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Some Properties of Composition Operator Acting Between General Hyperbolic Type Spaces

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Abstract
In this paper,we define generalized hyperbolic function classes, we study the composition operator C_ϕ from Bloch-type B_α spaces to $Q_{K,\omega}$ spaces and from B^*_α spaces to $Q^*_{K,\omega}$ spaces. The criteria for these operator to be bounded or compact and Lipschitz continuous are given. Our study also includes the corresponding hyperbolic spaces.

1. Introduction

Let ϕ be an analytic self-map of the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ denote the classes of analytic functions in the unit disc $\mathbb{D}$. The symbol ϕ induces a linear composition operator $C_\phi(f)=f\circ\phi$ from $H(\mathbb{D})$ or $B(\mathbb{D})$ into itself.

The hyperbolic function classes are subsets of the class $B(\mathbb{D})$ of all analytic functions f in the unit disc $\mathbb{D}$, such that $|f(z)|<1$ and the hyperbolic derivative of $f\in B(\mathbb{D})$ are defined by $f^*(z)=\frac{|f'(z)|}{1-|f(z)|^2}$, (cf. [7]). The hyperbolic derivative of the composition $f\circ\phi$ satisfies the equality $(f\circ\phi)^*(z)=f^*(\varphi(z))|\phi'(z)|$ which can be understood as a kind of chain rule. The study of composition operator C_ϕ acting on spaces of analytic functions has engaged many analysts for many years ([2, 3] and others).

The problem of boundedness and compactness of C_ϕ has been studied in many Banach spaces of analytic and hyperbolic functions and the study of such operators has recently attracted the most attention ([4, 5, 8, 14, 15] and others). If (X,d) is a metric space, we denote the open and closed balls with center x and radius $r>0$ by

$$B(x,r):=\{y\in X:d(y,x)<r\} \text{ and } \bar{B}(x,r):=\{y\in X:d(x,y)\le r\}, \text{ respectively.}$$

A function $f\in B(\mathbb{D})$ belongs to α-Bloch space $\mathcal{B}^\alpha, 0<\alpha<\infty$ if

$$\|f\|_{\mathcal{B}_\alpha}=\sup_{z\in D}(1-|z|^2)^\alpha|f'(z)|<\infty. \tag{1}$$

The little α-Bloch space $\mathcal{B}_{\alpha,0}$ consisting of all $f\in\mathcal{B}_\alpha$ such that

$$\lim_{|z|\to 1^-}(1-|z|^2)^\alpha |f'(z)|=0 \quad (see\,[17])$$

Definition 1.1 [9] For $0<\alpha\le 1$, a function $f\in B(\mathbb{D})$ is said to belong to the hyperbolic α-Bloch class $\mathcal{B}^*_\alpha$ if

$$\|f\|_{\mathcal{B}^*_\alpha}=\sup_{z\in \mathbb{D}} f^*(z)(1-|z|^2)^\alpha<\infty. \quad (2)$$

The little hyperbolic Bloch-type class $\mathcal{B}^*_{\alpha,0}$ consists of all $f\in\mathcal{B}^*_\alpha$ such that

$$\lim_{|z|\to 1} f^*(z)(1-|z|^2)^\alpha=0.$$

Note that when $\alpha=1,\mathcal{B}^*$ is the hyperbolic Bloch class and $\mathcal{B}^*_0$ is the little hyperbolic Bloch class [12]. The Schwarz-Pick lemma [11] implies $\mathcal{B}^*_\alpha=B(\mathbb{D})$ for all $\alpha\ge 1$ with $\|f\|_{\mathcal{B}^*_\alpha}\le 1$.

Let $0<\alpha<1$. We can find a natural metric on the hyperbolic (α-Bloch class $\mathcal{B}^*_\alpha$ by defining

$$d(f,g;\mathcal{B}^*{}_\alpha):=d_{\mathcal{B}^*_\alpha}(f,g)+\|f-g\|_{\mathcal{B}_\alpha}+|f(0)-g(0)|,$$

$$d_{\mathcal{B}^*_\alpha}(f,g):=\sup_{a\in\mathbb{D}}\left|\frac{f'(z)}{1-|f(z)|^2}-\frac{g'(z)}{1-|g(z)|^2}\right|(1-|z|^2)^\alpha. \quad (3)$$

2. $Q_{K,\omega}$ Spaces and Hyperbolic $Q^*_{K,\omega}$ Spaces

Definition 2.1([6, 10]) Let $K:[0,\infty)\to[0,\infty)$ be right-continuous and nondecreasing function, $0<p<\infty,-2<q<\infty$ and for given reasonable function $\omega:(0,1]\to(0,\infty)$, an analytic function f in $\mathbb{D}$ is said to belong to the space $Q_{K,\omega}(p,q)$ if

$$\|f\|_{Q_{K,\omega(p,q)}}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)<\infty,$$

If

$$\lim_{|a|\to 1^-}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^p\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)=0,$$

then $f\in Q_{K,\omega,0}(p,q)$. where

$$g(z,a)=\log\left|\frac{1-\bar{a}z}{z-a}\right|=\log\frac{1}{|\varphi_a(z)|},$$

the Green's function of $\mathbb{D}$ with logarithmic singularity at $a\in\mathbb{D}$.

Also,

$$\varphi_a(z)=\frac{a-z}{1-\bar{a}z},\quad for\ z\in\mathbb{D} \quad (see\,[1]),$$

$$1-|\phi_a(z)|^2=\frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} \quad (see\,[13]),$$

and the following identity is easily verified:

$$|\phi'_a(z)|=\frac{(1-|\phi_a(z)|^2)}{(1-|z|^2)}=\frac{(1-|a|^2)}{|1-\bar{a}z|^2} \quad (see\ [13]).$$

For $a\in\mathbb{D}$, the substitution $z=\varphi_a(w)$ results in the Jacobian change in measure given by $dA(w)=|\phi'_a(z)|^2\,dA(z)$.

Remark 1.1 If $\omega\equiv 1$, we obtain $Q_K(p,q)$ type spaces. If $q=p=2$, and $\omega(t)=t$, we obtain Q_K. If $q=p=2$, $\omega(t)=t$ and $K(t)=t^p$, we obtain Q_p. If $\omega\equiv 1$ and $K(t)=t^s$, then $Q_{K,\omega}=F(p,q,s)$ classes.

Definition 2.2Let $K:[0,\infty)\to[0,\infty)$ be right-continuous and nondecreasing function, $0<p<\infty,-2<q<\infty$ and for given reasonable function $\omega:(0,1]\to(0,\infty)$, an analytic function f in $\mathbb{D}$ is said to belong to the space $Q^*_{K,\omega}(p,q)$ if

$$\|f\|_{Q^*_{K,\omega(p,q)}}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(f^*(z))^p\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)<\infty. \quad (4)$$

If

$$\lim_{|a|\to 1^-}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(f^*(z))^p\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)=0,$$

then $f\in Q^*_{K,\omega,0}(p,q)$. For $2\le p<\infty,-2<q<\infty$, and $f,g\in Q^*_{K,\omega}(p,q)$, define their distance by

$$d(f,g;Q^*_{K,\omega}(p,q)) := d_{Q^*_{K,\omega}(p,q)}(f,g) + \| f-g \|_{Q_{K,\omega}(p,q)} + | f(0)-g(0) |,$$

where

$$d_{Q^*_{K,\omega}(p,q)}(f,g) := (\sup_{z\in D} \int_D | f^*_p(z) - g^*_p(z) |^p \, (1-|z|)^q \frac{K(g^s(z,a))}{\omega^p(1-|z|)} dA(z))^{\frac{1}{p}}. \tag{5}$$

Definition 2.3A composition operator $C_\phi : \mathcal{B}^*_\alpha \to Q^*_{K,\omega}(p,q)$ is said to be bounded, if there is a positive constant C such that $C_\phi f \|_{Q^*_{K,\omega}(p,q)} \le C \| f \|_{B^*_\alpha} \ \forall f \in \mathcal{B}^*_\alpha$.

Definition 2.4 A composition operator $C_\phi : \mathcal{B}^*_\alpha \to Q^*_{K,\omega}(p,q)$ is said to be compact, if it maps any ball in $\mathcal{B}^*_\alpha$ onto a pre-compact set in $Q^*_{K,\omega}(p,q)$.

Lemma 2.1 [17] Let $0<\alpha<\infty$, then there exist two holomorphic maps $f, g \in \mathcal{B}_\alpha$ such that for some constant C,

$$(f'(z)+g'(z))(1-|z|^2)^\alpha \approx C > 0, \quad \text{for each } z \in \mathbb{D}.$$

3. Main Results

Theorem 3.1 Let $0<\alpha<\infty, 0\le p<\infty, -2<q<\infty$. Then the following statements are equivalent:

(1) $C_\phi : \mathcal{B}_\alpha \to Q_{K,\omega}(p,q)$ is bounded;

(2) $C_\phi : \mathcal{B}_{\alpha,0} \to Q_{K,\omega}(p,q)$ is bounded.

(3) $\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^{\alpha p}} (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \infty.$

Moreover, if $0<\alpha\le 1$, then (1)-(3)are equivalent to

(4) $C_\phi : \mathcal{B}^*_\alpha \to Q^*_{K,\omega}(p,q)$ is bounded;

(5) $C_\phi : \mathcal{B}^*_{\alpha,0} \to Q^*_{K,\omega}(p,q)$ is bounded;

Proof: It is easy to see that (1) implies (2) since $\mathcal{B}_{\alpha,0} \subset \mathcal{B}_\alpha$.

Now we show that (2) implies (3). Let $f \in \mathcal{B}_\alpha$ and $f_t(z) = f(tz)$, $0<t<1$, then $f_t \in \mathcal{B}_{\alpha,0}$ and $\| f_t \|_{B_\alpha} \le \| f \|_{B_\alpha}$. We assume (2) holds, we have $\| C_\phi f \|_{K,\omega,p,q} \le \| C_\phi \| \| f \|_{B_\alpha}$ for all $f \in \mathcal{B}_{\alpha,0}$. Let f, g be two functions from Lemma 1.1, we obtain

$$(C_\phi f)'(z) + (C_\phi g)'(z))(1-|\phi(z)|^2)^\alpha \approx C |\phi'(z)| > 0, \quad \text{for each } z \in \mathbb{D}.$$

Thus

$$\begin{aligned}
&\int_{\mathbb{D}} \frac{|t\phi'(z)|^p}{(1-|t\phi(z)^2|)^{p\alpha}} \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a)) dA(z) \\
&\le C \int_{\mathbb{D}} ((C_\phi f)'(z))^p + ((C_\phi g)'(z))^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a)) dA(z) \\
&\le C(\| C_\phi f \|^p_{K,\omega,p,q} + \| C_\phi f \|^p_{K,\omega,p,q}) \\
&\le C \| C_\phi \|^p (\| f \|^p_{B_\alpha} + \| g \|^p_{B_\alpha}) < \infty.
\end{aligned} \tag{6}$$

This estimate together with the Fatou's lemma, implies that C_ϕ is bounded, so (3) is satisfied.

To prove that (3) implies (4). Let $0<\alpha\le 1$ and $f \in \mathcal{B}^*_\alpha$, Then

$$\| f\circ\phi \|^p_{Q^*_{K,\omega}(p,q)} = \int_D ((f\circ\phi)^*(z))^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a)) dA(z)$$

$$=\int_{\mathbb{D}}(f^*(\phi(z)))\,|\phi'(z)|^p\,\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)$$

$$\leq \|f\|^p_{\mathcal{B}^*_\alpha}\int_{\mathbb{D}}\frac{|\phi'(z)|^p}{(1-|\phi(z)^2|)^{p\alpha}}\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z),$$

and therefore $C_\phi:\mathcal{B}^*_\alpha\to Q^*_{K,\omega}(p,q)$ is bounded if (3) is satisfied.

Note that (4) implies (5) since $\mathcal{B}^*_{\alpha,0}\subset\mathcal{B}^*_\alpha$.

To prove that (5) implies (3). First let $\alpha=1$. Suppose that $C_\phi:\mathcal{B}^*_{\alpha,0}\to Q^*_{K,\omega}(p,q)$ is bounded and define $f_t(z):=tz, t\in\mathbb{D}$. Then $f_t^*(z)=\frac{|t|}{(1-|bz|^2)}$ and $f_t\in\mathcal{B}^*_{\alpha,0}$. Since (5)holds, there is a positive constant C such that

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|t\phi'(z)|^p}{(1-|t\phi(z)|^2)^p}\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)$$

$$\leq C\,\|f_t\|^p_{\mathcal{B}^*}\leq\infty. \tag{7}$$

This with fatou's lemma yields (3) with $\alpha=1$. If $0<\alpha\leq 1$, we use the fact that for each function $f\in B^*_\alpha$, the analytic function $C_\phi(f)\in Q^*_{K,\omega}(p,q)$. Then using the functions of Lemma 1.1

$$\{\|C_\phi f_1\|^p_{Q^*_{K,\omega}(p,q)}+\|C_\phi f_2\|^p_{Q^*_{K,\omega}(p,q)}\}$$

$$=\{\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}[((f_1\circ\phi)^*(z))^p+((f_2\circ\phi)^*(z))^p]\times\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)\}$$

$$\geq\{\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}[(f_1\circ\phi)^*(z)+(f_2\circ\phi)^*(z)]^p\times\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)\}$$

$$\geq\{\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}[(f_1^*(\phi))(z)+(f_2^*(\phi))(z)]^p\times|\phi'(z)|^p\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)\}$$

$$\geq C\{\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)\}. \tag{8}$$

Then, the condition (3) with $0<\alpha\leq 1$ follows.

We still have to show that (3) implies (1). Assume that (3) is holds, and $f\in B^*_\alpha$ with $\|f\|_{\mathcal{B}_\alpha}\leq 1$, we can see that

$$\|C_\phi f\|^p_{Q_{K,\omega}(p,q)}=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}((f\circ\phi)'(z))^p\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)$$

$$=\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(f'(\phi(z)))^p\,|\phi'(z)|^p\,\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)$$

$$\leq\|f\|^p_{\mathcal{B}_\alpha}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)<\infty. \tag{9}$$

Then (1) holds, and the proof is complete.

Theorem 3.2 Let $0<\alpha<\infty, 0\leq p<1, -2<q<\infty$. Then the following statements are equivalent:

(1) $C_\phi:\mathcal{B}^*_\alpha\to Q^*_{K,\omega}(p,q)$ is bounded;

(2) $C_\phi:\mathcal{B}^*_\alpha\to Q^*_{K,\omega}(p,q)$ is Lipschitz continuous;

(3) $\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^{\alpha p}}(1-|z|)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)<\infty.$

Proof:

Since (1) and (3) are equivalent by Theorem 2.1, it remains to prove that (2) $\Leftrightarrow$ (3).

Assume first that $C_\phi : \mathcal{B}^*_\alpha \to Q^*_{K,\omega}(p,q)$ is Lipschitz continuous, that is, there exists a positive constant C such that

$$d(f\circ\phi, g\circ\phi; Q^*_{K,\omega}(p,q)) \le C d(f,g;\mathcal{B}^*_\alpha), \quad \text{for all } f,g\in\mathcal{B}^*_\alpha.$$

Taking $g=0$, this implies

$$\|f\circ\phi\|_{Q^*_{K,\omega}(p,q)} \le C(\|f\|_{\mathcal{B}^*_\alpha} + \|f\|_{\mathcal{B}_\alpha} + |f(0)|), \quad \text{for all } f\in\mathcal{B}^*_\alpha. \tag{10}$$

The assertion (3) for $\alpha=1$, follows by choosing $f(z)=z$ in (10).

If $0<\alpha<1$, then

$$|f(z)| \le \left|\int_0^{|z|} |f'(s)|\,ds + |f(0)|\right|$$

$$\le \|f\|_{\mathcal{B}_\alpha} \int_0^{|z|} \frac{ds}{(1-s^2)^\alpha} + |f(0)|$$

$$\le C\frac{\|f\|_{\mathcal{B}_\alpha}}{1-\alpha} + |f(0)|,$$

this yields

$$|f(\phi(0)) - g(\phi(0))| \le C\frac{\|f-g\|_{\mathcal{B}_\alpha}}{(1-\alpha)} + |f(0)-g(0)|$$

Moreover, from Lemma 1.1, for $f,g\in\mathcal{B}^*_\alpha$, we deduce that

$$(f^*(z)+g^*(z))(1-|z|^2)^\alpha \ge C > 0, \quad \text{for all } z\in\mathbb{D}.$$

Therefore,

$$\|f\|_{\mathcal{B}^*_\alpha} + \|g\|_{\mathcal{B}^*_\alpha} + \|f\|_{\mathcal{B}_\alpha} + \|g\|_{\mathcal{B}_\alpha} + |f(0)| + |g(0)|$$

$$\ge C\int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}} \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))\,dA(z).$$

For which the assertion (3) follows.

Assume now that (3) is satisfied, we have

$$d(f\circ\phi, g\circ\phi; Q^*_{K,\omega}(p,q)) = d_{Q^*_{K,\omega}(p,q)}(f\circ\phi, g\circ\phi) + \|f\circ\phi - g\circ\phi\|_{Q_{K,\omega}(p,q)} + |f(\phi(0)) - g(\phi(0))|$$

$$\le d_{\mathcal{B}^*_\alpha}(f,g)\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}} \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))\,dA(z)\right)^{\frac{1}{p}}$$

$$+ \|f-g\|_{\mathcal{B}_\alpha}\left(\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}} \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))\,dA(z)\right)^{\frac{1}{p}}$$

$$+\frac{\|f-g\|_{\mathcal{B}_\alpha}}{1-\alpha} + |f(0)-g(0)| \le C d(f,g;\mathcal{B}^*_\alpha).$$

Thus $C_\phi : \mathcal{B}^*_\alpha \to Q^*_{K,\omega}(p,q)$ is Lipschitz continuous and the proof is established.

Theorem 3.3 Let $0<\alpha<\infty, 0\le p<\infty, -2<q<\infty$. Then the following statements are equivalent:

(1) $C_\phi : \mathcal{B}_\alpha \to Q_{K,\omega}(p,q)$ is compact;

(2) $C_\phi : \mathcal{B}_\alpha \to Q_{K,\omega}(p,q)$ is compact;

(3) $$\lim_{r\to 1^-}\sup_{a\in\mathbb{D}}\int_{|\phi(z)>r|} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^{\alpha p}} (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)}\,dA(z) < \infty.$$

Moreover, if $0<\alpha\le 1$, then (1)-(3) are equivalent to

(4) $C_\phi : \mathcal{B}^*_\alpha \to Q^*_{K,\omega}(p,q)$ is compact.

Proof:

The equivalence of (1)- (3) is similar to that proved in [17]. Hence it remains to show that these together are equivalent to (4).

We first assume that (3) holds. Let $B := \overline{B}(g,\delta) \subset \mathcal{B}_{\alpha}^{*}$, $g \in \mathcal{B}_{\alpha}^{*}$ and $\delta > 0$, be a closed ball, and let $(f_n)_{n=1}^{\infty} \subset B$ be any sequence. We show that its image has a convergent subsequence in $Q_{K,\omega}^{*}(p,q)$, which proves the compactness of C_ϕ by definition.

Again, $(f_n)_{n=1}^{\infty} \subset B(\mathbb{D})$, hence, there is a subsequence $(f_{n_j})_{j=1}^{\infty}$ which converges uniformly on the compact subsets of $\mathbb{D}$ to an analytic function f. By Cauchy formula for the derivative of an analytic function, also the sequence $(f'_{n_j})_{j=1}^{\infty}$ converges uniformly on the compact subsets of $\mathbb{D}$ to f'. It follows that also the sequences $(f_{n_j} \circ \phi)_{j=1}^{\infty}$ and $(f'_{n_j} \circ \phi)_{j=1}^{\infty}$ converge uniformly on the compact subsets of $\mathbb{D}$ to $f \circ \phi$ and $f' \circ \phi$, respectively. Moreover, $f \in B \subset \mathcal{B}_{\alpha}^{*}$ since for any fixed $R, 0 < R < 1$, the uniform convergence yield

$$\sup_{|z|\le R} \left| \frac{f'(z)}{1-|f(z)|^2} - \frac{g'(z)}{1-|g(z)|^2} \right| (1-|z|^2)^{\alpha}$$

$$+\sup_{|z|\le R} |f'(z)-g'(z)|(1-|z|^2)^{\alpha} + |f(0)-g(0)|$$

$$= \lim_{j\to\infty} \sup_{|z|\le R} \left| \frac{f'_{n_j}(z)}{1-|f_{n_j}(z)|^2} - \frac{g'(z)}{1-|g(z)|^2} \right| (1-|z|^2)^{\alpha}$$

$$+\lim_{j\to\infty} (\sup_{|z|\le R} |f'_{n_j}(z)-g'(z)|(1-|z|^2)^{\alpha} + |f_{n_j}(0)-g(0)|) < \delta.$$

Hence, $d(f,g;\mathcal{B}_{\alpha}^{*}) \le \delta$.

Let $\varepsilon > 0$. Since (3) is satisfied, we may fix $r, 0 < r < 1$, such that

$$\sup_{a\in\mathbb{D}} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}} \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z) \le \varepsilon.$$

By the uniform convergence, we may fix $N_1 \in \mathbb{N}$ such that

$$|f_{n_j} \circ \phi(0) - f \circ \phi(0)| \le \varepsilon, \quad \textit{for all } j \ge N_1. \tag{11}$$

The condition (3) is known to imply the compactness of $C_\phi : \mathcal{B}_\alpha \to Q_{K,\omega}(p,q)$, hence possibly to passing once more to a subsequence and adjusting the notations, we may assume that

$$\| f_{n_j} \circ \phi - f \circ \phi \|_{Q_{K,\omega}(p,q)} \le \varepsilon, \quad \textit{for all } j \ge N_2;\ N_2 \in \mathrm{N}. \tag{12}$$

Since $(f_{n_j})_{j=1}^{\infty} \subset B$ and $f \in B$, it follows that

$$\sup_{a\in\mathbb{D}} \int_{|\phi(z)|>r} [(f_{n_j}\circ\phi)^*(z) - (g\circ\phi)^*(z)]^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))\, dA(z)$$

$$\le \sup_{a\in\mathbb{D}} \int_{|\phi(z)|>r} Ł(f_{n_j}, g, \phi) \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))\, dA(z)$$

$$\le d_{\mathcal{B}_{\alpha}^{*}}(f_{n_j}, g) \sup_{a\in\mathbb{D}} \int_{|\phi(z)|>r} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}} \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))\, dA(z),$$

where

$$Ł(f_{n_j}, g, \phi) = \left| \frac{((f_{n_j}\circ\phi)'(z))}{1-|(f_{n_j}\circ\phi)(z)|^2} - \frac{(g\circ\phi)'(z)}{1-|(g\circ\phi)(z)|^2} \right|^p$$

hence,

$$\sup_{a\in\mathbb{D}}\int_{|\phi(z)|>r}[(f_{n_j}\circ\phi)^*(z)-(g\circ\phi)^*(z)]^p\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))\,dA(z)\le C\varepsilon. \tag{13}$$

On the other hand, by the uniform convergence on the compact disc $\mathbb{D}$, we can find an $N_3\in\mathbb{N}$ such that for all $j\ge N_3$,

$$Ł_1(f_{n_j},g,\phi)=\left|\frac{f'_{n_j}(\phi(z))}{1-|(f_{n_j}\circ\phi)(z)|^2}-\frac{g'_{n_j}(\phi(z))}{1-|(g\circ\phi)(z)|^2}\right|\le\varepsilon.$$

For all z with $|\phi(z)|\le r.$ Hence, for such j,

$$\sup_{a\in\mathbb{D}}\int_{|\phi(z)|\le r}[(f_{n_j}\circ\phi)^*(z)-(g\circ\phi)^*(z)]^p\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z)$$

$$\le\sup_{a\in\mathbb{D}}\int_{|\phi(z)|\le r}Ł_1(f_{n_j},g,\phi)|\phi'(z)|^p\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))\,dA(z)$$

$$\le\varepsilon(\sup_{a\in\mathbb{D}}\int_{|\phi(z)|\le r}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{\alpha p}}\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z))^{\frac{1}{p}}\le C\varepsilon, \tag{14}$$

where C is bounded which is obtained from Theorem 2.1. Combining (12), (13), (14) and (15) we deduce that $f_{n_j}\to f$ in $Q^*_{K,\omega}(p,q)$.

For the converse direction, let $f_n(z):=\frac{1}{2}n^{\alpha-1}z^n$ for all $n\in\mathbb{N}$, $n\ge2$.

$$\|\ f\|_{\mathcal{B}^*_\alpha}=\sup_{a\in\mathbb{D}}\frac{n^\alpha\,|z|^{n-1}}{1-2^{-2n}n^{2(\alpha-1)}\,|z|^{2n}}(1-|z|^2)^\alpha$$

$$\le3\sup_{a\in\mathbb{D}}n^\alpha\,|z|^{n-1}\,(1-|z|^2)^\alpha.$$

Then the sequence $(f_n)_{n=1}^\infty$ belongs to the ball $\overline{B}(0;3)\subset\mathcal{B}^*_\alpha$ (see [9]). We are assuming that C_ϕ maps the closed ball $\overline{B}(0;3)\subset\mathcal{B}^*_\alpha$ into a compact subset of $Q^*_{K,\omega}(p,q)$, hence, there exists an unbounded increasing subsequence $(n_j)_{j=1}^\infty$ such that the image subsequence $(C_\phi f_{n_j})_{n=1}^\infty$ converges with respect to the norm. Since, both $(f_n)_{n=1}^\infty$ and $(C_\phi f_{n_j})_{n=1}^\infty$ converge to the zero function uniformly on compact subsets of $\mathbb{D}$, the limit of the latter sequence must be 0. Hence,

$$\lim_{j\to\infty}\|n_j^{\alpha-1}\phi^{n_j}\|_{Q^*_{K,\omega}(p,q)}=0. \tag{15}$$

Now let $r_j=1-\frac{1}{n_j}$. For all numbers a, $r_j\le a<1$, we have the estimate

$$\frac{n_j^\alpha a^{n_j-1}}{1-a^{n_j}}\ge\frac{1}{e(1-a)^\alpha}\qquad(see\ 9). \tag{16}$$

Using (16) we deduce

$$\|n_j^{\alpha-1}\phi^{n_j}\|_{Q^*_{K,\omega}(p,q)}\ge\sup_{a\in\mathbb{D}}\int_{|\phi(z)|\ge r_j}\left|\frac{n_j^\alpha(\phi(z))^{n_j-1}|\phi'(z)|}{1-|\phi(z)^{n_j}|^2}\right|^p\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))\,dA(z)$$

$$\ge\frac{1}{4e^2}\sup_{a\in\mathbb{D}}\int_{|\phi(z)|>r_j}\frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}}\frac{(1-|z|)^q}{\omega^p(1-|z|)}K(g(z,a))dA(z). \tag{17}$$

From (16) and (17), the condition (3) follows. The proof is therefore completed.

Theorem 3.4 Let $0<\alpha<\infty, 0\le p<\infty,-2<q<\infty.$ Then the following statements are equivalent:

(1) $C_\phi:\mathcal{B}_\alpha\to Q_{K,\omega,0}(p,q)$ is bounded;

(2) $C_\phi : \mathcal{B}_\alpha \to Q_{K,\omega,0}(p,q)$ is compact;

(3) $\lim_{|a|\to 1}\int_{\mathbb{D}} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^2)^{\alpha p}}(1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) = 0.$

Moreover, if $0<\alpha\le 1$, then (1)-(3)are equivalent to

(4) $C_\phi : \mathcal{B}^*_\alpha \to Q^*_{K,\omega,0}(p,q)$ is bounded;

Proof: The equivalence of (1)- (3) is similar to that proved in [17]. It remains to prove the necessity and the sufficiency of (4).

The necessity of (4). Let $0<\alpha\le 1$ and $f \in \mathcal{B}^*_\alpha$, from (1) and Theorem 2.1 we have $C_\phi : \mathcal{B}^*_\alpha \to Q^*_{K,\omega}(p,q)$ is bounded. Furthermore, (3) implies that

$$\lim_{|a|\to 1}\int_{\mathbb{D}} ((f\circ\phi)^*(z))^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z)$$

$$= \lim_{|a|\to 1}\int_{\mathbb{D}} (f^*(z))^p |\phi'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z)$$

$$\le \|f\|^p_{\mathcal{B}^*_\alpha} \lim_{|a|\to 1}\int_{D} \frac{|\phi'(z)|^p}{(1-|\phi(z)^2|)^{p\alpha}} \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z) = 0. \tag{18}$$

That is, $C_\phi : \mathcal{B}^*_\alpha \to Q^*_{K,\omega,0}(p,q)$ is bounded.

The sufficincy of (4). Let $C_\phi : \mathcal{B}^*_\alpha \to Q^*_{K,\omega,0}(p,q)$ is bounded. Then using the functions of Lemma 1.1 as the proof of Theorem 2.1, we obtains

$$\lim_{|a|\to 1}\int_{\mathbb{D}} \frac{|\phi'(z)|^p}{(1-|\phi(z)|^2)^{p\alpha}} \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z)$$

$$\simeq \lim_{|a|\to 1}\int_{\mathbb{D}} ((C_\phi f_1)^*(z))^p + ((C_\phi f_2)^*(z))^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a)))dA(z)$$

$$\simeq \lim_{|a|\to 1}\int_{\mathbb{D}} (\sum_{i=1}^{2}((C_\phi f_i)^*(z))^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a)))dA(z) = 0. \tag{19}$$

Then, the condition (3) with $0<\alpha\le 1$ follows, and the proof is complete.

Theorem 3.5 *Let* $0<\alpha<\infty, 0\le p<\infty, -2<q<\infty.$ *Then the following statements are equivalent:*

(1) $C_\phi : \mathcal{B}_{\alpha,0} \to Q_{K,\omega,0}(p,q)$ is bounded;

(2) $C_\phi \in Q_{K,\omega,0}(p,q)$ and

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}} \frac{|\phi'(z)|^2}{(1-|\phi(z)|^p)^{\alpha p}}(1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} dA(z) < \infty.$$

Moreover, if $0<\alpha\le 1$, then (1)-(2)are equivalent to

(3) $C_\phi : \mathcal{B}^*_{\alpha,0} \to Q^*_{K,\omega,0}(p,q)$ is bounded.

Proof: The proof of (1) and (2) are equivalent is very similar to the proof in [17], it remains to prove the necessity and the sufficiency of (3). For every $\mathcal{B}^*_\alpha$, and $\varepsilon>0$ there exists $0<r<1$ such that

$$f^*(z)(1-|z|^2)^\alpha < \varepsilon.$$

For this fixed r, and when $|z|<r$, we have

$$\sup_{a\in\mathbb{D}}\int_{|\phi(z)|>r} ((f\circ\phi)^*(z)) \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z)$$

$$+\int_{|\phi(z)|\le r} ((f\circ\phi)^*(z)) \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z)$$

$$\leq \|f\|^p_{\mathcal{B}^*_\alpha} \int_{|\phi(z)|>r} \left(\frac{|\phi'(z)|^p}{(1-|\phi(z)^2|)^{p\alpha}} \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z)\right.$$

$$+\sup_{|w|\leq r} \frac{|f'(w)|^p}{(1-|f(w)|^2)^p} \int_{|\phi(z)|\leq r} |\phi'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z)$$

$$\leq \varepsilon^p M + \sup_{|w|\leq r} \frac{|f'(w)|^p}{(1-|f(w)|^2)^p} \int_{|\phi(z)|\leq r} |\phi'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z).$$

By (2), the right hand side tends to zero as $|z|\to 1^-$. Hence $C_\phi(f)\in Q^*_{K,\omega,0}(p,q)$, for every $f\in \mathcal{B}^*_{\alpha,0}$. The Sufficiency of (3). We show that (3) implies (2). From the condition (3), we have $C_\phi : \mathcal{B}^*_{\alpha,0} \to Q^*_{K,\omega}(p,q)$ is bounded, by Theorem 2.1

$$\sup_{a\in\mathbb{D}} \int_{|\phi(z)|>r} \left(\frac{|\phi'(z)|^p}{(1-|\phi(z)^2|)^{p\alpha}} \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z)\right) < \infty. \tag{20}$$

Also, where the function $f(z)=\dfrac{z}{2}\in \mathcal{B}^*_{\alpha,0}$, we have

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |\phi'(z)|^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z)$$

$$\leq 2^p \lim_{|a|\to 1^-} \int_{\mathbb{D}} ((f\circ\phi)^*(z))^p \frac{(1-|z|)^q}{\omega^p(1-|z|)} K(g(z,a))dA(z) = 0. \tag{21}$$

Then (2) is satisfied. The proof is complete.

4. Conclusions

Necessary and sufficient conditions are given for the composition operator C_ϕ to be Lipschitz continuous, bounded and compact composition operator from Bloch-type $\mathcal{B}_\alpha$ spaces to $Q_{K,\omega}$ spaces, from $\mathcal{B}^*_\alpha$ spaces to $Q^*_{K,\omega}$ spaces.

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