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On the solution to a fractional Black-Scholes equation for the price of an option

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Abstract
The purpose of this paper is to obtain a solution for a fractional Black- Scholes formula for the price of an option for every $t\epsilon[0,T]$. For this purpose, we first derive the Black- Scholes equation for a generic pay-off function whose value is equivalently the worth of the stock at time t. We further obtain the equilibrium price and growth rate of the stock that is priced in the market. An analysis of the stability and convergence of the solution is given in concrete setting.

1. Introduction

Fractional differential equation (FDE) can be extensively applied to various disciplines such asphysics, mechanics, chemistry and engineering, see [1-3]. Hence, in recent years, fractionaldifferential equations have been of great interest and there have been many results onexistence and uniqueness of the solutions of FDE, see [4-8], thus giving good motivation forfurther development of this topic. A fractional Black-Scholes formula for the price of an option for every $t\epsilon\ [0.T]$driven by a fractional Brownian motion is a family member of the FDE. Let$(\Omega, F, \mathbb{P})$be a complete probability space. A standard fractional Brownian motion (fBm)
$\{B_H(t), t \in \mathbb{R}\}$with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process with continuous sample paths such that

$$\mathbb{E}[\beta^H(t)\beta^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}) \qquad (1.1)$$

for $s, t \in \mathbb{R}$. It is clear that for $H = 1/2$, this process is a standard Brownian motion. In this paper, it is assumed that $H \in \left(\frac{1}{2}, 1\right)$.

This process has been introduced and studied by researchers [9 and the references therein]. Its self-similar and long-range dependence make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields.

Consider a time interval $[0, T]$ with arbitrary fixed horizon T and let
$\{\beta^H(t), t \in [0,T]\}$ the one-dimensional fractional Brownian motion with Hurst Parameter $H \in \left(\frac{1}{2}, 1\right)$. It is well known that β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K_H(t,s)d\beta(s), \qquad (1.2)$$

where $\beta = \{\beta(t): t \in [0,T]\}$ is a Wiener process, and $K_H(t,s)$ is the kernel given by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad (1.3)$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)}}$ and $\beta(,)$ denotes the Beta function. We put $K_H(t,s) = 0$ if $t \le s$.

Denote by $\mathcal{H}$ the reproducing kerrnel Hilbert space of the fBm, then $\mathcal{H}$ is the closure of the set of indicator functions $\{1_{[0;t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t;s).$$

The mapping $1_{[0,t]} \to \beta^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ by the previous isometry. Recall that for $\psi, \varphi \in \mathcal{H}$ their scalar product in $\mathcal{H}$ is given by

$$\langle \psi, \varphi \rangle_{\mathcal{H}} = H(2H-1) \int_0^T \int_0^T \psi(s)\varphi(t)|t-s|^{2H-2} ds dt. \quad (1.4)$$

Let us consider the operator K_H^* from $\mathcal{H}$ to $\mathbb{L}^2([0,T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi(r) \frac{\partial K}{\partial r}(r,s) dr. \quad (1.5)$$

K_H^* is an isometry between $\mathcal{H}$ and $\mathbb{L}^2([0,T])$. Moreover for any $\varphi \in \mathcal{H}$ we have

$$\beta^H(\varphi) = \int_0^T (K_H^* \varphi)(t) d\beta(t). \quad (1.6)$$

It follows from[10] that the elements of $\mathcal{H}$ may not be functions but distributions of negative order. In the case $H > \frac{1}{2}$, the second partial derivative of the covariance function

$$\frac{\partial R_H}{\partial t \partial s} = \alpha_H |t-s|^{2H-2}, \quad (1.7)$$

where $\alpha_H = H(2H-2)$, is integrable, and we can write

$$R_H(t,s) = \alpha_H \int_0^T \int_0^s |u-v|^{2H-2} du dv. \quad (1.8)$$

In order to obtain a space of functions contained in $\mathcal{H}$, we consider the linear space $|\mathcal{H}|$ generated by the measurable functions ψ such that

$$\|\psi\|_{|\mathcal{H}|}^2 := \alpha_H \int_0^T \int_0^T |\psi(s)||\varphi(t)||s-t|^{2H-2} ds dt < \infty, \quad (1.9)$$

where $\alpha_H = H(2H-1)$. The space $|\mathcal{H}|$ is a Banach space with the norm $\|\psi\|_{|\mathcal{H}|}$.

If $0 < H < 1$ the fractional Brownian motion (fBm) with Hurst parameter H is the continuous Gaussian process $\{B_H(t), t \in R\}, B_H(t) = 0$ with mean $E[B_H(t)] = 0$ and whose covariance is given as in equation (1.1), If $H = \frac{1}{2}$ then $B_H(t)$ coincides with the standard Brownian motion $B(t)$. The fractional Brownian motion is a self-similar process meaning that for any $\alpha > 0, B_H(\alpha t)$ has the same law as $\alpha^H B_H(t)$.

The constant H determines the sign of the covariance of the future and past increments. This covariance is positive when $H > \frac{1}{2}$, zero when $H = \frac{1}{2}$ and negative when $H < \frac{1}{2}$.

Another property of the fractional Brownian motion is that for $H > \frac{1}{2}$ it has long range dependence in the sense that if we put

$$r(n) = Cov\left(B_H(1), B_H(n+1) - B_H(n)\right) \quad (1.10)$$

then

$$\sum_{n=1}^{\infty} r(n) = \infty.$$

Our aim in this paper is to derive fractional Black-Scholes equation driven by a fractional Brownian motion $B_H(t), \frac{1}{2} < H < 1$. We also determine the equilibrium price and the market growth rate of shares and analyze the stability and convergence criteria of a solution of the fBm in a general case.

2. Derivation of the Fractional Black-Scholes Equation

We base our derivation on replicating portfolio that ensures that no arbitrage opportunities are allowed. As in the discrete case, consider a portfolio $\Lambda = \{\Lambda_t\}_{t>0}$, which is $\mathfrak{F}_t$- measurable(we can choose as we go, but any point in time the choice is deterministic) Λ_t denotes the proportion of shares invested at time t, the rest of the money is invested in the money market account, giving risk-free rate of return, r, say. In what follows, we state:

Lemma 1 (Fractional Ito formula): Consider the fractional differential equation

$$dX(t) = \mu(t,w)dt + \sigma(t,w)dB_H(t), \mu, \sigma \in L_\varphi^{1,2}$$

If $f \in C^2(R_+ \times R)$ then we have;

$$f(t,X(t)) = f(0,X(0)) + \int_0^t \frac{\partial f}{\partial S}(S,X(s))ds + \int_0^t \frac{\partial f}{\partial x}(S,X(S))\mu(S)\, dS + \int_0^t \frac{\partial f}{\partial x}(S,X(S))\sigma(S)dB_H(s) + \int_0^t \frac{\partial^2 f}{\partial x^2}(S,X(S))\sigma(S)V_S^{\emptyset}(S)dS.$$

Where $V_S^{\emptyset} S(\tau) = \sigma S(\tau) \int_0^\tau \emptyset(\tau,u)du = \sigma H S(\tau)\tau^{2H-1}$.

Theorem 1: Given a generic payoff function $G(t) = V(s,t)$, the PDE associated with the price of a derivative on the stock price is

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + H\sigma^2 S^2 t^{2H-1} \frac{\partial^2 V}{\partial S^2} = -rV \quad (2.1)$$

Proof: The stock price S_t follows the fractional Brownian motion process

$$\frac{dS}{S} = \mu dt + \sigma dB_H(t)\,,\ S(0) = s, \quad (2.2)$$

and the wealth of an investor X_t, follows a diffusion driven by (with time suppressed)

$$dX = \wedge dS + r(X - \wedge S)dt. \quad (2.3)$$

Putting equation (2.2) into equation (2.3) yields;

$$dX = \{rX - \wedge S(\mu - r)\}dt + \wedge S\sigma dW_H, \quad (2.4)$$

where $\mu - r$ is the risk premium.

Suppose that the value of this claim at time t is given by

$$G(t) = V(S,t),\ S = S_t. \quad (2.5)$$

Applying the fractional Ito's formula on equation (2.5) and using lemma 1, we have

$$dG = \left[\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial s} + H\sigma^2 S^2 t^{2H-1}\right]dt + \sigma S\frac{\partial V}{\partial s}dW_H. (2.6)$$

To track $G(t)$ at all times, we have under the assumption of complete market that

$$X_t = G(t) = V(S,t)\ \forall t \in\ [0,T]. \quad (2.7)$$

Thus

$$\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial s} + H\sigma^2 s^2 t^{2H-1}\frac{\partial^2 V}{\partial s^2} = rV + \wedge_t S(\mu - r) \quad (2.8)$$

and

$$\sigma S\frac{\partial V}{\partial s} = \wedge_t S\sigma. \quad (2.9)$$

Equation (2.9) gives the delta-hedging optimal (rule)

$$\wedge_t = \frac{\partial v}{\partial s}(s,t). \quad (2.10)$$

While equation (2.7) with $X(t) = V(s,t)$ gives

$$\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + H\sigma^2 S^2 t^{2H-1}\frac{\partial^2 V}{\partial S^2} = rV + S\mu\frac{\partial V}{\partial S} - Sr\frac{\partial V}{\partial S}$$

which implies equation (2.1) as required.

3. Solution to Equation (2.1)

For a European option with maturity date T, striking price K, and payoff function G, the value price $V = V(S,t)$ which satisfies the following fBm

$$\frac{\partial V}{\partial t} + Ht^{2H-1}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial v}{\partial s}rV = 0\,, (S,t) \in (0,\infty) \times (0,T)V(s,0) = h(s), \quad (3.1)$$

we set $S = e^x \Longrightarrow x = ln\, s\ , \mu(x,t) = V(e^x,t)$ and $h(e^x) = g(x)$(see Thapa, et al, 2012) to get

$$\frac{\partial u}{\partial t} + Ht^{2H-1}\sigma^2\left(\frac{\partial^2 u}{\partial x} - \frac{\partial u}{\partial x}\right) + r\frac{\partial u}{\partial x} - ru = 0$$

$$= \frac{\partial u}{\partial t}Ht^{2H-1}\sigma^2\frac{\partial^2 u}{\partial x^2} - (Ht^{2H-1}\sigma^2 - r)\frac{\partial u}{\partial x} - ru,$$

this implies that

$$\frac{\partial^2 u}{\partial x^2} - \left(1 - \frac{r}{H\sigma^2}t^{1-2H}\right)\frac{\partial u}{\partial x} - \frac{r}{H\sigma^2}t^{1-2H}u = -\frac{x}{H\sigma^2}t^{1-2H}. \quad (3.2)$$

By [11], equation (3,2) reduces to the following second other differential equation

$$u'' + \lambda u' + \alpha u = -\alpha x,\ \lambda = \alpha - 1. \quad (3.3)$$

We obtain the auxiliary solution of the homogenous part of equation (3.3) as (see [10])

$$\frac{u'}{u} = \left\{\begin{matrix} \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\alpha}}{2}\left(\frac{C_1\sin h\frac{\sqrt{\lambda^2-4\alpha}}{2}t + C_2\sin h\frac{\sqrt{\lambda^2-4\alpha}}{2}t}{C_1\cos h\frac{\sqrt{\lambda^2-4\alpha}}{2}t + C_2\sin h\frac{\sqrt{\lambda^2-4\alpha}}{2}t}\right),\ \lambda^2 > 4\alpha \\ \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\alpha}}{2}\left(\frac{-C_1\sin\frac{\sqrt{\lambda^2-4\alpha}}{2}t + C_2\sin\frac{\sqrt{\lambda^2-4\alpha}}{2}t}{C_1\cos\frac{\sqrt{\lambda^2-4\alpha}}{2}t + C_2\sin\frac{\sqrt{\lambda^2-4\alpha}}{2}t}\right), \lambda^2 < 4\alpha \\ \frac{\lambda}{2} + \frac{C_1}{C_1+C_2}\,, \qquad \lambda - 4\alpha = 0 \end{matrix}\right\}, \quad (3.4)$$

which is equivalent to

$$\frac{u'}{u} = \left\{\begin{matrix} \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\alpha}}{2}\left(\frac{\tan h\frac{\sqrt{\lambda^2-4\alpha}}{2}t + {C_2}/{C_1}}{1 + {C_1}/{C_2}\tan h\frac{\sqrt{\lambda^2-4\alpha}}{2}t}\right),\ \lambda^2 > 4\alpha \\ \frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4\alpha}}{2}\left(\frac{-\tan\frac{\sqrt{\lambda^2-4\alpha}}{2}t + {C_2}/{C_1}}{1 + {C_1}/{C_2}\tan\frac{\sqrt{\lambda^2-4\alpha}}{2}t}\right),\ \lambda^2 < 4\alpha \\ \frac{\lambda}{2} + \frac{C_1}{C_1+C_2}\,, \qquad \lambda - 4\alpha = 0 \end{matrix}\right\}, \quad (3.5)$$

with solution

$$u(x,t) = u_0 exp\left\{\begin{matrix} \left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\alpha}}{2}\left(C_1\sin h\frac{\sqrt{\lambda^2-4\alpha}}{2}t + C_2\cos h\frac{\sqrt{\lambda^2-4\alpha}}{2}\right)x,\ \lambda^2 > 4\alpha \\ \left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\alpha}}{2}\left(C_1\sin\frac{\sqrt{\lambda^2-4\alpha}}{2}t + C_2\cos\frac{\sqrt{\lambda^2-4\alpha}}{2}\right)x,\ \lambda^2 < 4\alpha \\ \left(-\frac{\lambda}{2} + \frac{C_1}{C_1+C_2}\right)x\,, \qquad \lambda^2 = 4\alpha \end{matrix}\right\}, \quad (3.6)$$

or equivalently

$$u(x,t) = u_0 e^{\xi x} . \quad (3.7)$$

Where

$\xi_1 = \left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\alpha}}{2}\left(C_1 \sin h\frac{\sqrt{\lambda^2-4\alpha}}{2}t + C_2 \cos h\frac{\sqrt{\lambda^2-4\alpha}}{2}\right)x,\right.$

$\xi_2 = \left(\frac{-\lambda}{2} + \frac{\sqrt{\lambda^2-4\alpha}}{2}\left(C_1 \sin\frac{\sqrt{\lambda^2-4\alpha}}{2}t + C_2 \cos\frac{\sqrt{\lambda^2-4\alpha}}{2}\right)x\right.$ and

$\xi_3 = \left(-\frac{\lambda}{2} + \frac{C_1}{C_1+C_2}\right)x$,are copies of ξ.

For the particular solution, it is not difficult to see that $u_p = \frac{1}{2}e^x$, so that the general solution becomes

$$u(x,t) = u_0 e^{\xi x} + \frac{1}{2}e^x .$$

Or

$$V(S,t) = u_0 s^{\xi} + \frac{S}{2}. \quad (3.8)$$

From equation (3.8) notice that $V(0) = 0$ and

$$0 = u_0 \xi \hat{s}^{\xi} + \frac{\hat{s}}{2}. \quad (3.9)$$

Under equilibrium condition, the discounted profit from a unity capacity at $\hat{s}$ must be equal to the expected unit cost$\bar{s}$of risky option. Therefore by (3.9), we have

$$\bar{S} = u_0 \hat{s}^{\xi} + \frac{\hat{s}}{2}. \quad (3.10)$$

Solving for u_0 in (3.9) and (3.10)and equating the results gives

$$\hat{S} = \frac{2\bar{S}}{1-\frac{1}{\xi}}, \quad (3.11)$$

which is the equilibrium price.

Alternatively one can solve equation (3.1) for stock which is already priced in the market. To do this, we remove the effect of the discount rate r by letting $\bar{V} = e^{-rt}V \Rightarrow V = \bar{V}e^{rt}$, $\bar{S} = e^{-rt}S \Rightarrow S = \bar{S}e^{rt}$, so that equation (3.1) becomes;

$$\frac{d\bar{V}}{dt} + H\sigma^2 t^{2H-1}\bar{S}^2 \frac{\partial^2 \bar{V}}{\partial \bar{S}^2} = 0. \quad (3.12)$$

Solution by variation of parameter is given by;

$$V(s,t) = \exp\left\{-\left(\frac{\alpha S^{-1}}{\sigma^2} + \frac{\alpha t^{2H}}{2}\right)\right\}\exp\{rt\}$$

$$= \exp\left\{-\left(\frac{2\alpha S^{-1} + \alpha\sigma^2 t^{2H}}{2\sigma^2}\right)\exp\{rt\}\right.$$

$$= \exp\left\{-\left(\frac{2\alpha S^{-1} + (\alpha\sigma^2 t^{2H} + rt)}{2\sigma^2}\right)\right\} \quad (3.13)$$

Equation (3.13) is now the growth rate of the worth of stock of an investor.

4. Stability and Convergence Criteria

We use the explicit method in order to ensure stability and convergence. By convergence, we mean that the results of the method approach the analytical values as Δt and Δs both approach zero. By stability, we mean that errors made at one stage of the calculations, do not cause increasingly large errors as the computations are continued, but rather damp out eventually[12].

Let $\rho = -H\sigma^2 t^{2H-1}\bar{S}^2$, then Eq. (3.12) gives

$$\frac{d\bar{V}}{dt} = \rho\frac{\partial^2 \bar{V}}{\partial \bar{S}^2}. \quad (4.1)$$

Let us the symbol use V to represent the exact solution to Eq.(4.1), and v to represent the numerical solution. Recall that in the implicit method, ρ must be $\frac{1}{2}$ or less. This condition is true only true if $S = \sigma \leq 1$and $H \leq \frac{1}{2}$.At the moment we assume that v is free of round-off, so the only difference between V and v is the error made by replacing Eq. (4.1) by the difference equation. Let $e_i^j = V_i^j - v_i^j$, at the point $S = S_i, t = t_j$. By the explicit method, Eq. (4.1) becomes

$$v_i^{j+1} = \rho\left(v_i^j + v_{i-1}^j\right) + (1-2\rho)v_i^j. \quad (4.2)$$

Substituting $v = V - e$ into Eq. (4.2), we get

$$e_i^{j+1} = \rho\left(e_{i+1}^j + e_{i-1}^j\right) + (1-2\rho)e_i^j - \rho\left(V_{i+1}^j + v_{i-1}^j\right) - (1-2\rho)V_i^j + V_i^{j+1}. \quad (4.3)$$

By using Taylor series expansions, we have

$$V_{i+1}^j = V_i^j + \left(\frac{\partial V}{\partial S}\right)_{i,j}\Delta S + \frac{(\Delta S)^2}{2}\frac{\partial^2 V(\xi_1,t_j)}{\partial S^2}, S_1 < \xi_1 < S_{i+1},$$

$$V_{i-1}^j = V_i^j + \left(\frac{\partial V}{\partial S}\right)_{i,j}\Delta x + \frac{(\Delta S)^2}{2}\frac{\partial^2 V(\xi_2,t_j)}{\partial S^2}, S_{i-1} < \xi_2 < S_i,$$

$$V_i^{j+1} = V_i^j + \Delta t\frac{\partial V(S_i,\eta)}{\partial t}, t_j < \eta < t_{j+1}.$$

Substituting these into Eq. (4.3) and simplifying, we get

$$e_i^{j+1} = \rho\left(e_{i+1}^j + e_{i-1}^j\right) + (1-2\rho)e_i^j - \Delta t\left[\frac{\partial V(S_i,\eta)}{\partial t} - \right.$$
∂2Vξ, tj∂S2, Si−1<ξ<Si+1. (4.4)

Let E^j be the magnitude of the maximum error in the row of calculations for $t = t_j$, and let $M > 0$ be an upper bound for the magnitude of the expression in brackets in Eq. (4.4). if $\rho \leq \frac{1}{2}$, all the coefficients in Eq. (4.4) are positive (or zero) and we may write the inequality

$$\left|e_i^{j+1}\right| \leq 2\rho E^j + (1-2\rho)E^j + M\,\Delta t = E^j + M\,\Delta t.$$

This is true for all the e_i^{j+1}at $t = t_{j+1}$, so

$$E^{j+1} \leq E^{j} + M\Delta t.$$

Since this is true at each time step,

$$E^{j+1} \leq E^{j} + M\,\Delta t \leq E^{j-1} + 2M\,\Delta t \leq \cdots$$

$$\leq E^{0} + (j+1)\Delta t = E^{0} + Mt_{j+1}$$

$$= Mt_{j+1},$$

because E^0, the errors at $t = 0$ are zero, since V is given by the initial conditions.

Now, as $\Delta S \to 0, \Delta t \to 0$ if $\rho \leq \frac{1}{2}$, and $M \to 0$, because, as both Δx and Δt get smaller,

$$\left[\frac{\partial V(S_i,\eta)}{\partial t} - \frac{\partial^2 V(\xi,t_j)}{\partial S^2}\right] \to \left(\frac{\partial V}{\partial t} - \frac{k}{cp}\frac{\partial^2 V}{\partial S^2}\right)_{i,j} = 0.$$

This last is by virtue of Eq. (4.1), of course. Consequently, we have shown that the explicit method is convergent for $\rho \leq \frac{1}{2}$, because the errors approach zero as Δt and Δx are made smaller.

5. Conclusion

We have obtained the solution of a fractional Black-Scholes formula of the price of an option. We also have shown that the explicit method is convergent for $\rho \leq \frac{1}{2}$. The fractional Brownian motion has a long memory. Therefore the growth rate of the worth of a stock no longer depend on time, $T - t$ but on the stock price, S_t. Notice from equation (3.13)that the growth rate $V(S,t)$ depends largely on how $S \to \infty$ or how $S \to 0$.

MSC: 35Q51, 35B30, 91B30

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