

This research paper has been able to showcase the fact that the Quasi-Newton Algorithm constructed via the high profile augmented lagrangian multiplier method formulated to solve the proportional control problem with and without delay by Olotu and dawodu [14, 15] for real coefficients can as well be adapted to the generalized optimal control system with vector and matrix coefficients. The algorithm is well-posed and generates the state and control variables that optimize the generalized objective function with an optimal proportional feedback law. The generated result of the hypothetical example using the algorithm was tested and it responded favourably when compared with the analytically known results.

Appendix: Flowchart



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