International Journal of Mathematical Analysis and Applications
2015; 2(3): 35-39
Published online April 30, 2015 (http://www.aascit.org/journal/ijmaa)
ISSN: 2375-3927

American Association for Science and Technology

Keywords
Multiplicative Difference Operators,
Nikolskii-Besov Spaces,
Norms Equivalent

Received: March 20, 2015
Revised: April 1, 2015
Accepted: April 2, 2015

On Equivalent Norms in Nikolskii - Besov's Isospaces Containing Multiplicative Differences of Fractional Order

M. U. Yakhshiboev

Samarkand branch of Tashkent University of Informational Technology, Samarkand, UZBEKIS

Email address
yahshiboev@rambler.ru

Citation
M. U. Yakhshiboev. On Equivalent Norms in Nikolskii - Besov's Isospaces Containing Multiplicative Differences of Fractional Order. *International Journal of Mathematical Analysis and Applications.* Vol. 2, No. 3, 2015, pp. 35-39.

Abstract
In this paper it is studied equivalent norms in the Nikolskii-Besov spaces containing multiplicative differences of fractional order.

1. Introduction

Properties of the Nikolskii-Besov spaces $B^l_{p,\theta}(R^n)$ are presented in details, e.g. in [1], as well as in [2]. Various problems related to application of multiplicative difference operators of fractional order, which are invariant with respect to strain [3], as well as differences of more general type in the theory of functional spaces were considered in [4]-[9].

Let us first give a definition of the Nikolskii-Besov space. Let $p,\theta \ge 1$ be real numbers, let $l,\sigma \ge 1$ and $m \ge 0$ be integer numbers such that

$$\sigma + m > l > m\,. \tag{1.1}$$

$B^l_{p,\theta}\left(\frac{1}{x}\right) := B^l_{p,\theta}\left(R^1_+, \frac{dx}{x}\right)$ is the Nikolskii-Besov space of all $f \in L_p\left(\frac{1}{x}\right) := L_p\left(R^1_+, \frac{dx}{x}\right)$ such that the m th order generalized derivative $D^m f$ exists and

$$\left\| f; B^l_{p,\theta}\left(\frac{1}{x}\right)\right\| = \left\| f; L_p\left(\frac{1}{x}\right)\right\| + \left\| f; b^l_{p,\theta}\left(\frac{1}{x}\right)\right\| < \infty\,, \tag{1.2}$$

where

$$\left\| f; L_p\left(1/x\right)\right\| = \begin{cases} \left(\int\limits_0^\infty |f(x)|^p \frac{dx}{x}\right)^{\frac{1}{p}}, & if \quad 1 \le p < \infty, \\ \sup\limits_{x\in R^1_+} vrai\left|f(x)\right|, & if \quad p = \infty \end{cases}$$

and

$$\left\|f;b_{p,\theta}^{l}\left(1/x\right)\right\|=\left\|\left|1-h\right|^{-(l-m)}\left\|\tilde{\Delta}_{h^{Sign(1-h)}}^{\sigma}D^{m}f;L_{p}\left(1/x\right)\right\|;L_{\theta}\left(1/\left|1-h\right|\right)\right\|=$$
$$=\left\{\int_{0}^{1}\left|\frac{\left\|\tilde{\Delta}_{h}^{\sigma}D^{m}f;L_{p}\left(1/x\right)\right\|}{(1-h)^{l-m}}\right|^{\theta}\frac{dh}{1-h}+\int_{1}^{\infty}\left|\frac{\left\|\tilde{\Delta}_{h^{-1}}^{\sigma}D^{m}f;L_{p}\left(1/x\right)\right\|}{(h-1)^{l-m}}\right|^{\theta}\frac{dh}{h-1}\right\}^{\frac{1}{\theta}}. \tag{1.3}$$

In terms of strain Π_h we introduce a magnitude

$$\left(\tilde{\Delta}_{h}^{\sigma}f\right)(x)=\left(\left(E-\Pi_{h}\right)^{\sigma}f\right)(x)=\sum_{k=0}^{\sigma}(-1)^{k}C_{\sigma}^{k}\left(\Pi_{h^{k}}f\right)(x)=\sum_{k=0}^{\sigma}(-1)^{k}C_{\sigma}^{k}f\left(x\cdot h^{k}\right) \tag{1.4}$$

for $h,x\in R_{+}^{1}$, where C_{σ}^{k} is a binomial coefficient, E is unit operator, Π_h is strain operator defined by $(\Pi_h f)(x):=f(x\cdot h)$. This magnitude is called final difference of order σ of function $f(x)$ with "multiplicative" step h.

Let now σ be arbitrary positive real number. Consider the difference -generally speaking, fractional order - determined by

$$\left(\tilde{\Delta}_{h}^{\sigma}f\right)(x)=\left(\left(E-\Pi_{h}\right)^{\sigma}f\right)(x)=\sum_{k=0}^{\infty}(-1)^{k}C_{\sigma}^{k}f\left(x\cdot h^{k}\right) \tag{1.5}$$

As (1.4) this is the bounded operator, which acts from $L_p(1/x)$ to $L_p(1/x)$. The series

$$C(\sigma):=\sum_{k=o}^{\infty}\left|C_{\sigma}^{k}\right|<\infty$$

converges due to $\left|C_{\sigma}^{k}\right|\le\frac{C}{k^{1+\sigma}}$, which implies that for any bounded function f the series in (1.5) converges absolutely and uniformly for all σ.

- We note that $C(\sigma)=2^{\sigma}$ for integer σ and for non-integer σ it holds $C(\sigma)=2^{[\sigma]+1}$.
- The difference (1.5) we call left-sided if $0<h<1$ and right-sided if $h>1$.
- Further, we consider the spaces $B_{p,\theta}^{l}(1/x)$ for any $\sigma>0$, which can be defined as above only replacing the multiplicative differences of integer order (1.4) by the multiplicative differences of arbitrary order (1.5).

2. Auxiliary Statements

Lemma 2.1. Let $\sigma>0$, then

$$\sum_{k=o}^{\infty}\left|C_{\sigma}^{k}\right|=\sum_{k=o}^{[\sigma]}\left(1+(-1)^{[\sigma]+k}\right)\cdot C_{\sigma}^{k}.$$

For the proof of the lemma see, e.g. [9].

Corollary 2.2. If $0<\sigma\le 1$, then

$$\sum_{k=o}^{\infty}\left|C_{\sigma}^{k}\right|=2$$

Corollary 2.3. The following estimates

$$2^{[\sigma]}\le\sum_{k=o}^{\infty}\left|C_{\sigma}^{k}\right|\le 2^{[\sigma]+1}$$

hold for all $\sigma>0$.

Lemma 2.4. For any $\alpha,\beta>0$ the following equalities

$$\tilde{\Delta}_{h}^{\alpha+\beta}=\tilde{\Delta}_{h}^{\alpha}\tilde{\Delta}_{h}^{\beta},\qquad \tilde{\Delta}_{2h}^{\alpha}=\left(\Pi_{\delta}+I\right)^{\alpha}\tilde{\Delta}_{h}^{\alpha}$$

Proof. These equalities follow from the relations

$$(E-A)^{\alpha+\beta}=(E-A)^{\alpha}(E-A)^{\beta};\qquad \left(E-A^{2}\right)^{\alpha}=(E+A)^{\alpha}(E-A)^{\alpha}$$

which can be proved by multiplication of the series with using the equalities for binomial coefficients, which follow from analogous relations

$$(1-x)^{\alpha+\beta}=(1-x)^{\alpha}(1-x)^{\beta};\qquad \left(1-x^{2}\right)^{\alpha}=(1+x)^{\alpha}(1-x)^{\alpha}$$

for real x.

Lemma 2.5. Let $1\le p\le\infty$ and $\sigma_2>\sigma_1\ge 0$ then the following estimate

$$\left\|\tilde{\Delta}_{h}^{\sigma_2}f;\ L_{p}(1/x)\right\|\le 2^{[\sigma_2-\sigma_1]+1}\left\|\tilde{\Delta}_{h}^{\sigma_1}f;\ L_{p}(1/x)\right\|$$

holds for all $f\in L_p(1/x)$.

Proof. According to Lemma lemma 2.4 we have

$$\tilde{\Delta}_{h}^{\sigma_2}=\tilde{\Delta}_{h}^{\sigma_2-\sigma_1}\tilde{\Delta}_{h}^{\sigma_1}=\sum_{k=o}^{\infty}(-1)^{k}C_{\sigma_2-\sigma_1}^{k}\Pi_{h^{k}}\tilde{\Delta}_{h}^{\sigma_1}$$

from which and Corollary 2.3 we obtain

$$\left\|\tilde{\Delta}_h^{\sigma_2} f;\ L_p\left(1/x\right)\right\| \le \sum_{k=o}^{\infty}\left(-1\right)^k\left|C_{\sigma_2-\sigma_1}^k\right|\ \left\|\tilde{\Delta}_h^{\sigma_1} f;\ L_p\left(1/x\right)\right\| \le 2^{[\sigma_2-\sigma_1]+1}\left\|\tilde{\Delta}_h^{\sigma_1} f;\ L_p\left(1/x\right)\right\|.$$

Lemma 2.6. For any $\sigma > 0$, $0 < h < 1$ or $h > 1$

$$\tilde{\Delta}_h^{\sigma} = 2^{-\sigma}\tilde{\Delta}_{2h}^{\sigma} + Q_h\tilde{\Delta}_h^{\sigma+1} \tag{2.1}$$

where

$$Q_h = -\sum_{k=0}^{\infty} C_{\sigma}^k 2^{-k}\tilde{\Delta}_k^{k-1} \tag{2.2}$$

and

$$\left\|Q_h\right\|_{L_p(1/x)\to L_p(1/x)} \le \frac{1}{2}\sum_{k=1}^{\infty}\left|C_{\sigma}^k\right|.$$

Proof. Equality (2.1) is operator analogue (after substitution of x by Π_h) of equality

$$(x-1)^{\sigma} = \frac{1}{2^{\sigma}}(x^2-1)^{\sigma} + p(x)(x-1)^{\sigma+1}$$

which is equivalent to

$$1 = 2^{-\sigma}(x+1)^{\sigma} + p(x)(x-1),$$

in which

$$p(x) = \frac{1-2^{-\sigma}(x+1)^{\sigma}}{x-1} = \frac{1}{x-1}\left[1-\left(1+\frac{x-1}{2}\right)^{\sigma}\right] =$$

$$= \frac{1}{x-1}\left(1-\sum_{k=0}^{\infty} C_{\sigma}^k\frac{(x-1)^k}{2^k}\right) = -\sum_{k=1}^{\infty} C_{\sigma}^k\frac{(x-1)^{k-1}}{2^k}.$$

In order to obtain (2.1) it is sufficient first of all to check that

$$E = 2^{-\sigma}(\Pi_h + E)^{\sigma} + Q_h\tilde{\Delta}_h^1 = \left(E + \frac{\tilde{\Delta}_h}{2}\right)^{\sigma} + Q_h\tilde{\Delta}_h^1,$$

and then to multiply this equality by $\tilde{\Delta}_h^{\sigma} = (E-\Pi_h)^{\sigma}$ and to use the Lemma 2.4.

The proof is complete.

3. Main Results and Proofs

Theorem3.1. Let $1 \le p,\theta \le \infty$ and $l > 0$. At different sets of natural numbers σ and integer number m satisfying (1.1), the spaces $B_{p,\theta}^l(1/x)$ coincide, and norms (1.2) are equivalent.

Theorem3.2. Let $1 \le p,\theta \le \infty$ and $l > 0$. At different sets of positive (not obligatory integer) numbers σ and integer numbers m satisfying (1.1), the spaces $B_{p,\theta}^l(1/x)$ coincide, and norms (1.2) are equivalent.

Proof of Theorem 3.1. For $k = 1,2$ we set

$$\left\|f;B_{p,\theta}^l\left(1/x\right)\right\|^{(\sigma_k,m_k)} = \left\|f;L_p\left(1/x\right)\right\| + \left\|f;b_{p,\theta}^l\left(1/x\right)\right\|^{(\sigma_k,m_k)},$$

where

$$\left\|f;b_{p,\theta}^l\left(1/x\right)\right\|^{(\sigma_k,m_k)} = \left\|\frac{\left\|\tilde{\Delta}_{h^{Sign(1-h)}}^{\sigma} D^m f;L_p\left(1/x\right)\right\|}{\left|1-h\right|^{l-m_k}};L_{\theta}\left(1/\left|h-1\right|\right)\right\|$$

σ_1,σ_2 are positive numbers and m_1,m_2 are non-negative integer numbers such that

$$\sigma_1 + m_1 > l > m_1 \ge 0.,\quad \sigma_2 + m_2 > l > m_2 \ge 0.$$

Let us suppose $\sigma_2 > \sigma_1$.

1.First of all we consider the case $m_1 = m_2 = 0$ and we use $\left\|f\right\|^{(\sigma_k)}$ instead of $\left\|f\right\|^{(\sigma_k,0)}$. The following estimate

$$\left\|f;b_{p,\theta}^l\left(1/x\right)\right\|^{(\sigma_2)} \le c_1\left\|f;b_{p,\theta}^l\left(1/x\right)\right\|^{(\sigma_1)} \tag{3.1}$$

with $c_1 = 2^{[\sigma_2-\sigma_1]+1}$ follows from the lemma 2.5.

Let $f \in L_p(1/x)$. We will prove the inequality

$$\left\|f;b_{p,\theta}^l\left(1/x\right)\right\|^{(\sigma_1)} \le c_2\left\|f;b_{p,\theta}^l\left(1/x\right)\right\|^{(\sigma_2)}. \tag{3.2}$$

For this purpose it is sufficient to prove that

$$\left\|f;b_{p,\theta}^l\left(1/x\right)\right\|^{(\sigma_1)} \le c_3\left\|f;b_{p,\theta}^l\left(1/x\right)\right\|^{(\sigma_1+1)} \tag{3.3}$$

for any $\sigma_1 > l$.

Indeed, let natural number s be chosen such that $\sigma_1 + s - 1 < \sigma_2 \le s + \sigma_1$. Then, by using inequality (3.3) consequently s times, and using (3.1) with $\sigma_1 + s$ instead of σ_2 and σ_2 instead of σ_1 we have (3.2).

According to Lemma 2.6, for $f \in L_p(1/x)$.

$$\tilde{\Delta}_h^{\sigma_1} f = 2^{-\sigma_1}\tilde{\Delta}_{2h}^{\sigma_1} f + Q_h\tilde{\Delta}_h^{\sigma_1+1} f$$

from which, taking into account (2.3) and Corollary 2.3, follows that

$$\left\|\tilde{\Delta}_h^{\sigma_1} f;L_p(1/x)\right\| \le 2^{-\sigma_1}\left\|\tilde{\Delta}_{2h}^{\sigma_1} f;L_p(1/x)\right\| + 2^{[\sigma_1]}\left\|\tilde{\Delta}_h^{\sigma_1+1} f;L_p(1/x)\right\|, \tag{3.4}$$

$$\left\|\tilde{\Delta}^{\sigma_1}_{h^{-1}} f; L_p(1/x)\right\| \le 2^{-\sigma_1} \left\|\tilde{\Delta}^{\sigma_1}_{(2h)^{-1}} f; L_p(1/x)\right\| + 2^{[\sigma_1]} \left\|\tilde{\Delta}^{\sigma_1+1}_{h^{-1}} f; L_p(1/x)\right\|. \tag{3.5}$$

By multiplying (3.4) by $(1-h)^{-l}$ and (3.5) by $(h-1)^{-l}$, and then by summing up these inequalities and applying the norm (1.3) we have

$$\left(\int_0^{1-\varepsilon} \left| \frac{\left\|\tilde{\Delta}^{\sigma_1}_{h} f; L_p(1/x)\right\|}{|1-h|^l} \right|^\theta \frac{dh}{1-h} + \int_{1+\eta}^{\infty} \left| \frac{\left\|\tilde{\Delta}^{\sigma_1}_{h^{-1}} f; L_p(1/x)\right\|}{|h-1|^l} \right|^\theta \frac{dh}{h-1} \right)^{\frac{1}{\theta}} \le$$

$$\le 2^{-\sigma_1} \left(\int_0^{1-\varepsilon} \left| \frac{\left\|\tilde{\Delta}^{\sigma_1}_{2h} f; L_p(1/x)\right\|}{|1-h|^l} \right|^\theta \frac{dh}{1-h} + \int_{1+\eta}^{\infty} \left| \frac{\left\|\tilde{\Delta}^{\sigma_1}_{(2h)^{-1}} f; L_p(1/x)\right\|}{|h-1|^l} \right|^\theta \frac{dh}{h-1} \right)^{\frac{1}{\theta}} + \tag{3.6}$$

$$+2^{[\sigma_1]} \left(\int_0^{1-\varepsilon} \left| \frac{\left\|\tilde{\Delta}^{\sigma_1+1}_{h} f; L_p(1/x)\right\|}{|1-h|^l} \right|^\theta \frac{dh}{1-h} + \int_{1+\eta}^{\infty} \left| \frac{\left\|\tilde{\Delta}^{\sigma_1+1}_{h^{-1}} f; L_p(1/x)\right\|}{|h-1|^l} \right|^\theta \frac{dh}{h-1} \right)^{\frac{1}{\theta}},$$

where $\varepsilon > 0, \eta > 0$.

Let us denote

$$\phi(\varepsilon,\eta) := \left(\int_0^{1-\varepsilon} \left| \frac{\left\|\tilde{\Delta}^{\sigma_1}_{h} f; L_p(1/x)\right\|}{|1-h|^l} \right|^\theta \frac{dh}{1-h} + \int_{1+\eta}^{\infty} \left| \frac{\left\|\tilde{\Delta}^{\sigma_1}_{h^{-1}} f; L_p(1/x)\right\|}{|h-1|^l} \right|^\theta \frac{dh}{h-1} \right)^{\frac{1}{\theta}}.$$

We note that

$$\phi(\varepsilon,\eta) \le 2^{\sigma_1} (\theta l)^{-\frac{1}{\theta}} (\varepsilon^{-l\theta} - 1 + \eta^{-l\theta}) \left\| f; L_p(1/x)\right\|. \tag{3.7}$$

After changing $2h$ by h in first member on the right side of (3.7) under the integral we will have it equal to $2^{l-\sigma_1} \phi(2\varepsilon, 2\eta)$. That is why from (3.6) follows that

$$\phi(\varepsilon;\eta) \le c\phi(2\varepsilon, 2\eta) + M, \tag{3.8}$$

where $c = 2^{l-\sigma_1} < 1$ and $M = 2^{[\sigma_1]} \left\| f; b^l_{p,\theta}(1/x)\right\|^{\sigma_1+1}$. By consequently applying the estimate in (3.8) to itself $(m-1)$ times we have that

$$\phi(\varepsilon;\eta) \le c(c\phi(4\varepsilon, 4\eta) + M \le \ldots \le c^m \phi(2^m\varepsilon, 2^m\eta) +$$

$$+(c^{m-1} + c^{m-2} + \ldots + c + 1)M = c^m \phi(2^m\varepsilon, 2^m\eta) + \frac{1-c^m}{1-c} M.$$

Since $f \in L_p(1/x)$, by transition to limit $m \to \infty$ and taking into account (3.7) we have that

$$\phi(\varepsilon;\eta) \le \frac{M}{1-c}, \tag{3.9}$$

and by transition here to limit $\varepsilon \to 0, \eta \to 0$ we have that

$$\left\| f; b^l_{p,\theta}(1/x)\right\|^{(\sigma_1)} \le \frac{2^{[\sigma_1]}}{1-2^{l-\sigma_1}} \left\| f; b^l_{p,\theta}(1/x)\right\|^{(\sigma_1+1)}, \tag{3.10}$$

$f \in L_p(1/x)$, that we were needed to prove.

2. Let now $m_1 > 0$ or $m_2 > 0$. We prove that the spaces $B^l_{p,\theta}(1/x)$ with parameters σ_k and m_k, $k = 1,2$ satisfying to (1.1), and with parameters $[\sigma_k]+1$ and m_k are coincident and

$$\left\| f; B^l_{p,\theta}(1/x)\right\|^{(\sigma_k, m_k)} \sim \left\| f; B^l_{p,\theta}(1/x)\right\|^{([\sigma_k]+1, m_k)}, k = 1,2. \tag{3.11}$$

Indeed, if $f \in B^l_{p,\theta}(1/x)$ with parameters σ_k and m_k, $k = 1,2$, then

$$\left\| f; b^l_{p,\theta}(1/x)\right\|^{(\sigma_k, m_k)} < +\infty,$$

and inequality

$$\begin{aligned} \left\| f; b^l_{p,\theta}(1/x)\right\|^{([\sigma_k]+1, m_k)} &= \left\| D^{m_k} f; b^l_{p,\theta}(1/x)\right\|^{([\sigma_k]+1,0)} \le \\ \le c_1 \left\| D^{m_k} f; b^{l-m_k}_{p,\theta}(1/x)\right\|^{(\sigma_k,0)} &= c_1 \left\| f; b^l_{p,\theta}(1/x)\right\|^{(\sigma_k, m_k)} \end{aligned} \tag{3.12}$$

follows from (3.1). That means that $f \in B^l_{p,\theta}(1/x)$ with parameters $[\sigma_k]+1, m_k$.

If $f \in B^l_{p,\theta}(1/x)$ with parameters $[\sigma_k]+1$, m_k, then (since the difference order is integer) according to well - known properties of spaces $B^l_{p,\theta}(1/x)$ the generalized derivative $D^m f \in L_p(1/x)$ (see. [1], [2]). That is why, according to (3.2)

$$\begin{aligned} \left\| f; b^l_{p,\theta}(1/x)\right\|^{(\sigma_k, m_k)} &= \left\| D^{m_k} f; b^{l-m_k}_{p,\theta}(1/x)\right\|^{(\sigma_k,0)} \le \\ \le c_2 \left\| D^{m_k} f; b^{l-m_k}_{p,\theta}(1/x)\right\|^{(\sigma_k,0)} &= c_2 \left\| f; b^l_{p,\theta}(1/x)\right\|^{([\sigma_1]+1, m_k)}. \end{aligned}$$

This and (3.12) and (3.13) imply (3.11), which completes the proof of Theorem 3.1.

Proof of Theorem 3.2. The proof of the theorem immediately follows from (3.11) and Theorem 3.1.

References

[1] S.M. Nikol'skii. Approximation of functions of many variables and embedding theorems, Nauka, Moscow (in Russian).

[2] O.V. Besov, V.P. Il'in, S.M. Nikol'skii.Integral Represention of Functions and Embedding Theorems. Nauka, Moscow (1975)(in Russian).

[3] S.G. Samko, M.U. Yahshiboev. Multi – dimensional fractional differentiation, invariant with respect to strain in terms of fractional differences. Rostov na Donu, 1991. p. 55. Dep. VINIII 29.12.91, N 4901 - B 91.

[4] H.S. Shapir. A Taubertion theorem related to approximate on theory. Acta Math., 1968, v. 120, p. 279 - 292.

[5] J. Boman, H.S. Shapiro. Comparison theorema for a generalizad modulus of continuity. Ball. Amer. Math - Soc., 1969, v. 75, N 6, p. 1266 - 1268.

[6] P.L. Butrer, U. Westphal. An access to fractional differentiation via fractional difference quotients. In "Fractional calculus and its applications" Lecture Notes in Math, 1975, N 475, p. 116-145.

[7] Ya.S. Bugorov. Fractional difference operators and function classes, Proc. MIAS USSR, 1985, v. 1970 (in Russian).

[8] K.K. Golovnin. On equivalent normalization of fractional spaces. Proc. MIAS USSR, 1962, v. 66, pp. 364-383, (in Russian).

[9] V.I. Burenkov, Sh.D. Sobnak. On equivalent norms in Nikolskii – Besov spaces containing differences of fractional order. Boundary problems of mathematical physics and some problems of the theory of functional spaces. Collected papers, Editor V. N. Maslennikova, Moscow, UDN, 1985, p. 142.