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The Numerical Solution of System of Fractional Partial Differential Equations

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Abstract
In this paper, numericalsolution of fractional partial differential equations is obtained byfractional differential transform method. The fractional derivatives are described using Caputo sense; the method provides the solution in the form of a rapidly convergent series. From the result, it has been seen that the proposed method is very effective.

1. Introduction

Various phenomena in physics, like diffusion in a disordered or fractal medium, or in image analysis, or in risk management have been modeled by means of fractional partial differential equations. The fractional partial differential equations appear very frequently in physical sciences. Literatures (Momani and Odibat, 2007) discuss the linear and nonlinear partial differential equations of fractional order. Numbers of physical phenomena are governed by such equations (Podlubny, 1999; Rossikhin and Shitikova, 1997; Mohyud-Din and Noor 2008; Mohy0000ud-Din et al., 2009, 2010). Several techniques including decomposition, variational iteration, homotopy analysis, variation of parameters have been applied to solve such problems (Rossikhin and Shitikova, 1997; Mohyud-Din and Noor, 2008; Mohyud-Din, et al., 2009, 2010). The differential transform method was first introduced by Zhou (1986) who solved linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in form of polynomial expressions such as Taylor series expansion. But procedure is easier than the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally expensive for higher orders. Arikoglu and Ozkol implement a new analytical technique for the field of fractional calculus, for solving fractional type differential equations that will be named as Fractional Differential Transform Method (FDTM).

In this study, we will use FDTM to solve the fractional partial differential equations (FPDEs) of the form

$$AD_t^{\alpha}u(t,x)+BL_xu(t,x)+Cu(t,x)=f(t,x),$$

Where$\propto$is a parameter describing the fractionalderivative and$t\in(0,t_e), 0<\propto\leq 1$ and $x\in(-l,l)\subset R,\ \ A,B,C\in R^{n\times n}$, are constant matrices, $u,f:[0,t_e]\ \times[-l,l]\rightarrow R^n$. we are interested in cases where at least one of the matrices$A$or$B$is singular. The two special cases$A=0$ or $B=0$lead to ordinary differential equations or FDAEs which are not considered here. Therefore in this paper we assume that none of the matrices A or B is the zero matrixes.

2. Basic Definitions

We give some basic definitions and properties of the fractional calculus theory (Caputo, 1967) which are used further in this paper.

Definition 1. A real function $f(x), x > 0$ is said to be in the space $C_\mu, \mu \epsilon R$ if there existsa real number $P > \mu$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \epsilon C[0,\infty)$. Clearly $C_\mu < C_\beta$ if $\mu < \beta$.

Definition 2. A function $f(x), x > 0$ is said to be in the spaceC_μ^m, $m \epsilon N \cup \{0\}$ if $f^{(m)} \in C_\mu$.

*Definition 3.*The Riemann-Liouville fractional integral operator of order a $\propto > 0$of a function, $f \in C_\mu, \mu \geq -1$ is defined as [31]

$$\left({}_aJ^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \; x > a \tag{1}$$

$$\left({}_aJ^0 f\right)(x) = f(x). \tag{2}$$

Properties of the operator $J^\propto$ can be found in (Caputo, 1967), we mention only the following:

For$f \in C_\mu,\ \mu \geq -1, \propto, \beta \geq 0$, and $\Upsilon > -1$

$$\text{a)} \left({}_aJ^\alpha\, {}_aJ^\beta f\right)(x) = \left({}_aJ^{\alpha+\beta} f\right)(x) \tag{3}$$

$$\text{b)} \left({}_aJ^\alpha\, {}_aJ^\beta f\right)(x) = \left({}_aJ^\beta\, {}_aJ^\alpha f\right)(x) \tag{4}$$

$$\text{c)}\ {}_aJ^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma} \tag{5}$$

The Riemann–Lowville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator $D^\propto$ proposed by Caputo (1967) inhis work on the theory of viscoelasticity.

Definition 4. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$\left({}_aD^\propto f\right)(x) = \left({}_aJ^{m-\propto} D^m f\right)(x) = \frac{1}{\Gamma(m-a)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt \tag{6}$$

for $m-1 < \propto < m, m \in N, x > 0, f \epsilon C_{-1}^m$.

Lemma 1. If $-1 < \propto < m$, $m \in N$ and $f \epsilon C_\mu^m$, $\mu \geq -1$, then

$$\text{a)} \left({}_aJ^\alpha\, {}_aD^\alpha f\right)(x) = f(x) - \sum_{k=0}^{m-1} f^k(a) \frac{(x-a)^k}{k!}, \quad a \geq 0 \tag{7}$$

$$\text{b)} \left({}_aD^\alpha\, {}_aJ^\alpha f\right)(x) = f(x) \tag{8}$$

3. Fractional Two-Dimensional Differential Transform Method

DTM is an analytic method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylorseries method requires symbolic computation. However, the DTM obtains a polynomial series solution by means of an iterative procedure. The proposed method is based on the combination of the classical two dimensional DTM and generalized Taylor's Table 1 formula. Consider a function of two variables$u(x,y)$and suppose that it can be represented as a product of two single-variable functions, that is,$u(x,y) = f(x)g(y)$ Based on the properties of fractional two-dimensional differential transform (Jang, 2001; Kangalgil, 2009; Ravi, 2009; Arikoglu, 2007), the function$u(x,y)$can be represented as:

$$u(x,y) = \sum_{k=0}^{\infty} F_\propto(k)(x-x_0)^{k\alpha} \sum_{h=0}^{\infty} G_\beta(h)(y-y_0)^{h\beta}$$

$$= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} U_{\propto,\beta}(k,h)\,(x-x_0)^{k\alpha}\,(y-y_0)^{h\beta} \tag{9}$$

Where $0 < \propto$, $\beta \leq 1$, $U_{\propto,\beta}(k,h) = F_\propto(k) G_\beta(h)$, is called the spectrum of $u(x,y)$. The fractional two-dimensional differential transform of the function $u(x,y)$ is given by

$$U_{\propto,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)} \left[\left(D_{x_0}^\alpha\right)^k \left(D_{y_0}^\beta\right)^h u(x,y)\right]_{(x_0,y_0)} \tag{10}$$

Where $\left(D_{x_0}^\alpha\right)^k = \underbrace{D_{x_0}^\alpha \cdot D_{x_0}^\alpha \cdots D_{x_0}^\alpha}_{k}$

In case of $\propto = 1$ and $\beta = 1$ the Fractional two dimensional differential transform (9) reduces to the classical two-dimensional differential transform. Let $U_{\propto,\beta}(k,h), w_{\propto,\beta}(k,h)$ and $V_{\propto,\beta}(k,h)$ are the differential transformations of the functions $u(x,y), w(x,y)$ and $v(x,y)$, from Equations (9) and (10), some basic properties of the two-dimensional differential transform are introduced in Table 1.

Table 1. *The operations for the two-dimensional differential transform method.*

Original function	Transformed function
$u(x,y) = v(x,y) \pm w(x,y)$	$U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$
$u(x,y) = \lambda v(x,y)$	$U_{\alpha,\beta}(k,h) = \lambda\, V_{\alpha,\beta}(k,h)$
$u(x,y) = v(x,y)w(x,y)$	$U_{\propto,\beta}(k,h) = \sum_{r=0}^{k} \sum_{s=0}^{h} V_{\propto,\beta}(r,h-s)\, W_{\propto,\beta}(k-r,s)$

Original function	Transformed function
$u(x,y)=(x-x_0)^{m\propto}(y-y_0)^{n\beta}$	$U_{\propto,\beta}(k,h)=\delta(k-n)\delta(h-m)\ =\begin{cases}1, & k=n,h=m\\0, & k\neq n,h\neq m\end{cases}$
$u(x,y)=D_{x_0}^{\propto}v(x,y)$	$U_{\propto,\beta}(k,h)=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}V_{\propto,\beta}(k+1,h)$
$u(x,y)=D_{y_0}^{\beta}v(x,y)$	$U_{\propto,\beta}(k,h)=\frac{\Gamma(\beta(h+1)+1)}{\Gamma(\beta h+1)}V_{\propto,\beta}(k,h+1)$
$u(x,y)=D_{x_0}^{\propto}D_{y_0}^{\beta}v(x,y)$	$U_{\propto,\beta}(k,h)=\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}\frac{\Gamma(\beta(h+1)+1)}{\Gamma(\beta h+1)}$ $V_{\propto,\beta}(k+1,h+1),\quad 0<\propto,\beta\leq 1$
$u(x,y)=v(x,y)w(x,y)q(x,y)$	$U_{\propto,\beta}(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}V_{\alpha,\beta}(r,h-s-p)\,W_{\alpha,\beta}(t,s)Q_{\alpha,\beta}(k-r-t,p)$

Then, the fractional differential transform (10) becomes;

$$U_{\propto,\beta}(k,h)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}\left[D_{x_0}^{\alpha k}\left(D_{y_0}^{\beta}\right)^{h}u(x,y)\right]_{(x_0,y_0)} \quad (11)$$

4. Numerical Example

Here, FDTM will be applied for solving fractional partial differential equation. The results reveal that the method is very effective and simple.

Example. Consider the fractional partial differential equation (FPDE)

$$\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}D_t^{\propto}u\\D_t^{\propto}v\end{pmatrix}+\begin{pmatrix}-1 & 0\\-1 & 0\end{pmatrix}\begin{pmatrix}u_{xx}\\v_{xx}\end{pmatrix}+\begin{pmatrix}1 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}f_1\\f_2\end{pmatrix}, \quad (12)$$

$$x\in[-1,1],\qquad t\in[0,\infty),$$

With initial values

$$u(0,x)=x^3-x,\quad v(0,x)=x^4-1\,. \quad (13)$$

Where

$$f_1=(x^4-1)\cos(t)-6xe^{-t},$$
$$f_2=-6xe^{-t}-2(x^3-x)e^{-t}, \quad (14)$$

With the exact solution

$$\begin{pmatrix}u(t,x)\\v(t,x)\end{pmatrix}=\begin{pmatrix}(x^3-x)e^{-t}\\(x^4-1)cost\end{pmatrix}, \quad (15)$$

Equivalently, Equation (12) can be written as

$$D_t^{\propto}u-u_{xx}+u+v=f_1,$$
$$D_t^{\propto}u-u_{xx}-u=f_2\,. \quad (16)$$

By using the basic definition of the two-dimensional differential transform and taking the transform of Equation (16) can obtain that

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}U_{\propto,1}(k+1,h)-(h+1)(h+2)U_{\propto,1}(k,h+2)+U_{\propto,1}(k,h)+V_{\propto,1}(k,h)=F_1(k,h)\,,$$
$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}U_{\propto,1}(k+1,h)-(h+1)(h+2)U_{\propto,1}(k,h+2)-U_{\propto,1}(k,h)=F_2(k,h)\,, \quad (17)$$

The Taylor series of functions f_1 and f_2 about $x=0,t=0$ are

$$f_1=-1-6x+\frac{1}{2}t^2+6tx-3xt^2+xt^3-\frac{1}{24}t^4+x^4-\frac{1}{4}xt^4+\frac{1}{720}t^6-\frac{1}{2}x^4t^2+\frac{1}{20}xt^5$$
$$-\frac{1}{120}xt^6+\frac{1}{840}xt^7-\frac{1}{40320}t^8+\frac{1}{24}x^4t^4-\frac{1}{6720}xt^8+\cdots,$$
$$f_2=-4x+4xt-2xt^2-2x^3+\frac{2}{3}xt^3+2x^3t-\frac{1}{6}xt^4-x^3t^2+\frac{1}{30}xt^5+\frac{1}{3}x^3t^3-\frac{1}{180}xt^6-\frac{1}{12}x^3t^4$$
$$+\frac{1}{1260}xt^7+\frac{1}{60}x^3t^4-\frac{1}{10080}xt^8-\frac{1}{360}x^3t^6+\cdots. \quad (18)$$

From the initial condition given by Equation (13), we obtained:

$$U_{\propto,1}(0,h)=\delta(0,h-3)-\delta(0,h-1)=\begin{cases}1, & h=3\\-1, & h=1\\0, & h\neq 1,3\end{cases},$$
$$V_{\propto,1}(0,h)=\delta(0,h-4)-\delta(0,h)=\begin{cases}1, & h=4\\-1, & h=0\\0, & h\neq 0,4\end{cases}. \quad (19)$$

For each k,h, substituting equations (19) into equation (17) and by recursive method, the values $U(k,h)$ and $V(k,h)$ can be evaluated as

$$U(0,1)=-1,\quad U(0,3)=1,\quad V(0,0)=-1,\quad V(0,4)=1,\quad U(0,h)=0\ forh\neq 1,3\ ,$$

$$V(0,h)=0\ forh\neq 0,4\ ,\quad U(1,0)=0,\quad U(1,1)=\frac{1}{\Gamma(\alpha+1)}\ ,\quad U(1,2)=0,$$

$$U(1,3)=\frac{-1}{\Gamma(\alpha+1)}\ ,\quad U(1,4)=0,\ U(1,5)=0,\ U(1,6)=0,\ U(1,7)=0,\ U(1,8)=0,$$

$$U(1,9)=0, U(1,10)=0,\quad U(1,11)=0,\quad U(1,12)=0,\quad U(1,13)=0,\quad V(1,0)=0,$$

$$U(2,0)=0,\quad V(1,1)=\frac{-2+2\Gamma(\alpha+1)}{\Gamma(\alpha+1)}\ ,\quad U(2,1)=\frac{-5+4\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\ ,\quad V(1,2)=0,$$

$$U(2,2)=0,\quad U(2,3)=\frac{-1+2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\ ,\quad V(1,3)=\frac{2-2\Gamma(\alpha+1)}{\Gamma(\alpha+1)},\quad U(2,4)=0,$$

$$V(1,4)=0,\quad U(2,5)=0,\quad V(1,5)=0,\quad U(2,6)=0,\quad V(1,6)=0,\quad U(2,7)=0,$$

$$V(1,7)=0,\quad U(2,8)=0,\quad V(1,8)=0,\quad U(2,9)=0,\quad V(1,9)=0,\quad U(2,10)=0,$$

$$V(1,10)=0,\quad U(2,11)=0,\quad V(1,11)=0,\quad U(3,0)=0,\quad V(2,0)=\frac{1}{2},\quad V(2,1)=0,$$

$$V(2,2)=0,\quad U(3,2)=0,\quad V(2,3)=\frac{2-4\Gamma(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma(2\alpha+1)},\quad V(2,4)=\frac{-1}{2},$$

$$U(3,1)=\frac{-11+16\Gamma(\alpha+1)-2\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)},\quad U(3,3)=\frac{-\Gamma(2\alpha+1)-1+2\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}, U(3,4)=0,$$

$$V(2,4)=\frac{-1}{2},\quad U(3,4)=0,\quad V(2,5)=0,\quad U(3,5)=0,\quad V(2,6)=0,\quad U(3,6)=0,$$

$$V(2,7)=0,\quad U(3,7)=0,\quad V(2,8)=0,\quad U(3,8)=0,\quad V(2,9)=0,\quad U(3,9)=0,$$

$$U(4,0)=0,\quad V(3,0)=0,\quad U(4,1)=\frac{2\Gamma(3\alpha+1)-24\Gamma(2\alpha+1)-51+84\Gamma(\alpha+1)}{3\Gamma(4\alpha+1)},$$

$$V(3,1)=\frac{\Gamma(3\alpha+1)+12\Gamma(2\alpha+1)-96\Gamma(\alpha+1)+66}{3\Gamma(3\alpha+1)},\quad U(4,2)=0,\quad V(3,2)=0,$$

$$U(4,3)=\frac{\Gamma(3\alpha+1)-3\Gamma(2\alpha+1)+6\Gamma(\alpha+1)-3}{3\Gamma(4\alpha+1)},\quad U(4,4)=0,\quad V(3,4)=0,$$

$$V(3,3)=\frac{-\Gamma(3\alpha+1)+6\Gamma(2\alpha+1)-12\Gamma(\alpha+1)+6}{3\Gamma(3\alpha+1)},\quad U(4,5)=0,\quad V(3,5)=0,$$

$$U(4,6)=0,\quad V(3,6)=0,\quad U(4,7)=0,\quad V(3,7)=0,\quad U(5,0)=0,\quad V(4,0)=\frac{-1}{24},$$

$$U(5,1)=\frac{-\Gamma(4\alpha+1)+16\Gamma(3\alpha+1)-84\Gamma(2\alpha+1)+240\Gamma(\alpha+1)-138}{6\Gamma(5\alpha+1)},\quad U(5,2)=0,$$

$$V(4,1)=\frac{-\Gamma(4\alpha+1)-16\Gamma(3\alpha+1)+192\Gamma(2\alpha+1)-672\Gamma(\alpha+1)+408}{12\Gamma(4\alpha+1)},\quad V(4,2)=0,$$

$$U(5,3)=\frac{-\Gamma(4\alpha+1)+4\Gamma(3\alpha+1)-12\Gamma(2\alpha+1)+24\Gamma(\alpha+1)-12}{12\Gamma(5\alpha+1)},\quad U(5,4)=0,$$

$$V(4,3)=\frac{\Gamma(4\alpha+1)-8\Gamma(3\alpha+1)+24\Gamma(2\alpha+1)-48\Gamma(\alpha+1)+24}{12\Gamma(4\alpha+1)},\quad V(4,4)=\frac{1}{24},$$

$$U(5,5)=0,\quad V(4,5)=0,\quad U(6,0)=0,\quad V(5,0)=0,\quad U(6,2)=0,\quad V(5,2)=0,$$

$$U(6,1)=\frac{\Gamma(5\alpha+1)-20\Gamma(4\alpha+1)+140\Gamma(3\alpha+1)-600\Gamma(2\alpha+1)+1560\Gamma(\alpha+1)-870}{30\Gamma(6\alpha+1)},$$

$$V(5,1)=\frac{\Gamma(5\alpha+1)+20\Gamma(4\alpha+1)-320\Gamma(3\alpha+1)+1680\Gamma(2\alpha+1)-4800\Gamma(\alpha+1)+2760}{60\Gamma(5\alpha+1)},$$

$$U(6,3)=\frac{12\Gamma(5\alpha+1)-60\Gamma(4\alpha+1)+240\Gamma(3\alpha+1)-720\Gamma(2\alpha+1)+1440\Gamma(\alpha+1)-720}{720\Gamma(6\alpha+1)},$$

$$V(5,3)=\frac{-12\Gamma(5\alpha+1)+120\Gamma(4\alpha+1)-480\Gamma(3\alpha+1)+1440\Gamma(2\alpha+1)-2880\Gamma(\alpha+1)+1440}{720\Gamma(5\alpha+1)},\ldots\ .$$

Substituting all $U(k,h)$ and $V(k,h)$ into Equation (9), the series solution form will be obtained:

$$u(t,x)=(-1+\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+\frac{-5+4\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{-11+16\Gamma(\alpha+1)-2\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}$$

$$+\frac{2\Gamma(3\alpha+1)-24\Gamma(2\alpha+1)-51+84\Gamma(\alpha+1)}{3\Gamma(4\alpha+1)}t^{4\alpha}$$

$$+\frac{-\Gamma(4\alpha+1)+16\Gamma(3\alpha+1)-84\Gamma(2\alpha+1)+240\Gamma(\alpha+1)-138}{6\Gamma(5\alpha+1)}t^{5\alpha}$$

$$+\frac{\Gamma(5\alpha+1)-20\Gamma(4\alpha+1)+140\Gamma(3\alpha+1)-600\Gamma(2\alpha+1)+1560\Gamma(\alpha+1)-870}{30\Gamma(6\alpha+1)}t^{6\alpha}+\ldots)x$$

$$+(1-\frac{1}{\Gamma(\alpha+1)}t^{\alpha}+\frac{-1+2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{-\Gamma(2\alpha+1)-1+2\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}$$

$$+\frac{\Gamma(3\alpha+1)-3\Gamma(2\alpha+1)+6\Gamma(\alpha+1)-3}{3\Gamma(4\alpha+1)}t^{4\alpha}$$

$$+\frac{-\Gamma(4\alpha+1)+4\Gamma(3\alpha+1)-12\Gamma(2\alpha+1)+24\Gamma(\alpha+1)-12}{12\Gamma(5\alpha+1)}t^{5\alpha}$$

$$+\frac{12\Gamma(5\alpha+1)-60\Gamma(4\alpha+1)+240\Gamma(3\alpha+1)-720\Gamma(2\alpha+1)+1440\Gamma(\alpha+1)-720}{720\Gamma(6\alpha+1)}t^{6\alpha}+\cdots)x^3,$$

$$v(t,x)=-1+\frac{1}{2}t^{2\alpha}-\frac{1}{24}t^{4\alpha}+(\frac{-2+2\Gamma(\alpha+1)}{\Gamma(\alpha+1)}t^{\alpha}+\frac{\Gamma(3\alpha+1)+12\Gamma(2\alpha+1)-96\Gamma(\alpha+1)+66}{3\Gamma(3\alpha+1)}t^{3\alpha}$$

$$+\frac{-\Gamma(4\alpha+1)-16\Gamma(3\alpha+1)+192\Gamma(2\alpha+1)-672\Gamma(\alpha+1)+408}{12\Gamma(4\alpha+1)}t^{4\alpha}$$

$$+\frac{\Gamma(5\alpha+1)+20\Gamma(4\alpha+1)-320\Gamma(3\alpha+1)+1680\Gamma(2\alpha+1)-4800\Gamma(\alpha+1)+2760}{60\Gamma(5\alpha+1)}t^{5\alpha}+\cdots)x$$

$$+(\frac{2-2\Gamma(\alpha+1)}{\Gamma(\alpha+1)}t^{\alpha}+\frac{2-4\Gamma(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma(2\alpha+1)}t^{2\alpha}+\frac{-\Gamma(3\alpha+1)+6\Gamma(2\alpha+1)-12\Gamma(\alpha+1)+6}{3\Gamma(3\alpha+1)}t^{3\alpha}$$

$$+\frac{\Gamma(4\alpha+1)-8\Gamma(3\alpha+1)+24\Gamma(2\alpha+1)-48\Gamma(\alpha+1)+24}{12\Gamma(4\alpha+1)}t^{4\alpha}$$

$$+\frac{-12\Gamma(5\alpha+1)+120\Gamma(4\alpha+1)-480\Gamma(3\alpha+1)+1440\Gamma(2\alpha+1)-2880\Gamma(\alpha+1)+1440}{720\Gamma(5\alpha+1)}t^{5\alpha}$$

$$+\cdots)x^3+(1-\frac{1}{2}t^{2\alpha}+\frac{1}{24}t^{4\alpha}+\cdots)x^4+\cdots\ .$$

For special case $\propto=1$,the solution will be as follows:

$$u(t,x) = (-1 + t + \frac{-1}{2}t^2 + \frac{1}{6}t^3 + \frac{-1}{24}t^4 + \frac{1}{120}t^5 + \frac{-1}{720}t^{6\alpha} + \ldots)x$$

$$+(1 - t + \frac{1}{2}t^2 + \frac{-1}{6}t^3 + \frac{1}{24}t^4 + \frac{-1}{120}t^5 + \frac{1}{720}t^6 + \cdots)x^3$$

$$= (x - x^3)(-1 + t + \frac{-1}{2}t^2 + \frac{1}{6}t^3 + \frac{-1}{24}t^4 + \frac{1}{120}t^5 + \frac{-1}{720}t^{6\alpha} + \cdots) = (x^3 - x)e^{-t},$$

$$v(t,x) = (-1 + \frac{1}{2}t^2 - \frac{1}{24}t^4 + \cdots) + \left(1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \cdots\right)x^4$$

$$= \left(-1 + \frac{1}{2}t^2 - \frac{1}{24}t^4 + \cdots\right) + \left(1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \cdots\right)x^4$$

$$= (x^4 - 1)\left(1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 + \cdots\right) = (x^4 - 1)cost\,.$$

This is exact solution.

$u(t,x)$ and $v(t,x)$ are calculated for different values of $\propto$ Numerical comparisons are given in Table 2 and Table 3.

Table 2. *Numerical solution of* $u(t,x)$.

x	t	$\propto = 0.5$	$\propto = 0.75$	$\propto = 1$	
		u_{FDTM}	u_{FDTM}	u_{FDTM}	u_{Exact}
0.01	0.01	−0.009006957831	−0.009664785809	−0.009899508287	−0.009899508287
0.02	0.02	−0.01733003467	−0.01889021902	−0.01959613188	−0.01959613188
0.03	0.03	−0.02527362310	−0.02777269783	−0.02908716398	−0.02908716398
0.04	0.04	−0.03293599892	−0.03635611297	−0.03837008704	−0.03837008704
0.05	0.05	−0.04036801891	−0.04466865941	−0.04744256754	−0.04744256754
0.06	0.06	−0.04759969185	−0.05273070525	−0.05630245088	−0.05630245088
0.07	0.07	−0.05464965291	−0.06055793814	−0.06494775631	−0.06494775631
0.08	0.08	−0.06152952538	−0.06816294476	−0.07337667215	−0.07337667215
0.09	0.09	−0.06824624532	−0.07555611495	−0.08158755085	−0.08158755085
0.1	0.1	−0.07480342684	−0.08274620662	−0.08957890439	−0.08957890439

Table 3. *Numerical solution of* $v(t,x)$.

x	t	$\propto = 0.5$	$\propto = 0.75$	$\propto = 1$	
		v_{FDTM}	v_{FDTM}	v_{FDTM}	v_{Exact}
0.01	0.01	−0.9952761913	−0.9995558781	−0.9999499904	−0.9999499904
0.02	0.02	−0.9908314500	−0.9987747038	−0.9997998467	−0.9997998467
0.03	0.03	−0.9866205581	−0.9977883191	−0.9995492242	−0.9995492242
0.04	0.04	−0.9826403488	−0.9966434019	−0.9991975487	−0.9991975487
0.05	0.05	−0.9788962184	−0.9953670138	−0.9987440182	−0.9987440182
0.06	0.06	−0.9753969431	−0.9939774958	−0.9981876033	−0.9981876033
0.07	0.07	−0.9721529476	−0.9924883569	−0.9975270492	−0.9975270492
0.08	0.08	−0.9691755603	−0.9909100699	−0.9967608777	−0.9967608777
0.09	0.09	−0.9664766420	−0.9892510351	−0.9958873893	−0.9958873893
0.1	0.1	−0.9640683877	−0.9875181495	−0.9949046663	−0.9949046663

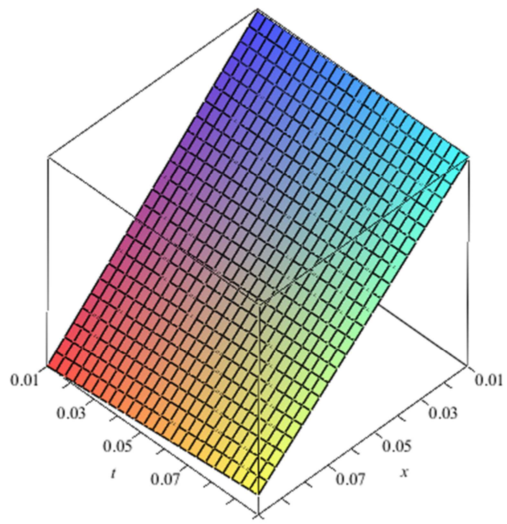

Figure 1. *Values of exact solution* $u(t,x)$.

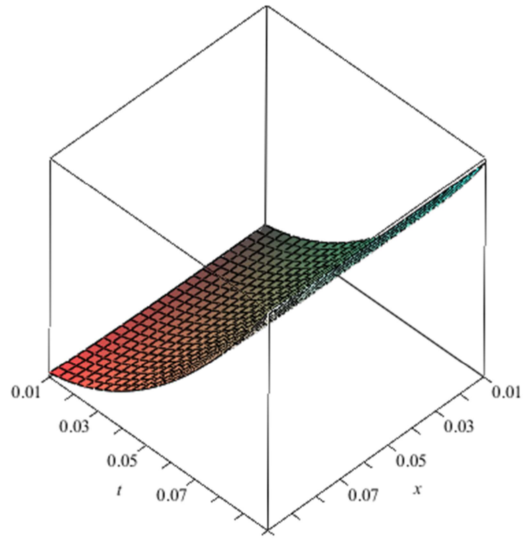

Figure 4. *For* $\propto = 1$, *values of* $v(t,x)$ *by FDTM.*

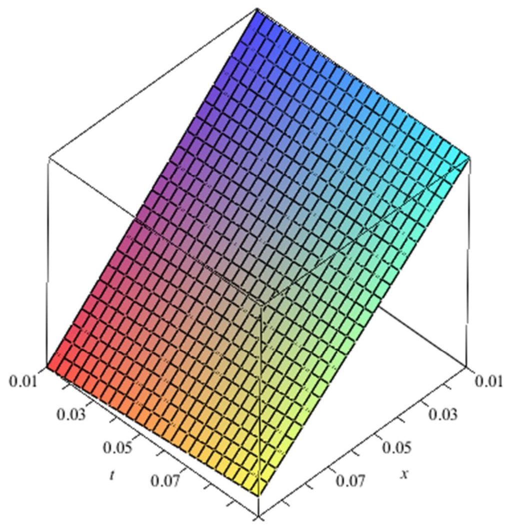

Figure 2. *For* $\propto = 1$, *values of* $u(t,x)$ *by FDTM.*

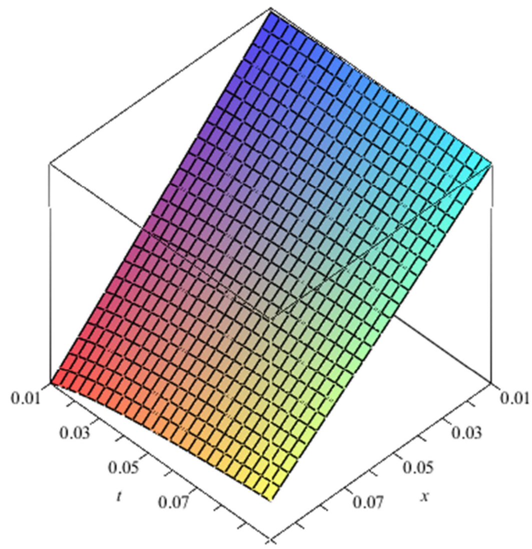

Figure 5. *For* $\propto = 0.5$, *values of* $u(t,x)$ *by FDTM.*

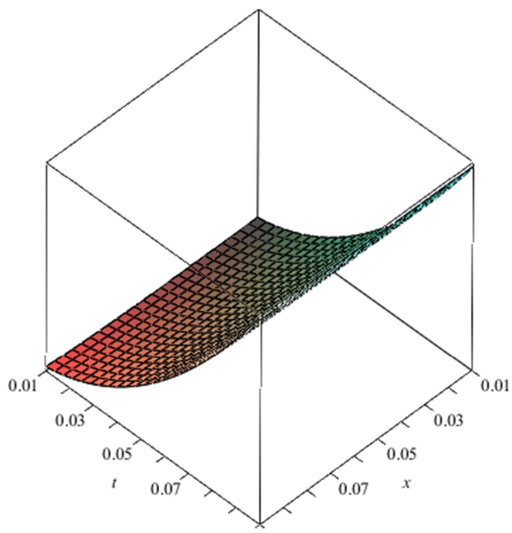

Figure 3. *Values of exact solution* $v(t,x)$.

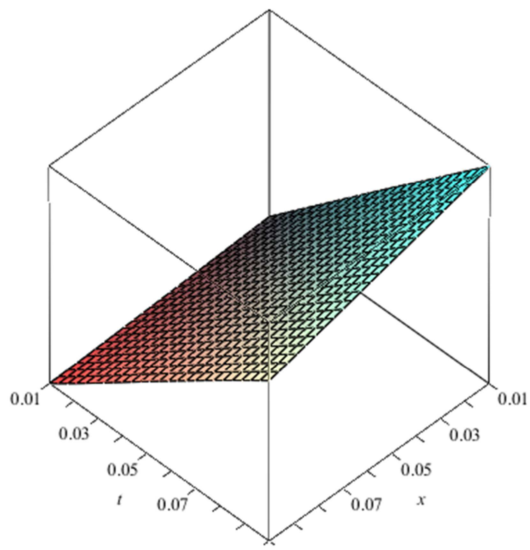

Figure 6. *For* $\propto = 0.5$, *values of* $v(t,x)$ *by FDTM.*

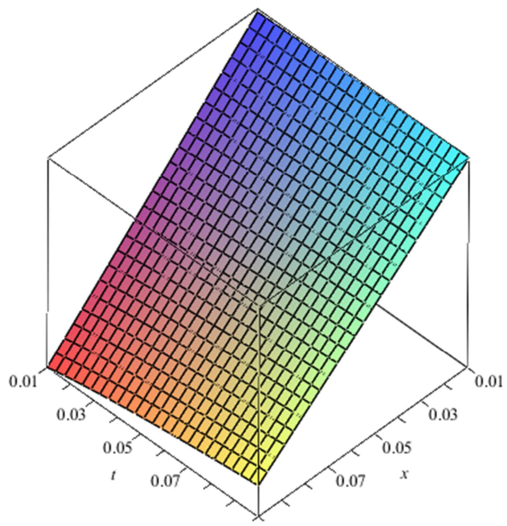

Figure 7. *For* $\propto = 0.75$, *values of* $u(t,x)$ *by FDTM.*

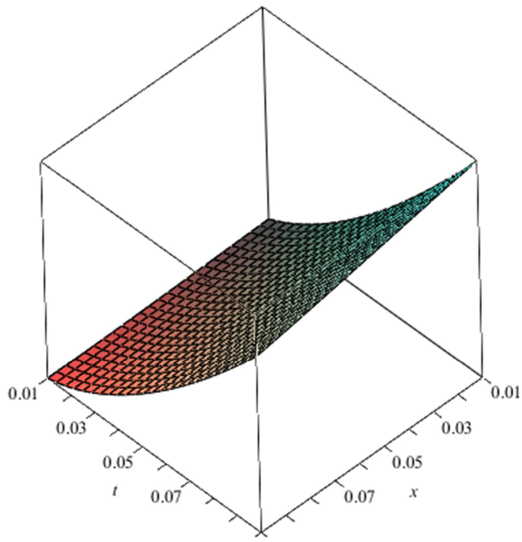

Figure 8. *For* $\propto = 0.75$, *values of* $v(t,x)$ *by FDTM.*

5. Conclusion

In this study, application of FDTM to fractional partial differential equation has been presented successfully. The results show thatFDTM is a powerful and efficient technique for finding analytic solutions for partial differential equations of fractional order. The obtained results reinforce the conclusions made by many researchers about the efficiency of FDTM.

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