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Feasible length of Taylor polynomial on given interval and application to find the number of roots of equation

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Abstract
It is very necessary to represent arbitrary function as a polynomial in many situations because polynomial has many valuable properties. Fortunately, any analytic function can be approximated by Taylor polynomial. The higher the degree of Taylor polynomial is, the better the approximation is gained. There is problem that how to achieve optimal approximation with restriction that the degree is not so high because of computation cost. This research proposes a method to estimate feasible degree of Taylor polynomial so that it is likely that Taylor polynomial with degree being equal to or larger than such feasible degree is good approximation of a function in given interval. The feasible degree is called the feasible length of Taylor polynomial. The research also introduces an application that combines Sturm theorem and the method to approximate a function by Taylor polynomial with feasible length in order to count the number of roots of equation in given interval.

1. Introduction to Taylor Approximation

Suppose one variable function f whose n+1th order derivative exists and is bounded on given interval $[a, b]$.

$$\left|f^{(n+1)}(x)\right| \leq M, \forall x \in [a, b]$$

Where $f^{(n+1)}(x)$ denotes the n+1$^{\text{th}}$ order derivative of $f(x)$.

Let

$$x_0 = \frac{(a+b)}{2}$$

Taylor series of function f at x_0 is:

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)dt = f(x_0) + \int_{x_0}^{x} f'(t)d(t-x)$$

Expending the inner integral [Rosenberg 2006] [Wikipedia 2014a], we have:

$$\int_{x_0}^{x} f'(t)dt = f'(t)(t-x)\Big|_{x_0}^{x} - \frac{1}{2}\int_{x_0}^{x} f''(t)d(t-x)^2 = f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \frac{1}{6}\int_{x_0}^{x} f'''(t)d(t-x)^3$$

By induction [Rosenberg 2006] [Wikipedia 2014a], we have:

$$\int_{x_0}^{x} f'(t)dt = f'(x_0)(x-x_0) + \frac{1}{2}f''(t)(x-x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n + \frac{(-1)^n}{(n+1)!}\int_{x_0}^{x} f^{(n+1)}(t)d(t-x)^{n+1}$$

Let $P_n(x)$ and $R_n(x)$ be Taylor polynomial and remainder of function $f(x)$, respectively, we have:

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

$$R_n(x) = \frac{(-1)^n}{(n+1)!}\int_{x_0}^{x} f^{(n+1)}(t)d(t-x)^{n+1} = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt$$

If n approaches $+\infty$, then

$$f(x) = P_n(x) + R_n(x)$$

Where n is defined the length of the Taylor polynomial $P_n(x)$ with note that n is known as the degree of $P_n(x)$. The function $f(x)$ is approximated by the Taylor polynomial $P_n(x)$ when $R_n(x)$ is truncation error [Burdden 2011 p. 11]. The quality of approximation depends on two following factors:

- The larger (longer) the length of $P_n(x)$ is, the better the approximation is.
- The smaller the truncation error $R_n(x)$ is, the better the approximation is.

Existing methods that improve the quality of approximation focus on minimizing the truncation error although lengthening $P_n(x)$ also makes the truncation error $R_n(x)$ in decrease. Please see the method "least squares approximation" and other ones inside chapter 8 of the book "Numerical Analysis" [Burdden 2011 pp. 497-558] for more details about approximation theory. Additionally, the square error [Callahan 2008 p. 669] between two Taylor polynomials is minimized so as to enhance the quality of approximation. By another way, this research focuses on how to determine the length of $P_n(x)$ so that it is possible to achieve good approximation but keep such length as small as possible. This length is called *feasible length*. In other words, the issued problem is how to find out the feasible length so that it is likely that $P_n(x)$ is the good approximation of $f(x)$. Given the feasible length n^*, the larger than n^* the length n is, the better the approximation $P_n(x)$ is. Hence, the method to estimate feasible length is proposed in next section.

2. Estimating Feasible Length of Taylor Polynomial

Given the n+1th order derivative $f^{(n+1)}(x)$ is bounded, the remainder is also bounded. We have [Rosenberg 2006] [Wikipedia 2014a]:

$$R_n(x) \leq \frac{M}{n!}\int_{x_0}^{x}(x-t)^n dt = -\frac{M}{(n+1)!}\left((x-t)^{n+1}\Big|_{x_0}^{x}\right) = M\frac{(x-x_0)^{n+1}}{(n+1)!}$$

It implies that

$$|R_n(x)| \leq M\frac{|x-x_0|^{n+1}}{(n+1)!}$$

Using Stirling approximation [Wikipedia 2014b] for factorial as below:

$$n! \cong \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

We have:

$$|R_n(x)| \leq M\frac{|x-x_0|^{n+1}}{\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1}}$$

Let $g(z)$ be function defined as below:

$$\frac{|x-x_0|^z}{\sqrt{2\pi z}\left(\frac{z}{e}\right)^z}$$

Where,

$$z \geq 1$$

Note that z is real variable and $x \neq x_0$ and it implies that g is function of n when n is considered as real variable instead of integer number as aforementioned. Now we find out the maximum point z_0 of $g(z)$. The natural logarithm of $g(z)$ is:

$$\log(g(z)) = z\log|x-x_0| - \frac{1}{2}\log(2\pi) - \frac{1}{2}\log(z) - z\log(z) + z = (\log|x-x_0|+1)z - z\log(z) - \frac{1}{2}\log(z) - \frac{1}{2}\log(2\pi)$$

The first derivative of $\log(g(z))$ with regard to z is:

$$\big(\log(g(z))\big)' = \log|x-x_0| - \log(z) - \frac{1}{2z}$$

That the function $g(z)$ gets maximal is equivalent to that its logarithm $\log(g(z))$ gets maximal and so the maximum point z^* is found out by setting the first derivative of $\log(g(z))$ to be zero. We have:

$$\log|x-x_0| - \log(z) - \frac{1}{2z} = 0 \Leftrightarrow \log(z) = \log|x-x_0| - \frac{1}{2z}$$

$$\Rightarrow \log|x-x_0| - 0.5 \leq \log(z) \leq \log|x-x_0|$$

Because the number $\frac{1}{2z}$ is smaller than 0.5 and approaches 0 when z approaches $+\infty$, we make an approximation of $\log(z)$ as below:

$$\log(z) \approx \log|x-x_0|$$

It implies that

$$z^* \approx |x-x_0|$$

When $z^* \approx |x - x_0|$ is the peak of function $g(z)$, there is comment that the larger the distance $|x - x_0|$ is, the more symmetric (more like bell-shape) the curve of function $g(z)$ is. Figure 1 depicts curves of function $g(z)$ with $|x - x_0| = 2.5, 3, 3.5, 4$.

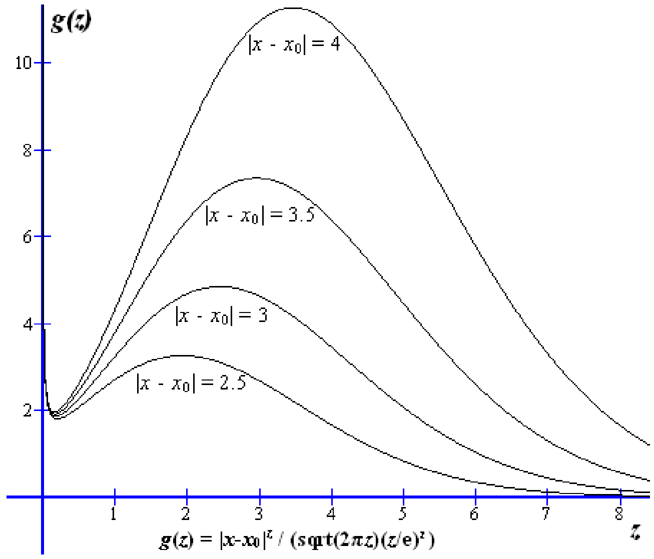

Figure 1. *Curves of function g(z)*

The point far by double distance $|x - x_0|$ results out possibly small value for the function $g(z)$ and moreover, $g(z)$ decreases more slowly when $z > |x - x_0|$. Therefore, the feasible length n^* is approximated as follows:

$$n^* = 2[\![z^*]\!] \approx 2[\![|x - x_0|]\!]$$

Where $[\![.]\!]$ denotes integer part which is the integer number that is nearest to given number, for example, $[\![1.5]\!] = 2$ and $[\![1.4]\!] = 1$. Figure 2 depicts the curve $g(z)$ and the feasible length n^* corresponding to distance $|x - x_0| = 3$.

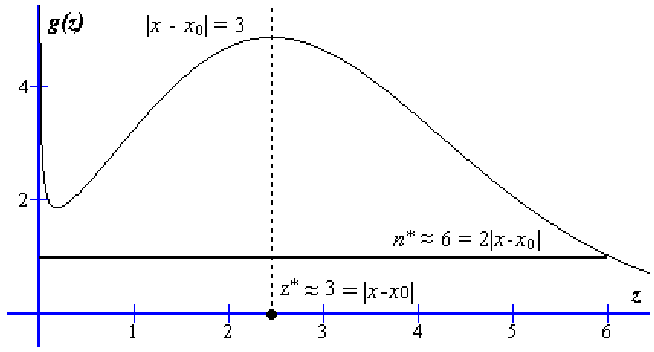

Figure 2. *Feasible length of g(z) with $|x - x_0| = 3$*

If $x = a$ or $x = b$ with note that a and b are end points of the pre-defined interval $[a, b]$, we have:

$$n^* \approx 2[\![|a - x_0|]\!] = 2[\![|b - x_0|]\!] = 2\left[\!\!\left[\frac{b-a}{2}\right]\!\!\right] \approx [\![b - a]\!]$$

The above formula can be interpreted that it requires at least $[\![b - a]\!]$ degrees or length of $[\![b - a]\!]$ for Taylor polynomial to reach feasibly good approximation of function $f(x)$ in given interval $[a, b]$. In general, we have a conclusion that:

Given pre-defined interval $[a, b]$, the feasible length of Taylor polynomial which results out the possibly good approximation of function $f(x)$ in interval $[a, b]$ is approximated by the distance $b–a$ with attention that the Taylor polynomial is constructed at central point $x_0 = \frac{a+b}{2}$.

$$n^* \approx [\![b - a]\!]$$

3. Application to Find the Number of Roots of Equation

The issued problem is to find out the number of roots of equation $f(x) = 0$ in given interval $[a, b]$. The problem is solved by 2-step task:

1. Constructing Taylor polynomial $P_n(x)$ to approximate $f(x)$ at central point $x_0 = \frac{a+b}{2}$. Because n will approach $+\infty$, we use the feasible length $n = [\![b - a]\!]$.
2. Applying Sturm theorem [Wikipedia 2014c] [Ta 2014] into counting the number of roots of equation $f(x) = 0$ in given interval $[a, b]$.

Suppose the feasible length $n = [\![b - a]\!]$, the optimal Taylor polynomial $P(x)$ is:

$$P(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n$$

Where,

$$x_0 = \frac{a+b}{2}$$

In step 2, we construct Sturm sequence [Wikipedia 2014c] [Ta 2014] as follows:

$P_0(x) = P(x)$

$P_1(x) = P'(x)$ ($P_1(x)$ is derivative of $P(x)$)

$P_2(x) = P_1(x)Q_0(x) - P_0(x)$

$P_3(x) = P_2(x)Q_1(x) - P_1(x)$

…

$P_m(x) = P_{m-1}(x)Q_{m-2}(x) - P_{m-2}(x)$

$0 = P_m(x)Q_{m-1}(x) - P_{m-1}(x)$

It is easy to infer that $P_m(x)$ is the opposite of the remainder of polynomial division of $P_{m-2}(x)$ by $P_{m-1}(x)$. Substituting the end point a into Sturm sequence and evaluating signs of such sequence ($P_0(a)$, $P_1(a)$,…, $P_m(a)$), we get a sign sequence and count the number of sign changes of this sign sequence. We denote $W(a)$ as this number of sign changes at point a. Similarly, let $W(b)$ the number of sign changes at point b. According to Sturm theorem, the number of roots of equation $f(x) = 0$ in the half-open interval $(a, b]$ is

$W(a) - W(b)$. For example, we count the number of roots of following equation in interval [1, 3].

$$f(x) = \log(x) + \frac{1}{2x} - \log(2) = 0$$

When the feasible length is $n = [\![3-1]\!] = 2$ and the optimal Taylor polynomial of $f(x)$ at central point $\frac{3+1}{2} = 2$ as follows:

$$P(x) = f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 = -\frac{1}{16}x^2 + \frac{5}{8}x - \frac{3}{4}$$

The Sturm sequence is determined as follows:

$$P_0(x) = -\frac{1}{16}x^2 + \frac{5}{8}x - \frac{3}{4}$$

$$P_1(x) = -\frac{1}{8}x + \frac{5}{8}$$

$$P_2(x) = \frac{1}{2}x - \frac{5}{2}$$

$$P_3(x) = \frac{13}{16}$$

The Sturm sequence evaluated at end point $x = 1$ is $P_0(1) = -3/16$, $P_1(1) = 1/2$, $P_2(1) = -2$, $P_3(1) = 13/16$ and so there are 3 sign changes, we have $W(1) = 3$. The Sturm sequence evaluated at end point $x = 3$ is $P_0(3) = 9/16$, $P_1(3) = 1/4$, $P_2(3) = -1$, $P_3(3) = 13/16$ and so there are 2 sign changes, we have $W(3) = 2$. Absolutely, the number of roots in half-open interval (1, 3] is $1 = W(1) - W(3)$. Figure 3 depicts approximated Taylor polynomial in this example:

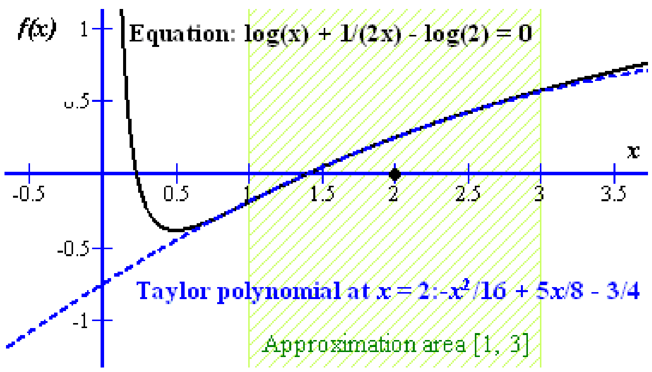

Figure 3. *Approximated Taylor polynomial*

Note that Taylor polynomial curve is drawn as dash line, which is the good approximation to the equation $\log(x) + \frac{1}{2x} - \log(2) = 0$ at point $x = 2$. It is easy to see in figure 3 that there is only one root in half-open interval (1, 3] for both the equation $\log(x) + \frac{1}{2x} - \log(2)$ and approximated polynomial $-\frac{1}{16}x^2 + \frac{5}{8}x - \frac{3}{4} = 0$.

4. Conclusion

The feasible length is essentially the possible lower bound of degree of Taylor polynomial when the best degree is infinite and so "feasible length" is a heuristic concept. It is likely that Taylor polynomial with degree being equal to or larger than feasible length is good approximation of a function in given interval and otherwise. The wider the interval is, the longer (larger) the feasible length is. But given a central point where Taylor polynomial is expended, there is situation that a point far from central point requires smaller feasible length to achieve good approximation than another point near to central point does. The reason is that the remainder is dependent on both the degree of polynomial and high order derivatives; hence, that high order derivatives may get small values at such far point results out small remainder. But the proposed method is based on assumption that the $n+1^{th}$ order derivative is bounded with the bound which is fixed in the formula to estimate the feasible length. In other words, high order derivatives are not accounted in the formula to estimate the feasible length. It means that feasible length is the estimated value and you can specify any degree larger than it. However it is very useful to apply feasible length into Sturm theorem so as to find out the number of roots in given equation $f(x) = 0$ because the given function $f(x)$ does not vary so much usually in relatively small interval and it is possible to gains optimal approximation of $f(x)$ in such interval; although Taylor polynomial can produce or eliminate roots unpredictably outside the interval, the number of roots inside relatively small interval is always counted exactly.

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