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The quasi-squares and their limit curve

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Abstract
The main theme in the book “Introduction to Quasi-Quadrilaterals” is that of quasi-quadrilaterals and their properties. The first topic is about quasi-square represented by the equation $x^{2n} + y^2 = 1 \ or \ x^2 + y^{2n} = 1$.It is well-defined: the curve lies between the unit circle and the specified square which has its center at the origin of the Cartesian system and, sides of length 2 which are parallel to the coordinative axes. These type of closed curves do not represent squares but for values of n larger than 100 they are almost squares. From this phenomenon derives the name “quasi-square”.Also, it is proved that the curve, represented by such equation,perfectly fits to the sides of the specified square as n increases beyond bound. In this paper we present a more general case of the sequence of the quasi-squares and confirm the above fitness by proving that there exists the limit curve of such a sequence. Other subsidiary theorems are proved as well.

1. Introduction

The quasi-square is a closed curve and has only four points in common with the unit circle and the specified square. They are the intersection points of the curves with the coordinative axes. An observation done (using GeoGebra software or other) is that the region between two consecutive curves corresponding to the equations $x^{2n} + y^2 = 1$ and $x^{2(n+1)} + y^2 = 1$, respectively , diminishes as n increases. This region loses of sight as n grows beyond bound, in other words, it completely disappears as $n \to \infty$. The geometrical interpretation of this fact is that for extremely large natural values of n the areas of two neighbor quasi-squares are almost equal or, the ratio of the mentioned areas is 1. This phenomenon is proved by the theorem at the end in which is considered a quasi-square represented by a more general equation:

$$x^{2n} + y^{2n} = 1$$

This equation represents the two cases mentioned above. The content of the theorem and its proof relate to the distance between any point on the quasi-square represented by the equation

$$x^{2n} + y^{2n} = 1$$

and the respective point on the neighbor quasi-square represented by the equation

$$x^{2(n+1)} + y^{2(n+1)} = 1$$

2. Existence of the Limit Curve

Let be given the quasi-square represented by the equation

$$x^{2n}+y^{2n}=1,\ n\in\mathbb{N}; x,y\in\mathbb{R}$$

Let M= (X,Y) be any point on the curve. Then its coordinates satisfy the equation:

$$X^{2n}+Y^{2n}=1$$

We use polar coordinates for a point M instead of rectangular or x-y coordinates. They are determined by the distance of point M from the origin O (denoted by r) and by the angle the line OM forms with the positive x-axis (denoted by θ). Then, the coordinates of any point on the quasi-square are: $\left((sin\theta)^{\frac{1}{n}},(cos\theta)^{\frac{1}{n}}\right)$. The distance of the point M from the origin is:

$$D(M,O)=\sqrt{\left((sin\theta)^{\frac{1}{n}}\right)^2+\left((cos\theta)^{\frac{1}{n}}\right)^2}=\sqrt{(sin^2\theta)^{\frac{1}{n}}+(cos^2\theta)^{\frac{1}{n}}}$$

The coordinates of the respective point N on the quasi-square represented by the equation

$$x^{2(n+1)}+y^{2(n+1)}=1$$

are $\left((sin\theta)^{\frac{1}{n+1}},(cos\theta)^{\frac{1}{n+1}}\right)$, and its distance from the origin is:

$$D(N,O)=\sqrt{\left((sin\theta)^{\frac{1}{n+1}}\right)^2+\left((cos\theta)^{\frac{1}{n+1}}\right)^2}=\sqrt{(sin^2\theta)^{\frac{1}{n+1}}+(cos^2\theta)^{\frac{1}{n+1}}}$$

The points O, M and N are not on a straight line, but this statement is another topic - not part of this paper. M and N are two respective points belonging to two neighbor or consecutive curves, respectively. As mentioned above, the areas of two neighbor quasi-squares are almost equal, or their ratio is 1, for extremely large natural values of n. This statement is equivalent to the statement that the distance between two respective points of two consecutive curves (that is, corresponding to the values n and n+1 of the exponent) becomes smaller and smaller beyond bound. The points O, M and N are not on a straight line (this can be easily seen for the first few values of n), but the points M and N are so close for large values of n, and merge in one point for extremely large values of n, hence it is not a significant error to estimate the distance between them using the difference $D(N,O)-D(M,O)$ or their ratio: $D(N,O)/D(M,O)$. The error done for large values of nis insignificant. The error continually decreases as n is increased, it is almost 0. The above statement is equivalent to the statement that $D(N,O)-D(M,O)=0$ or $D(N,O)/D(M,O)=1$. To prove the above statement, for simplicity, we have chosen the second option. So, we prove that

$$\lim_{n\to\infty}\frac{D(N,O)}{D(M,O)}=\lim_{n\to\infty}\frac{\sqrt{(sin^2\theta)^{\frac{1}{n+1}}+(cos^2\theta)^{\frac{1}{n+1}}}}{\sqrt{(sin^2\theta)^{\frac{1}{n}}+(cos^2\theta)^{\frac{1}{n}}}}=1$$

which is equivalent to the statement:

$$\lim_{n\to\infty}\frac{(sin^2\theta)^{\frac{1}{n+1}}+(cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}}+(cos^2\theta)^{\frac{1}{n}}}=1$$

Consequently, we prove the following statement.

Theorem1 Let be given the trigonometric function

$$f(\theta,n)=\frac{(sin^2\theta)^{\frac{1}{n+1}}+(cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}}+(cos^2\theta)^{\frac{1}{n}}}$$

determined on $\mathbb{R}$ ($n\in\mathbb{N}$, $\theta\in\mathbb{R}$, where θ is expressed in radian - $\mathbb{R}$ is the set of real numbers). Then, this function is non-increasing on the respective domain with respect to the variable n.

Proof: For two consecutive values of n consider the difference of the respective function values (denoted as below):

$$\Delta=f(n)-f(n+1)=\frac{(sin^2\theta)^{\frac{1}{n+1}}+(cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}}+(cos^2\theta)^{\frac{1}{n}}}-\frac{(sin^2\theta)^{\frac{1}{n+2}}+(cos^2\theta)^{\frac{1}{n+2}}}{(sin^2\theta)^{\frac{1}{n+1}}+(cos^2\theta)^{\frac{1}{n+1}}}=$$

$$=\frac{\left[(sin^2\theta)^{\frac{1}{n+1}}+(cos^2\theta)^{\frac{1}{n+1}}\right]^2-\left[(sin^2\theta)^{\frac{1}{n+2}}+(cos^2\theta)^{\frac{1}{n+2}}\right]\cdot\left[(sin^2\theta)^{\frac{1}{n}}+(cos^2\theta)^{\frac{1}{n}}\right]}{\left[(sin^2\theta)^{\frac{1}{n+1}}+(cos^2\theta)^{\frac{1}{n+1}}\right]\cdot\left[(sin^2\theta)^{\frac{1}{n}}+(cos^2\theta)^{\frac{1}{n}}\right]}\qquad(1)$$

Since we have trigonometric functions then the following inequality implications hold true:

$$0\le sin^2\theta\le 1\Longrightarrow 0\le(sin^2\theta)^{\frac{1}{n}}\le(sin^2\theta)^{\frac{1}{n+1}}\le(sin^2\theta)^{\frac{1}{n+2}}\qquad(2)$$

$$0 \leq cos^2\theta \leq 1 \Longrightarrow 0 \leq (cos^2\theta)^{\frac{1}{n}} \leq (cos^2\theta)^{\frac{1}{n+1}} \leq (cos^2\theta)^{\frac{1}{n+2}}$$

It follows that

$$(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}} \leq (sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}} and$$
$$(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}} \leq (sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}} \quad (3)$$

From (3) it follows that

$$\begin{cases} (sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}} = (sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}} + \alpha(\alpha \geq 0) \\ (sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}} = (sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}} - \beta \quad (\beta \geq 0) \end{cases} \quad (4)$$

Multiplying side by side in (4) we have

$$\left[(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}\right]^2 = \left[(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}\right] \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right] +$$
$$+\alpha \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right] - \beta \cdot \left[(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}\right] - \alpha \cdot \beta \quad (5)$$

But,

$$\alpha \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right] - \beta \cdot \left[(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}\right] - \alpha \cdot \beta =$$
$$= \alpha \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right] - \beta \cdot \left\{\left[(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}\right] + \alpha\right\} =$$
$$= \alpha \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right] - \beta \cdot \left[(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}\right] \quad (6)$$

Denoting by

$$\varphi(n) = (sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}} - (sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}$$

(4) can be written as following:

$$\begin{cases} \alpha = \varphi(n) \\ \beta = \varphi(n+1) \end{cases}$$

Can be easily proved (by differentiating or by subtracting) that the functional sequence $\{\varphi(n)\}$ is a strictly decreasing one with respect to n. That is:

$$n < n+1 \Longrightarrow \varphi(n) > \varphi(n+1)$$

Consequently, $\alpha > \beta$.

From (3) we have:

$$(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}} \leq (sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}$$

Hence,

$$\beta \cdot \left[(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}\right] \leq \alpha \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right] \Leftrightarrow$$

$$\alpha \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right] - \beta \cdot \left[(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}\right] \geq 0$$

Turning back to (6) we get,

$$\alpha \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right] - \beta \cdot \left[(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}\right] - \alpha \cdot \beta \geq 0$$

Therefore, considering (5), it is true that

$$\left[(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}\right]^2 \geq \left[(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}\right] \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right] \quad (7)$$

Thus way, in (1) the nominator is non-negative, whereas the denominator is positive since the expression $p(\theta, n) = (sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}$ takes only positive values for every $\theta \in \mathbb{R}$. For values of θ such that $\theta = k \cdot \pi + \frac{\pi}{2}$ or $\theta = k \cdot \pi (k \in \mathbb{Z})$ we have:

$$p\left(\theta = k \cdot \pi + \frac{\pi}{2}, n\right) = p(\theta = k \cdot \pi, n) = 1$$

For the other values of θ we observe that

$$p(\theta, 1) = sin^2\theta + cos^2\theta = 1$$

$$n \geq 1 \Longrightarrow (sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}} \geq sin^2\theta + cos^2\theta = 1 \quad (8)$$

Comparing (1) with results in (7) and (8) conclude that

$$f(n) - f(n+1) \geq 0 \Leftrightarrow f(n) \geq f(n+1)$$

This finishes the proof of the theorem. □

Note: The equality stands only for values of $\theta\ satisfying\ the\ conditions$:

$\theta = k \cdot \pi + \frac{\pi}{2}$or $\theta = k \cdot \pi (k \in \mathbb{Z})$. For θ satisfying the condition $\theta \neq k \cdot \pi + \frac{\pi}{2}$ and $\theta \neq k \cdot \pi (k \in \mathbb{Z})$ we have a strict inequality:

$$\left[(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}\right]^2 > \left[(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}\right] \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right]$$

Hence, the given function is strictly decreasing with respect to n. That is,

$$f(n) > f(n+1)$$

Since, $(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}} \geq 1 > 0$, $\forall\, n \in \mathbb{N}$, inequality (7) can be written as

$$\frac{\left[(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}\right] \cdot \left[(sin^2\theta)^{\frac{1}{n+2}} + (cos^2\theta)^{\frac{1}{n+2}}\right]}{\left[(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}\right]^2} \leq 1 \quad (9)$$

Corollary: Let be n (natural number) and $\theta \in \mathbb{R}$ (θ is expressed in radian where $\mathbb{R}$ is the set of real numbers). Then

1) The sequence of functions $p(\theta, n) = (sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}$ is non-decreasing with respect to n and bounded on $\mathbb{R}$.

2) The sequence of functions

$$f(\theta, n) = \frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}}$$

is non-increasing with respect to n and bounded in $\mathbb{R}$.

The first part is straightforward:

$$1 = sin^2\theta + cos^2\theta \leq (sin^2\theta)^{\frac{1}{2}} + (cos^2\theta)^{\frac{1}{2}} \leq \cdots \leq (sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}} \leq \cdots < 2$$

The equalities stand for values of variable θ such that $\theta = k \cdot \pi\ or\ \theta = k \cdot \pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$.

From (2) can be seen that

$$(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}} < 2\ for\ \forall\, n \in \mathbb{N}\ and\ \forall\, \theta \in \mathbb{R}$$

Notice that $p(\theta, n) = (sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}} < 2$ since for no value of θ we have simultaneously $sin\theta = 1\ and\ cos\theta = 1$.

The second part derives from the theorem just proved. So, we can write (starting with n=1):

$$\frac{(sin^2\theta)^{\frac{1}{2}} + (cos^2\theta)^{\frac{1}{2}}}{sin^2\theta + cos^2\theta} \geq \frac{(sin^2\theta)^{\frac{1}{3}} + (cos^2\theta)^{\frac{1}{3}}}{(sin^2\theta)^{\frac{1}{2}} + (cos^2\theta)^{\frac{1}{2}}} \geq \cdots \geq \frac{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}}{(sin^2\theta)^{\frac{1}{n-1}} + (cos^2\theta)^{\frac{1}{n-1}}} \geq \cdots$$

As in the first part, the equalities stand for values of variable θ such that $\theta = k \cdot \pi\ or\ \theta = k \cdot \pi + \frac{\pi}{2}$, $k \in \mathbb{Z}$.

Let find the maximum value of the first term using the differentiation technique.

Denote by

$$f(\theta) = \frac{(sin^2\theta)^{\frac{1}{2}} + (cos^2\theta)^{\frac{1}{2}}}{sin^2\theta + cos^2\theta} = \sqrt{sin^2\theta} + \sqrt{cos^2\theta}\, , \theta \in \mathbb{R}\ .$$

$$f'(\theta) = \frac{2sin\theta \cdot cos\theta}{2\sqrt{sin^2\theta}} - \frac{2sin\theta \cdot cos\theta}{2\sqrt{cos^2\theta}} = \frac{\sin(2\theta) \cdot \left(\sqrt{cos^2\theta} - \sqrt{sin^2\theta}\right)}{2 \cdot \sqrt{sin^2\theta} \cdot \sqrt{cos^2\theta}}$$

The domain of the derivative is $\theta \in \mathbb{R}\ \ and\ \ \theta \neq k \cdot \pi and \theta \neq k \cdot \frac{\pi}{2}$

$$f'(\theta) = 0 \Leftrightarrow sin\theta = cos\theta \Leftrightarrow tan\theta = 1 \Rightarrow$$

$$\theta = k \cdot \pi + \frac{\pi}{4}$$

The case $\sin(2\theta) = 0$ is excluded because:

$\sin(2\theta) = 0 \Leftrightarrow 2\theta = k \cdot \pi \Leftrightarrow \theta = k \cdot \frac{\pi}{2}$ but $\theta \neq k \cdot \frac{\pi}{2}$?!

It follows, then, that

$$f'(\theta) = \sqrt{cos^2\theta} - \sqrt{sin^2\theta} = 0 \Rightarrow cos\theta = sin\theta \Rightarrow tan\theta = 1 \Rightarrow$$

$$\theta = k \cdot \pi + \frac{\pi}{4}$$

Having into consideration the expression for θ, which is the set of the roots of the equation $f'(\theta) = 0$, we determine the signs of $f'(\theta)$ in the following intervals. They include all possible situations and any one can confirm the results by using the unit circle for trigonometric functions.

2.1. First Case

$$\theta \in \left]2k \cdot \pi - \frac{\pi}{4}, 2k \cdot \pi + \frac{\pi}{4}\right[, \theta \neq 2k \cdot \pi \text{or} \theta \in \left]2k \cdot \pi + \frac{3\pi}{4}, 2k \cdot \pi + \frac{5\pi}{4}\right[, \theta \neq (2k+1) \cdot \pi$$

$$\left(\frac{\sqrt{2}}{2} < cos\theta < 1 \ and -\frac{\sqrt{2}}{2} < sin\theta < \frac{\sqrt{2}}{2}\right) \Rightarrow \left(\frac{1}{2} < cos^2\theta < 1 \ and \ 0 < sin^2\theta < \frac{1}{2}\right) or$$

$$\left(-1 < cos\theta < -\frac{\sqrt{2}}{2} and -\frac{\sqrt{2}}{2} < sin\theta < \frac{\sqrt{2}}{2}\right) \Rightarrow \left(\frac{1}{2} < cos^2\theta < 1 \ and \ 0 < sin^2\theta < \frac{1}{2}\right)$$

In the two intervals it is true the implication: $cos^2\theta > sin^2\theta \Rightarrow \sqrt{cos^2\theta} - \sqrt{sin^2\theta} > 0$.

2.2. Second Case

$$\theta \in \left]2k\cdot\pi + \frac{\pi}{4}, 2k\cdot\pi + \frac{3\pi}{4}\right[, \theta \neq 2k\cdot\pi + \frac{\pi}{2} \text{or} \theta \in \left]2k\cdot\pi + \frac{5\pi}{4}, 2k\cdot\pi + \frac{7\pi}{4}\right[\theta \neq 2k\cdot\pi + 3\cdot\frac{\pi}{2}$$

$$\left(\frac{\sqrt{2}}{2} < sin\theta < 1 \ and -\frac{\sqrt{2}}{2} < cos\theta < \frac{\sqrt{2}}{2}\right) \Rightarrow \left(\frac{1}{2} < sin^2\theta < 1 \ and \ 0 < cos^2\theta < \frac{1}{2}\right) \textbf{or}$$

$$-1 < sin\theta < -\frac{\sqrt{2}}{2} and -\frac{\sqrt{2}}{2} < cos\theta < \frac{\sqrt{2}}{2} \Rightarrow \frac{1}{2} < sin^2\theta < 1 \ and \ 0 < cos^2\theta < \frac{1}{2}$$

On the two intervals it is true the implication: $cos^2\theta < sin^2\theta \Rightarrow \sqrt{cos^2\theta} - \sqrt{sin^2\theta} < 0$.

Identical results can be obtained on infinitely many other intervals, so we summarize the above results in the following table. Because of the limited space on the table we study the sign of derivative in four intervals (even six), and we denote the respective roots of the derivative and some important values by:

$$\theta_1 = 2k\cdot\pi + \frac{\pi}{4}, \theta_2 = 2k\cdot\pi + \frac{\pi}{2}, \theta_3 = 2k\cdot\pi + \frac{3\pi}{4}, \theta_4 = 2k\cdot\pi + \pi, \theta_5 = 2k\cdot\pi + \frac{5\pi}{4}$$

$\boldsymbol{\theta}$		$\boldsymbol{\theta_1}$	$\boldsymbol{\theta_1 < \theta < \theta_2}$	$\boldsymbol{\theta_2}$	$\boldsymbol{\theta_2 < \theta < \theta_3}$	$\boldsymbol{\theta_3}$	$\boldsymbol{\theta_3 < \theta < \theta_4}$	$\boldsymbol{\theta_4}$	$\boldsymbol{\theta_4 < \theta < \theta_5}$	$\boldsymbol{\theta_5}$	
$\sqrt{cos^2\theta} - \sqrt{sin^2\theta}$	+	0	-		-	0	+	0	+	0	-
sin(2θ)	+		+	0	-		-	0	+		+
$f'(\theta)$	+	0	-	?	+	0	-	?	+	0	-
$f(\theta)$	↗		↘		↗		↘		↗		↘

From the table can be understood that at the points θ_1, θ_3 and θ_5 the given function attains maximum. At the points θ_2 and θ_4 the given function does not attain minimum because its derivative is undefined at these points.

Calculate the maximum:

$$f_{max} = f\left(\theta = 2k\cdot\pi + \frac{\pi}{4}\right) = f\left(\theta = 2k\cdot\pi + \frac{3\pi}{4}\right) = f\left(\theta = 2k\cdot\pi + \frac{5\pi}{4}\right) =$$

$$= \sqrt{sin^2\left(2k\cdot\pi + \frac{\pi}{4}\right)} + \sqrt{cos^2\left(2k\cdot\pi + \frac{\pi}{4}\right)} = \sqrt{sin^2\left(\frac{\pi}{4}\right)} + \sqrt{cos^2\left(\frac{\pi}{4}\right)} = \sqrt{2}$$

Conclude, this way, that the lowest upper bound of the sequence of functions

$$f(\theta, n) = \frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}}$$

is $\sqrt{2}$. On the other side we have:

$$(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}} \geq (sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}} \Rightarrow$$

$$\frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}} \geq 1$$

Definitely, we have:

$$\sqrt{2} \geq \frac{(sin^2\theta)^{\frac{1}{2}} + (cos^2\theta)^{\frac{1}{2}}}{sin^2\theta + cos^2\theta} \geq \frac{(sin^2\theta)^{\frac{1}{3}} + (cos^2\theta)^{\frac{1}{3}}}{(sin^2\theta)^{\frac{1}{2}} + (cos^2\theta)^{\frac{1}{2}}} \geq \cdots \geq \frac{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}}{(sin^2\theta)^{\frac{1}{n-1}} + (cos^2\theta)^{\frac{1}{n-1}}} \geq \cdots \geq 1$$

The sequence of the above functions is bounded on both sides.

Prove now the next theorem related to the uniform convergence of the sequence of the above functions.

*Theorem 2:*Let be given the trigonometric function

$$f(\theta, n) = \frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}}$$

determined on $\mathbb{R}$ ($n \in \mathbb{N}$, $\theta \in \mathbb{R}$ where θ is expressed in radian and $\mathbb{R}$ is the set of real numbers). Then, the sequence of given functions converges uniformly to 1 on the respective domain as n increases beyond bound.

Proof For values of θ such that $\theta = k\cdot\pi + \frac{\pi}{2}$ or $\theta = k\cdot\pi (k \in \mathbb{Z})$ we have:

$$f(\theta, n) = 1$$

Consider values of θ such that $\theta \in \mathbb{R}$ and$\theta \neq k\cdot\pi + \frac{\pi}{2}$ or $\theta \neq k\cdot\pi (k \in \mathbb{Z})$. There are two general cases to treat with the respect to the values of $sin^2\theta$ and $cos^2\theta$.

Case 1 Values of θ such that $0 < |sin\theta| \leq \frac{\sqrt{2}}{2}$ hence $\frac{\sqrt{2}}{2} \leq |cos\theta| < 1$. This means that $0 < sin^2\theta \leq \frac{1}{2}$ and $\frac{1}{2} \leq cos^2\theta < 1$.

It follows that $sin^2\theta \le cos^2\theta \Longrightarrow (sin^2\theta)^{\frac{1}{n}} \le (cos^2\theta)^{\frac{1}{n}} \Longrightarrow$

$$(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}} \ge 2 \cdot (sin^2\theta)^{\frac{1}{n}} \Longrightarrow \frac{1}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}} \le \frac{1}{2 \cdot (sin^2\theta)^{\frac{1}{n}}} \Longrightarrow$$

$$\frac{(sin^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}} \le \frac{(sin^2\theta)^{\frac{1}{n+1}}}{2 \cdot (sin^2\theta)^{\frac{1}{n}}}$$

On the other side, $(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}} \le 2 \cdot (cos^2\theta)^{\frac{1}{n}} \Longrightarrow$

$$\frac{1}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}} \ge \frac{1}{2 \cdot (cos^2\theta)^{\frac{1}{n}}} \Longrightarrow$$

$$\frac{(sin^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}} \ge \frac{(sin^2\theta)^{\frac{1}{n+1}}}{2 \cdot (cos^2\theta)^{\frac{1}{n}}}$$

Definitely,

$$\frac{(sin^2\theta)^{\frac{1}{n+1}}}{2 \cdot (cos^2\theta)^{\frac{1}{n}}} \le \frac{(sin^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}} \le \frac{(sin^2\theta)^{\frac{1}{n+1}}}{2 \cdot (sin^2\theta)^{\frac{1}{n}}} \quad (10)$$

Similarly we have:

$$\frac{1}{2 \cdot (cos^2\theta)^{\frac{1}{n}}} \le \frac{1}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}} \le \frac{1}{2 \cdot (sin^2\theta)^{\frac{1}{n}}} \Longrightarrow$$

$$\frac{(cos^2\theta)^{\frac{1}{n+1}}}{2 \cdot (cos^2\theta)^{\frac{1}{n}}} \le \frac{(cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}} \le \frac{(cos^2\theta)^{\frac{1}{n+1}}}{2 \cdot (sin^2\theta)^{\frac{1}{n}}} \quad (11)$$

Adding up side by side (10) and (11) we obtain:

$$\frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{2 \cdot (cos^2\theta)^{\frac{1}{n}}} \le \frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}} \le \frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{2 \cdot (sin^2\theta)^{\frac{1}{n}}}$$

But,

$$\frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{2 \cdot (sin^2\theta)^{\frac{1}{n}}} \le \frac{2 \cdot (cos^2\theta)^{\frac{1}{n+1}}}{2 \cdot (sin^2\theta)^{\frac{1}{n}}} \, and \, \frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{2 \cdot (cos^2\theta)^{\frac{1}{n}}} \ge \frac{2 \cdot (sin^2\theta)^{\frac{1}{n+1}}}{2 \cdot (cos^2\theta)^{\frac{1}{n}}}$$

It follows that

$$\frac{(sin^2\theta)^{\frac{1}{n+1}}}{(cos^2\theta)^{\frac{1}{n}}} \le \frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}} \le \frac{(cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}}} \quad (12)$$

We show now that the functions $R_n(\theta) = \frac{(cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}}}$ (on the right side) and $L_n(\theta) = \frac{(sin^2\theta)^{\frac{1}{n+1}}}{(cos^2\theta)^{\frac{1}{n}}}$ (on the left side) converge uniformly on the given interval.

Consider $ln\big(R_n(\theta)\big) = \frac{1}{n+1} \cdot ln(cos^2\theta) - \frac{1}{n} \cdot ln(sin^2\theta) \xrightarrow{n\to\infty} 0 - 0 = 0$

It follows that,

$$R_n(\theta) \xrightarrow{n\to\infty} 1$$

In the same way is proved that

$$L_n(\theta) \xrightarrow{n\to\infty} 1$$

Applying the squeeze theorem is obtained that the sequence of functions

$$f(\theta, n) = \frac{(sin^2\theta)^{\frac{1}{n+1}} + (cos^2\theta)^{\frac{1}{n+1}}}{(sin^2\theta)^{\frac{1}{n}} + (cos^2\theta)^{\frac{1}{n}}}$$

converges uniformly to 1.

Case 2: Values of θ such that $0 < |cos\theta| \le \frac{\sqrt{2}}{2}$ hence $\frac{\sqrt{2}}{2} \le |sin\theta| < 1$. This means that $0 < cos^2\theta \le \frac{1}{2}$ and $\frac{1}{2} \le sin^2\theta < 1$. As in the previous case, following the same steps of estimations, it is proved again that the sequence of given functions converges uniformly to 1. This concludes the theorem.

Theorem 3 There exists the limit curve of the quasi-square represented by the equation $x^{2n} + y^{2n} = 1$ ($n \in \mathbb{N}$; $x, y \in \mathbb{R}$) as n increases beyond bound.

Proof: The quasi-square is a closed curve lied between the unit circle and the specified square: square with center at the origin of the Cartesian system and sides of length 2 which are parallel to the coordinative axes. Indeed, the equation shows that the coordinates of a point M = (X,Y) on the quasi-square satisfy the condition:

$$|X| \le 1 \; and |Y| \le 1$$

and, there is no case that $|X| = |Y| = 1$. The condition of inequalities implies that the points of the quasi-square are within the specified square.

On the other hand, the points of the unit circle have a distance of 1 unit from the center of the coordinative system. Compare the distance of any points on the quasi-square from the origin with the radius of the unit circle, by calculating the difference. The distance of any points of the quasi-square from the origin is:

$$D = \sqrt{X^2 + Y^2} = \sqrt{X^2 + (1 - X^{2n})^{\frac{1}{n}}} \Longrightarrow$$

$$D-1=\sqrt{X^2+(1-X^{2n})^{\frac{1}{n}}}-1$$
$$=\frac{X^2+(1-X^{2n})^{\frac{1}{n}}-1}{\sqrt{X^2+(1-X^{2n})^{\frac{1}{n}}}+1}$$

But, $|X|\leq 1 \Rightarrow X^{2n}\leq 1 \Rightarrow 0\leq 1-X^{2n}\leq 1 \Rightarrow (1-X^{2n})^{\frac{1}{n}}\geq 1-X^{2n} \Rightarrow$

$$(1-X^{2n})^{\frac{1}{n}}-1\geq 1-X^{2n}-1=-X^{2n}$$

On the other side, $|X|\leq 1 \Rightarrow X^{2n}\leq X^2 \Rightarrow X^2+(1-X^{2n})^{\frac{1}{n}}-1\geq X^2-X^{2n}\geq 0$

$$\Rightarrow D-1\geq 0 \;\Rightarrow D\geq 1$$

The obtained result shows that the points of the quasi-square are outside the unit circle, or on it.

Also, in the first part of the corollary it is proved that the sequence of functions $p(\theta,n)=(sin^2\theta)^{\frac{1}{n}}+(cos^2\theta)^{\frac{1}{n}}$ is non-decreasing with respect to n and bounded on $\mathbb{R}$. This means that the distance of any points on the quasi-square, which is $\sqrt{(sin^2\theta)^{\frac{1}{n}}+(cos^2\theta)^{\frac{1}{n}}}$, is continually growing as n increases but it is smaller than $\sqrt{2}$ (the sequence is bounded). In addition, there exists the limit of the ratio of the distances from the origin of the respective points on the neighbor curves as n grows beyond bound, fact which is supported by theorems 1 and 2:

$$\lim_{n\to\infty}\frac{D(N,O)}{D(M,O)}=\lim_{n\to\infty}\frac{\sqrt{(sin^2\theta)^{\frac{1}{n+1}}+(cos^2\theta)^{\frac{1}{n+1}}}}{\sqrt{(sin^2\theta)^{\frac{1}{n}}+(cos^2\theta)^{\frac{1}{n}}}}=1$$

It is a uniform convergence (by theorem 2). The point chosen in the above arguments is arbitrary and it has a limit position. Follows that there exists the limit curve of the sequence of the quasi-squares represented by the equation $x^{2n}+y^{2n}=1$ as n increases beyond bound.

3. Conclusion

Paper results are of special interest: firstly, for mathematicians in studying the properties of the above family of closed curves by the use of analytical tools and methods and exploring other properties of such curves; secondly, for engineers and architects for whom the most of designs appear to be quasi-rectangles, or parts of quasi-rectangles. Knowing the algebraic representations of such curves it is easier for them, using software, to construct their geometric representations.

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