International Journal of Mathematical Analysis and Applications
2015; 2(3): 47-61
Published online May 10, 2015 (http://www.aascit.org/journal/ijmaa)
ISSN: 2375-3927

American Association for Science and Technology

Keywords
Gamma Function,
Beta Function,
Hypergeometric Function,
Confluent Hypergeometric Function

Received: March 23, 2015
Revised: April 7, 2015
Accepted: April 8, 2015

Notes on Generalized Hypergeometric and Confluent Hypergeometric Functions

S. I. El-Soubhy

Department of Mathematics, College of Science, Taibah University, Saudi Arabia

Email address
soone1@windowslive.com

Citation
S. I. El-Soubhy. Notes on Generalized Hypergeometric and Confluent Hypergeometric Functions. *International Journal of Mathematical Analysis and Applications.* Vol. 2, No. 3, 2015, pp. 47-61.

Abstract
Recently, some generalizations of the generalized Gamma, Beta, Gauss hypergeometric and confluent hypergeometric functions have been introduced in literature. The main object of this paper is to express the nth derivative of $z^s F_p^{(\alpha,\beta)}(a,b;c;z)$ with respect to the variable z in a closed formula of hypergeometric function itself. Moreover, some new integral involving of the above-mentioned functions are obtained and many important results are noted.

1. Introduction

In mathematics, there are several special functions that are of particular significance and are used in many applications [1, 2,3]. In addition, some of special functions find applications in such diverse areas as astrophysics, fluid dynamics and quantum physics [2, 3, 4, 5, 6,7]. Examples of such well-known functions are the Gamma, Beta and hypergeometric functions [1-11]. Next, extensions of Gamma, Beta, Gauss hypergeometric function (GHF) and confluent hypergeometric function (CHF) have been extensively studied in the recent past by inserting a regularization factor $e^{\frac{-p}{t}}$ [5].

The following extension of the gamma function is introduced by Chaudhry and Zubair [5]:

$$\Gamma_p(x) = \int_0^\infty t^{x-1}\exp\left(-t-\frac{p}{t}\right)dt, Re(p) > 0. \tag{1}$$

The extension of Euler's beta function is considered by Chaudhry et al. [8] in the following form

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(\frac{-p}{t(1-t)}\right)dt,$$
$$Re(p) > 0, Re(x) > 0, Re(y) > 0, \tag{2}$$

and they proved that this extension has connections with the Macdonald, error and Whittaker functions; and as a result

$$\Gamma_0(x) = \Gamma(x) \text{ and } \beta_0(x,y) = \beta(x,y).$$

Following this, Chaudhry et al. [2] used $\beta_p(x,y)$ to extend the hypergeometric function, known as the extended Gauss hypergeometric function (EGHF), as follows:

$$F_p(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{\beta_p(b+n,c-b)}{\beta(b,c-b)}\frac{(z)^n}{n!},$$
$$p\geq 0, Re(c)>Re(b)>0, \tag{3}$$

where $(a)_n$ denotes the Pochhammer symbol defined by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$$
$$=\begin{cases}1, n=0; a\in\mathbb{C}/\{0\}\\ a(a+1)(a+2)\dots(a+n-1), \quad n\in\mathbb{N}, a\in\mathbb{C}.\end{cases}$$

In addition, the integral representation of Euler's type function is

$$F_p(a,b;c;z)=\frac{1}{\beta(\mathrm{b},c-b)}\int_0^1 \mathrm{t}^{\mathrm{b}-1}(1-t)^{\mathrm{c}-b-1}(1-zt)^{-a}$$
$$\times exp\left(\frac{-p}{t(1-t)}\right)dt,$$
$$p\geq 0\text{and}|\arg(1-z)|<\pi<p; Re(c)>Re(b)>0. \tag{4}$$

Also, the extended confluent hypergeometric function (ECHF) is defined as

$$\varphi_p(b;c;z)=\sum_{n=0}^{\infty}\frac{\beta_p(b+n,c-b)}{\beta(b,c-b)}\frac{(z)^n}{n!},$$
$$p\geq 0, Re(c)>Re(b)>0. \tag{5}$$

The differentiation properties, the Mellin transforms of $F_p(a,b;c;z)$ were also considered in [2]. The transformation formulas, recurrence relations, summation and asymptotic formulas were obtained for this function. Very recently, some new representations of these extended functions were reported in terms of finite number of well-known higher transcendental functions [3,7].

The following generalized Euler's gamma function (GEGF) is defined in [3] as

$$\Gamma_{\mathrm{p}}{}^{(\alpha,\beta)}(x)=\int_0^{\infty}t^{x-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)dt,$$
$$Re(\alpha)>0, Re(\beta)>0, Re(p)>0, Re(x)>0. \tag{6}$$

While, the generalized Euler's beta function (GEBF) is given by

$$\beta_{\mathrm{p}}{}^{(\alpha,\beta)}(x\,,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt,$$
$$Re(p)>0, Re(x)>0, Re(y)>0, Re(\alpha)>0, Re(\beta)>0. \tag{7}$$

It is obvious from (1) and (6), (2) and (7) that,

$$\Gamma_p^{(\alpha,\alpha)}(x)=\Gamma_p(x), \Gamma_0{}^{(\alpha,\alpha)}(x)=\ \Gamma(x),$$
$$\beta_p^{(\alpha,\alpha)}(x,y)=\ \beta_p(x,y), \text{and}\beta_0{}^{(\alpha,\beta)}(x,y)=\beta(x,y).$$

Now, and in view of (7), the generalized (Gauss, resp., confluent) hypergeometric functions (GGHF, resp., GCHF) are defined by,

$$F_p{}^{(\alpha,\beta)}(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{\beta_p{}^{(\alpha,\beta)}(b+n,c-b)}{\beta(\mathrm{b},c-b)}\frac{\mathrm{z}^{\mathrm{n}}}{\mathrm{n}!}, \tag{8}$$

and

$$ {}_1F_1{}^{(\alpha,\beta;p)}(b;c;z) = \sum_{n=0}^{\infty} \frac{\beta_p{}^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)} \frac{z^n}{n!}, \tag{9} $$

and their corresponding integral representations are given by [3]:

$$ F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{\beta(b,c-b)} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)(1-zt)^{-a}dt, $$

$$ Re(p) \ \ge 0, \text{and } |\arg(1-z)| < \pi < p; Re(c) > Re(b) > 0. \tag{10} $$

In addition,

$$ {}_1F_1{}^{(\alpha,\beta;p)}(b;c;z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt, $$

$$ p \ge 0, Re(c) > Re(b) > 0. \tag{11} $$

It is to be noted here that [2],

$$ F_p^{(\alpha,\alpha)}(a,b;c;z) = F_p(a,b;c;z), $$

$$ F_0{}^{(\alpha,\beta)}(a,b;c;z) = {}_2F_1(a,b;c;z), $$

further,

$$ {}_1F_1{}^{(\alpha,\alpha;p)}(b;c;z) = {}_1F_1{}^{p}(b;c;z) = \varphi_p(b;c;z), $$

$$ {}_1F_1{}^{(\alpha,\beta;0)}(b;c;z) = {}_1F_1(b;c;z). $$

The generalized hypergeometric function with p numerator and q denominator parameters is defined by [1]

$$ {}_pF_q(a_1,a_2,\dots,a_p;b_1,b_2,\dots,b_q;z) $$

$$ = {}_pF_q\left(\begin{matrix} a_1,a_2,\dots,a_p \\ b_1,b_2,\dots,b_q \end{matrix};z\right) = \sum_{r=0}^{\infty} \frac{(a_1)_r(a_2)_r \dots (a_p)_r}{(b_1)_r(b_2)_r \dots (b_q)_r} \frac{z^r}{r!}, $$

$$ = \frac{\Gamma(b_1)\dots\Gamma(b_q)}{\Gamma(a_1)\dots\Gamma(a_p)} \sum_{r=0}^{\infty} \frac{\Gamma(a_1+r)\Gamma(a_2+r)\dots\Gamma(a_p+r)}{\Gamma(b_1+r)\Gamma(b_2+r)\dots\Gamma(b_q+r)} \frac{z^r}{r!}, $$

where $z \in \mathbb{C}, p \le q, a_i, b_j \in \mathbb{C}, b_j \ne 0,-1,-2,\dots, i = 1,2,\dots,p, j = 1,2,\dots,q.$

The nth derivative of $z^sF_p^{(\alpha,\beta)}(a,b;c;z)$ with respect to the variable z will be derived in the next Section. Further, new integral representations of (GGHF), (GCHF) with some useful new results, of the above-mentioned functions, are given in Section 3. Finally, some conclusion remarks are noted in Section 4.

2. The Derivatives of Generalized Gauss Hypergeometric (GGHF) and Confluent Hypergeometric Function (GCHF)

The generalization of beta function (7) and generalized Gauss hypergeometric function (GGHF) (8) or (10), in addition to confluent hypergeometric function (GCHF) (9) or (11), will be used to derive the nth derivative of $z^sF_p^{(\alpha,\beta)}(a,b;c;z)$ with respect to the variable z. The next theorem summarizes these relationships. Furthermore, some useful results are also considered in this Section.

Theorem 2.1. For the generalized Gauss hypergeometric function, we have

$$ \frac{d^n}{dz^n}\{z^sF_p{}^{(\alpha,\beta)}(a,b;c;z)\} = (-1)^s \frac{(a)_n(b)_n}{(c)_n} \sum_{w=0}^{\infty} (a+n)_w \frac{1}{(1-a-w-n)_s} $$

$$ \times \frac{1}{(w+n+1)_{-s}} \frac{\beta_p{}^{(\alpha,\beta)}(b+w+n-s,c-b)}{\beta(b+n,c-b)} \frac{z^w}{w!}, |z| < 1, n \in \mathbb{N}_0, \tag{12} $$

and in the integral form

$$\frac{d^n}{dz^n}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\}$$
$$= (-1)^s \frac{(a)_n (b)_n}{(c)_n} \frac{1}{\beta(b+n,c-b)}$$
$$\times \sum_{w=0}^{\infty} (a+n)_w \frac{1}{(1-a-w-n)_s} \frac{1}{(w+n+1)_{-s}}$$
$$\times \int_0^1 t^{b+n-s-1}(1-t)^{c-b-1} {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\frac{(zt)^w}{w!}\,dt,$$
$$|z|<1, n\in\mathbb{N}_0, \tag{13}$$

Proof. Direct Substitution of (8) into the left hand side of (12), yields

$$\frac{d^n}{dz^n}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\}$$
$$= \frac{d^n}{dz^n}\{z^s \sum_{r=0}^{\infty} (a)_r \frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{z^r}{r!}\}$$
$$= \sum_{r=0}^{\infty} (a)_r \frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{d^n}{dz^n}\frac{z^{r+s}}{r!}$$
$$= \sum_{r+s=n}^{\infty} (a)_r \frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}$$
$$\times (r+s)(r+s-1)\dots(r+s-n+1)\frac{z^{r+s-n}}{r!}$$
$$= \sum_{r+s=n}^{\infty} (a)_r \frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\left\{\frac{(r+s)!}{(r+s-n)!}\right\}\frac{z^{r+s-n}}{r!}.$$

Writing $r+s-n=w$, gives

$$\frac{d^n}{dz^n}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\}$$
$$= \sum_{w=0}^{\infty} (a)_{w+n-s} \frac{\beta_p^{(\alpha,\beta)}(b+w+n-s,c-b)}{\beta(b,c-b)}\left\{\frac{(w+n)!}{(w+n-s)!}\right\}\frac{z^w}{w!}. \tag{14}$$

In view of

$$(a)_{w+n} = (a)_n (a+n)_w \quad , \quad (a)_{n-k} = \frac{(-1)^k (a)_n}{(1-a-n)_k} \quad ,$$

and

$$(a)_{w+n-s} = \frac{(-1)^s (a)_{n+w}}{(1-a-w-n)_s} \quad ,$$

we get

$$\frac{d^n}{dz^n}\left\{z^s F_p^{(\alpha,\beta)}(a,b;c;z)\right\}$$
$$= (-1)^s \sum_{w=0}^{\infty} \frac{(a)_n (a+n)_w}{(1-a-w-n)_s} \frac{\beta_p^{(\alpha,\beta)}(b+w+n-s,c-b)}{\beta(b,c-b)}$$

$$\times \frac{\Gamma(w+n+1)}{\Gamma(w+n-s+1)} \frac{z^w}{w!}$$

$$= (-1)^s (a)_n \sum_{w=0}^{\infty} \frac{(a+n)_w}{(1-a-w-n)_s} \frac{\beta_p{}^{(\alpha,\beta)}(b+w+n-s, c-b)}{\beta(b, c-b)}$$

$$\times \frac{\Gamma(w+n+1)}{\Gamma(w+n-s+1)} \frac{z^w}{w!}.$$

By making use of the formula $\beta(b, c-b) = \frac{(c)_n}{(b)_n} \beta(b+n, c-b)$ [3], we have

$$\frac{d^n}{dz^n}\left\{z^s F_p{}^{(\alpha,\beta)}(a, b; c; z)\right\}$$

$$= (-1)^s \frac{(a)_n (b)_n}{(c)_n} \sum_{w=0}^{\infty} \frac{(a+n)_w}{(1-a-w-n)_s} \frac{1}{(w+n+1)_{-s}}$$

$$\times \frac{\beta_p{}^{(\alpha,\beta)}(b+w+n-s, c-b)}{\beta(b+n, c-b)} \frac{z^w}{w!}, \tag{15}$$

and this completes the proof of the first part of Theorem 2.1.

The second part of Theorem 2.1, can be easily proved in a similar way like that used for proving the first part.

The particular expressions for the derivatives of generalized Gauss hypergeometric function (GGHF) and confluent hypergeometric function (GCHF) may be obtained as special cases from formulas (12) and (13). These are given in the following five corollaries:

Corollary 1. Substituting of $p = 0$ into (12) or (13), yields the nth derivative of the classical (GHF) as.

$$\frac{d^n}{dz^n}\left\{z^s {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right)\right\} = (s-n+1)_n z^{s-n} {}_3F_2\left(\begin{matrix} a, b, s+1 \\ c, s-n+1 \end{matrix}; z\right). \tag{16}$$

Corollary 2. If we put $n = 1$ and $s = 0$, into (12) or (13), then we get immediately the first derivative of (GGHF),

$$\frac{d}{dz}\{F_p{}^{(\alpha,\beta)}(a, b; c; z)\} = \frac{ab}{c} F_p{}^{(\alpha,\beta)}(a+1, b+1; c+1; z). \tag{17}$$

Corollary 3. Substitution of $s = 0$, into (12) or (13), yields nth derivative of (GGHF) as,

$$\frac{d^n}{dz^n}\left\{F_p{}^{(\alpha,\beta)}(a, b; c; z)\right\} = \frac{(a)_n (b)_n}{(c)_n} F_p{}^{(\alpha,\beta)}(a+n, b+n; c+n; z) \tag{18}$$

Corollary 4. One can easily show that, for (GCHF),

$$\frac{d^n}{dz^n} \varphi_p{}^{(\alpha,\beta)}(b; c; z) = \frac{(b)_n}{(c)_n} \varphi_p{}^{(\alpha,\beta)}(b+n; c+n; z), \tag{19}$$

$$= \frac{(b)_n}{(c)_n} {}_1F_1{}^{(\alpha,\beta;p)}(b+n; c+n; z).$$

Corollary 5. Finally, the nth derivative of the generalized hypergeometric function ${}_pF_q$ is given by

$$\frac{d^n}{dz^n}\left\{z^s {}_pF_q\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}; z\right)\right\}$$

$$= (s-n+1)_n z^{s-n} \times {}_{p+1}F_{q+1}\left(\begin{matrix} a_1, a_2, \dots, a_p, s+1 \\ b_1, b_2, \dots, b_q, s-n+1 \end{matrix}; z\right),$$

$$z \in \mathbb{C}; p \leq q; a_i, b_j \in \mathbb{C}; b_j \neq 0, -1, -2, \dots; i = 1,2, \dots, p; j = 1,2, \dots, q. \tag{20}$$

It is to be noted here that the two results (17) and (18) are in complete agreement with those given in [3].

3. Some Integral Formulas of the Generalized Gauss Hypergeometric (GGHF) and Confluent Hypergeometric Function (GCHF)

Theorem 3.1. For the generalized Gauss hypergeometric function (GGHF), we have the following integral

$$\int_0^1 x^{n-1}(1-x)^{m-1} F_p^{(\alpha,\beta)}(a,b;c;kx)dx$$

$$= \beta(n,m)\sum_{r=0}^{\infty}\frac{(a)_r(n)_r}{(n+m)_r}\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}, \tag{21}$$

***Proof*.** Making use of relation (8) with relation (21), gives

$$\int_0^1 x^{n-1}(1-x)^{m-1}\sum_{r=0}^{\infty}(a)_r\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{(kx)^r}{r!}dx$$

$$=\sum_{r=0}^{\infty}(a)_r\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}\int_0^1 x^{n+r-1}(1-x)^{m-1}dx,$$

$$=\sum_{r=0}^{\infty}(a)_r\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}\beta(n+r,m),$$

$$=\sum_{r=0}^{\infty}(a)_r\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}\left[\frac{\Gamma(n+r)\Gamma(m)}{\Gamma(m+n+r)}\right],$$

$$=\beta(n,m)\sum_{r=0}^{\infty}(a)_r\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}\left[\frac{\Gamma(n+r)}{\Gamma(n+m+r)}\frac{\Gamma(m+n)}{\Gamma(n)}\right],$$

$$=\beta(n,m)\sum_{r=0}^{\infty}\frac{(a)_r(n)_r}{(n+m)_r}\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}.$$

This complete the proof of the Theorem.
Special cases of formula (21)are given in the following corollary.
Corollary 6.
(i) The integral of the classical (GHF), which is obtained by taking $p=0$, is .

$$\int_0^1 x^{n-1}(1-x)^{m-1}\,{}_2F_1(a,b;c;kx)dx = \beta(n,m)\,{}_3F_2\left({a,b,n \atop n+m,c};k\right). \tag{22}$$

This result is in complete agreement with the result given in [9, p.273].
(ii) the integral of (GCHF) is given by,

$$\int_0^1 x^{n-1}(1-x)^{m-1}\varphi_p^{(\alpha,\beta)}(b;c;kx)dx$$

$$=\sum_{r=0}^{\infty}\beta(n+r,m)\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{k^r}{r!}. \tag{23}$$

(iii) Moreover, for the generalized hypergeometric function ${}_pF_q$, we have

$$\int_0^1 x^{n-1}(1-x)^{m-1}\,{}_pF_q\left({a_1,a_2,\dots,a_p \atop b_1,b_2,\dots,b_q};kx\right)dx$$

$$=\beta(n,m)\,{}_{p+1}F_{q+1}\left({a_1,a_2,\dots,a_p,n \atop b_1,b_2,\dots,b_q,n+m};k\right),$$

$$x\in\mathbb{C};p\le q;a_i,b_j\in\mathbb{C};b_j\ne 0,-1,-2,\dots;i=1,2,\dots,p;j=1,2,\dots,q. \tag{24}$$

The following theorem gives new integral formulas for the generalized Gauss hypergeometric function (GGHF).
Theorem 3.2. For the generalized Gauss hypergeometric function (GGHF), we have the following integral,

$$\int_0^\infty e^{-kx} x^{m-1} F_p{}^{(\alpha,\beta)}(a,b;c;nx)dx$$

$$= \sum_{r=0}^{\infty} \frac{\Gamma(m)}{k^m}(a)_r(m)_r \frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{r!\beta(b,c-b)}\left(\frac{n}{k}\right)^r, k \neq 0. \tag{25}$$

Proof. Direct calculations using (10) yield

$$F_p^{(\alpha,\beta)}(a,b;c;nx) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}$$

$$\times (1-nxt)^{-a} {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt. \tag{26}$$

Because, $(1-nxt)^{-a} = \sum_{r=0}^{\infty} \frac{(a)_r}{r!}(nxt)^r$, we have

$$\int_0^\infty e^{-kx} x^{m-1} F_p{}^{(\alpha,\beta)}(a,b;c;nx)dx$$

$$= \frac{1}{\beta(b,c-b)}\int_0^\infty \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{-kx} x^{m-1} {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)$$

$$\times \sum_{r=0}^{\infty} \frac{(a)_r}{(r!)}(nxt)^r dxdt$$

$$= \frac{1}{\beta(b,c-b)}\sum_{r=0}^{\infty}(a)_r \frac{n^r}{r!}\int_0^\infty \int_0^1 t^{b+r-1}(1-t)^{c-b-1} e^{-kx} x^{m+r-1}$$

$$\times {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dxdt$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{\beta(b,c-b)}\frac{n^r}{r!}\int_0^\infty e^{-kx}x^{m+r-1}dx \int_0^1 t^{b+r-1}(1-t)^{c-b-1}$$

$$\times {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{\beta(b,c-b)}\frac{n^r}{r!}\int_0^\infty e^{-kx}x^{m+r-1}dx\beta_p^{(\alpha,\beta)}(b+r,c-b).$$

Because,

$$\frac{1}{k}\int_0^\infty e^{-w}\left(\frac{w}{k}\right)^{m+r-1}dw = \frac{1}{k^{m+r}}\int_0^\infty e^{-w}w^{m+r-1}dw,$$

then

$$\int_0^\infty e^{-kx} x^{m-1} F_p{}^{(\alpha,\beta)}(a,b;c;nx)dx$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{\beta(b,c-b)}\frac{\Gamma(m)}{k^{m+r}}\frac{n^r}{r!}\frac{\Gamma(m+r)}{\Gamma(m)}\beta_p^{(\alpha,\beta)}(b+r,c-b),$$

$$= \sum_{r=0}^{\infty} \frac{\Gamma(m)}{k^m}\frac{(a)_r(m)_r}{r!}\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\left(\frac{n}{k}\right)^r.$$

This complete the proof of the Theorem.

Three important special cases of Theorem 3.2 are noted in the following corollary.

Corollary 7.

(i)

$$\int_0^\infty e^{-kx}x^{m-1}\,{}_2F_1(a,b;c;nx)dx$$

$$=\sum_{r=0}^{\infty}\frac{\Gamma(m)}{k^m}\frac{(a)_r(m)_r}{r!}\frac{\beta(b+r,c-b)}{\beta(b,c-b)}(\frac{n}{k})^r,$$

$$=\frac{\Gamma(m)}{k^m}\sum_{r=0}^{\infty}\frac{(a)_r(b)_r(m)_r}{(c)_r r!}(\frac{n}{k})^r\ ,$$

$$=\frac{\Gamma(m)}{k^m}\,{}_3F_1\left(\begin{matrix}a,b;m\\ c\end{matrix};\frac{n}{k}\right),k\neq 0. \tag{27}$$

(ii)

$$\int_0^\infty e^{-kx}x^{m-1}\varphi_p{}^{(\alpha,\beta)}(b;c;nx)dx$$

$$=\sum_{r=0}^{\infty}\frac{\Gamma(m+r)}{k^m}\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{r!\,\beta(b,c-b)}(\frac{n}{k})^r,k\neq 0.$$

This result is in complete agreement with the result given in [9, p. 98].

(iii)

$$\int_0^\infty e^{-kx}x^{m-1}\,{}_pF_q\left(\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q\end{matrix};nx\right)dx$$

$$=\frac{\Gamma(m)}{k^m}\,{}_{p+1}F_q\left(\begin{matrix}a_1,a_2,\dots,a_{p,},m\\ b_1,b_2,\dots,b_q\end{matrix};\frac{n}{k}\right)dx,k\neq 0,$$

$$x\in\mathbb{C};p\leq q;a_i,b_j\in\mathbb{C};b_j\neq 0,-1,-2,\dots;i=1,2,\dots,p;j=1,2,\dots,q. \tag{28}$$

Theorem 3.3. For the generalized Gauss hypergeometric function (GGHF), we have the following integral

$$\int_0^\infty e^{-m^2x^2}x^{\mu-1}F_p{}^{(\alpha,\beta)}\left(a,b;c;\pm n^2x^2\right)dx$$

$$=\frac{\Gamma(\frac{\mu}{2})}{2m^\mu}\sum_{r=0}^{\infty}(a)_r\left(\frac{\mu}{2}\right)_r\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{(\pm\frac{n^2}{m^2})^r}{r!}. \tag{29}$$

Proof. In a similar manner, by using (10), we have

$$\int_0^\infty e^{-m^2x^2}x^{\mu-1}F_p{}^{(\alpha,\beta)}\left(a,b;c;\pm n^2x^2\right)dx$$

$$=\frac{1}{\beta(b,c-b)}\int_0^\infty\int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1-(\pm n^2x^2)t\right)^{-a}e^{-m^2x^2}x^{\mu-1}$$

$$\times\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dxdt.$$

However, $(1-(\pm n^2x^2)t)^{-a}=\sum_{r=0}^{\infty}\frac{(a)_r}{r!}(\pm n^2x^2t)^r$, so

$$\int_0^\infty e^{-m^2x^2} x^{\mu-1} F_p^{(\alpha,\beta)}(a,b;c;\pm\ n^2x^2)dx$$

$$= \frac{1}{\beta(b,c-b)} \sum_{r=0}^{\infty} \frac{(a)_r}{r!} \int_0^\infty \int_0^1 t^{b+r-1}(1-t)^{c-b-1}(\pm 1)^r n^{2r} e^{-m^2x^2}$$

$$\times x^{\mu+2r-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dxdt,$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r}{\beta(b,c-b)} \int_0^\infty e^{-m^2x^2} x^{\mu+2r-1} dx \beta_p^{(\alpha,\beta)}(b+r,c-b) \frac{\left(\pm\ n^2\right)^r}{r!}.$$

Using the definition of Gamma function, $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$, and letting $m^2x^2 = w$, we have,

$$\int_0^\infty e^{-m^2x^2} x^{\mu+2r-1} dx = \frac{1}{2m} \int_0^\infty e^{-w} (\frac{\sqrt{w}}{m})^{\mu+2r-1} \frac{1}{\sqrt{w}} dw,$$

$$= \frac{\frac{1}{2}}{m^{\mu+2r}} \Gamma(\frac{\mu}{2}+r\ ),$$

and accordingly,

$$\int_0^\infty e^{-m^2x^2} x^{\mu-1} F_p^{(\alpha,\beta)}(a,b;c;\pm n^2x^2)dx$$

$$= \frac{1}{2} \sum_{r=0}^{\infty} \frac{(a)_r}{\beta(b,c-b)} \frac{1}{m^{\mu+2r}} \Gamma\left(\frac{\mu}{2}+r\right) \beta_p^{(\alpha,\beta)}(b+r,c-b) \frac{(\pm n^2)^r}{r!},$$

$$= \frac{1}{2} \sum_{r=0}^{\infty} \frac{(a)_r}{\beta(b,c-b)} \frac{1}{m^{\mu+2r}} (\frac{\mu}{2})_r \Gamma(\frac{\mu}{2}\ ) \beta_p^{(\alpha,\beta)}(b+r,c-b) \frac{(\pm n^2)^r}{r!},$$

$$= \frac{1}{2} \sum_{r=0}^{\infty} \frac{(a)_r}{m^\mu} (\frac{\mu}{2})_r \Gamma(\frac{\mu}{2}) \frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)} \frac{\left(\frac{\pm n^2}{m^2}\right)^r}{r!},$$

$$= \frac{\Gamma(\ \frac{\mu}{2})}{2m^\mu} \sum_{r=0}^{\infty} (a)_r (\frac{\mu}{2})_r \frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)} \frac{\left(\frac{\pm n^2}{m^2}\right)^r}{r!}.$$

This completes the proof of the Theorem.

Again, Three important special cases of Theorem 3.3 are given in the following corollary

Corollary 8.

(i)

$$\int_0^\infty e^{-m^2x^2} x^{\mu-1}\ {}_2F_1(a,b;c;\pm n^2x^2)dx$$

$$= \frac{\Gamma\left(\frac{\mu}{2}\right)}{2m^\mu} \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r} \left(\frac{\mu}{2}\right)_r \frac{(\pm \frac{n^2}{m^2})^r}{r!},$$

$$= \frac{\Gamma\left(\frac{\mu}{2}\right)}{2m^\mu}\ {}_3F_1\left(\begin{matrix} a,b;\frac{\mu}{2};\pm\frac{n^2}{m^2} \\ c \end{matrix}\right). \tag{30}$$

(ii)

$$\int_0^\infty e^{-m^2x^2}x^{\mu-1}\varphi_p{}^{(\alpha,\beta)}(b;;c;\pm\ n^2x^2)dx$$

$$=\frac{1}{2m^\mu}\sum_{r=0}^\infty \Gamma\left(\frac{\mu}{2}+r\right)\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{(\pm\frac{n^2}{m^2})^r}{r!}. \tag{31}$$

(iii)

$$\int_0^\infty e^{-m^2x^2}x^{\mu-1}\ {}_pF_q\left(\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q\end{matrix};\pm\ n^2x^2\right)dx$$

$$=\frac{\Gamma\left(\frac{\mu}{2}\right)}{2m^\mu}\ {}_{p+1}F_1\left(\begin{matrix}a_1,a_2,\dots,a_p,\dfrac{\mu}{2};\pm\dfrac{n^2}{m^2}\\ b_1,b_2,\dots,b_q\end{matrix}\right)dx,$$

$$x\in\mathbb{C};p\le q;a_i,b_j\in\mathbb{C};b_j\neq 0,-1,-2,\dots;i=1,2,\dots,p;j=1,2,\dots,q. \tag{32}$$

Theorem 3.4. For the generalized Gauss hypergeometric function (GGHF), we have the following integral

$$\int_0^1 x^{n-1}(1-x)^{m-1}F_p{}^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\sum_{r=0}^\infty (a)_r\beta(n,m+r)\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{(\frac{1}{2})^r}{r!}. \tag{33}$$

Proof. Because,

$$\left(1-\frac{(1-x)}{2}t\right)^{-a}=\sum_{r=0}^\infty\frac{(a)_r}{r!}\left(\frac{1-x}{2}\right)^r t^r,$$

and using (10), we obtain

$$F_p{}^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}$$

$$\times\sum_{r=0}^\infty\frac{(a)_r}{r!}\left(\frac{1-x}{2}\right)^r t^r\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\mathrm{d}t,$$

and accordingly,

$$\int_0^1 x^{n-1}(1-x)^{m-1}F_p{}^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\frac{1}{\beta(b,c-b)}\sum_{r=0}^\infty\frac{(a)_r}{r!}\int_0^1\int_0^1 x^{n-1}(1-x)^{m-1}t^{b+r-1}(1-t)^{c-b-1}$$

$$\times\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\left(\frac{1-x}{2}\right)^r dxdt,$$

$$=\frac{1}{\beta(b,c-b)}\sum_{r=0}^\infty\frac{(a)_r}{2^r r!}\int_0^1\int_0^1 x^{n-1}(1-x)^{m+r-1}t^{b+r-1}(1-t)^{c-b-1}$$

$$\times\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dxdt,$$

$$=\frac{1}{\beta(b,c-b)}\sum_{r=0}^\infty\frac{(a)_r}{2^r r!}\int_0^1 x^{n-1}(1-x)^{m+r-1}dx\int_0^1 t^{b+r-1}(1-t)^{c-b-1}$$

$$\times\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt.$$

Making use of (7), enables one to get

$$\int_0^1 t^{b+r-1}(1-t)^{c-b-1}\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt = \beta_p{}^{(\alpha,\beta)}(b+r,c-b),$$

which immediately yields

$$\int_0^1 x^{n-1}(1-x)^{m-1}F_p{}^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\frac{1}{\beta(b,c-b)}\sum_{r=0}^{\infty}(a)_r\beta(n,m+r)\beta_p{}^{(\alpha,\beta)}(b+r,c-b)\frac{1}{r!}\left(\frac{1}{2}\right)^r.$$

Special cases from formula (33) are obtained for the particular expressions of the generalized Gauss hypergeometric function (GGHF) and confluent hypergeometric function (GCHF), as given in the following corollary

Corollary 9.

(i)

$$\int_0^1 x^{n-1}(1-x)^{m-1}\ {}_2F_1\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\ \beta(n,m)\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}\frac{(m)_r}{(n+m)_r}\frac{\left(\frac{1}{2}\right)^r}{r!},$$

$$=\beta(n,m)\ {}_3F_2\left({a,b,m \atop n+m,c};\tfrac{1}{2}\right), \tag{34}$$

(ii)

$$\int_0^1 x^{n-1}(1-x)^{m-1}\varphi_p{}^{(\alpha,\beta)}\left(b;c;\frac{1-x}{2}\right)dx$$

$$=\sum_{r=0}^{\infty}\beta(n,m+r)\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{\left(\frac{1}{2}\right)^r}{r!}, \tag{35}$$

(iii)

$$\int_0^1 x^{n-1}(1-x)^{m-1}\ {}_pF_q\left({a_1,a_2,\dots,a_p \atop b_1,b_2,\dots,b_q};\frac{1-x}{2}\right)dx$$

$$=\beta(n,m)\ {}_{p+1}F_{q+1}\left({a_1,a_2,\dots,a_p,m \atop b_1,b_2,\dots,b_q,n+m};\tfrac{1}{2}\right). \tag{36}$$

where $z\in\mathbb{C}; p\le q; a_i,b_j\in\mathbb{C}; b_j\neq 0,-1,-2,\dots; i=1,2,\dots,p; j=1,2,\dots,q.$.

Theorem 3.5. For the generalized Gauss hypergeometric function (GGHF) we have the following integral

$$\int_0^1 (1-x^2)^{m-1}F_p{}^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\sum_{r=0}^{\infty}2^{2m+r-2}\beta(m+r,m)\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{(a)_r}{r!}\left(\frac{1}{2}\right)^r, \tag{37}$$

Proof. Making use of (10), yields

$$F_p{}^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}$$

$$\times_1 F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\sum_{r=0}^{\infty}\frac{(a)_r}{r!}(\frac{1-x}{2})^r t^r dt,$$

and this gives

$$\int_0^1 (1-x^2)^{m-1} F_p^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\frac{1}{\beta(b,c-b)}\int_0^1\int_0^1 (1-x^2)^{m-1}t^{b-1}(1-t)^{c-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)$$

$$\times\sum_{r=0}^{\infty}\frac{(a)_r}{r!}\left(\frac{1-x}{2}\right)^r t^r dtdx.$$

Let $x=\cos\theta, 1-x=2\sin^2\left(\frac{\theta}{2}\right)$, and $(1+x)=2\cos^2\left(\frac{\theta}{2}\right)$. Then, in this case, the integration takes the form

$$\int_0^1 (1-x^2)^{m-1} F_p^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\frac{1}{\beta(b,c-b)}\sum_{r=0}^{\infty}\frac{(a)_r}{2^r r!}\int_0^{\frac{\pi}{2}}\int_0^1\left(2\sin^2\left(\frac{\theta}{2}\right)\right)^r\left(2\sin^2\left(\frac{\theta}{2}\right)2\cos^2\left(\frac{\theta}{2}\right)\right)^{m-1}\sin\theta$$

$$\times t^{b+r-1}(1-t)^{c-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)d\theta dt,$$

$$=\frac{1}{\beta(b,c-b)}\sum_{r=0}^{\infty}\int_0^{\frac{\pi}{2}}\int_0^1 2^{2m+r-1}sin^{2m+2r-1}\left(\frac{\theta}{2}\right)cos^{2m-1}\left(\frac{\theta}{2}\right)$$

$$\times t^{b+r-1}(1-t)^{c-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\frac{(a)_r}{2^r r!}d(\frac{\theta}{2})dt,$$

using (2) and (7), gives

$$\int_0^1 (1-x^2)^{m-1} F_p^{(\alpha,\beta)}\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\sum_{r=0}^{\infty}2^{2m+r-2}\beta(m+r,m)\frac{\beta_p^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{(a)_r}{r!}\left(\frac{1}{2}\right)^r.$$

Corollary 10.
(i) Substituting $p=0$ into (37) gives the integral

$$\int_0^1 (1-x^2)^{m-1}\,{}_2F_1\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\tfrac{1}{2}\beta(\tfrac{1}{2},m)\,{}_3F_2\left(\begin{matrix}a,b;m\\c;2m\end{matrix};1\right). \qquad (38)$$

Proof. Setting $p=0$ in (37) and using (7), along with the properties of the Gamma function, yields

$$\int_0^1 (1-x^2)^{m-1}\,{}_2F_1\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\sum_{r=0}^{\infty}2^{2m+r-2}\frac{\Gamma(m+r)\Gamma(m)}{\Gamma(2m+r)}\frac{\Gamma(b+r)}{\Gamma(c+r)}\frac{\Gamma(c)}{\Gamma(b)}\frac{(a)_r}{r!}\left(\frac{1}{2}\right)^r,$$

$$=\sum_{r=0}^{\infty} 2^{2m+r-2}\frac{(a)_r(b)_r}{(c)_r}\frac{\Gamma(m+r)\Gamma(m)}{\Gamma(2m+r)}\frac{1}{r!}\left(\frac{1}{2}\right)^r,$$

$$=\sum_{r=0}^{\infty} 2^{2m+r-2}\frac{(a)_r(b)_r}{(c)_r}\frac{(m)_r}{(2m)_r}\beta(m,m)\frac{1}{r!}\left(\frac{1}{2}\right)^r.$$

By using the Legendre duplication formulae

$$\Gamma(2m)=\frac{1}{\sqrt{2\pi}}2^{2m-\frac{1}{2}}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right),$$

then,

$$\int_0^1 (1-x^2)^{m-1}\,{}_2F_1\left(a,b;c;\frac{1-x}{2}\right)dx$$

$$=\sum_{r=0}^{\infty}\frac{1}{2}\frac{(a)_r(b)_r}{(c)_r}\frac{(m)_r}{(2m)_r}\beta(\frac{1}{2},m)\ \frac{(1)^r}{r!},$$

$$=\frac{1}{2}\beta\left(\frac{1}{2},m\right)\,{}_3F_2\left(\begin{matrix}a,b;m\\ c;2m\end{matrix};1\right).$$

(ii)

$$\int_0^1 (1-x^2)^{m-1}\varphi_p{}^{(\alpha,\beta)}\left(b;c;\frac{1-x}{2}\right)dx$$

$$=\sum_{r=0}^{\infty} 2^{2m+r-2}\beta(m+r,m)\frac{\beta_p{}^{(\alpha,\beta)}(b+r,c-b)}{\beta(b,c-b)}\frac{1}{r!}\left(\frac{1}{2}\right)^r. \tag{39}$$

(iii)

$$\int_0^1 (1-x^2)^{m-1}\,{}_pF_q\left(\begin{matrix}a_1,a_2,\dots,a_p\\ b_1,b_2,\dots,b_q\end{matrix};\frac{1-x}{2}\right)dx$$

$$=\beta(\tfrac{1}{2},m)\,{}_{p+1}F_{q+1}\left(\begin{matrix}a_1,a_2,\dots,a_{p,},m\\ b_1,b_2,\dots,b_q,2m\end{matrix};1\right) \tag{40}$$

where $x\in\mathbb{C}; \mathrm{p}\le q; a_i,b_j\in\mathbb{C}; b_j\neq 0,-1,-2,\dots; i=1,2,\dots,p; j=1,2,\dots,q.$

The next theorem considers the behaviour of the generalized Gauss hypergeometric function (GGHF) using the gamma function.

Theorem 3.6.

$$\lim_{\gamma\to -n}(\Gamma(\gamma))^{-1}F_p{}^{(\alpha,\beta)}(a,b;\gamma;x)$$

$$=\frac{x^{n+1}}{\Gamma(b)\Gamma(-b-n)}\sum_{s=0}^{\infty}\frac{(a+n+s)!}{\Gamma(a)(n+s+1)!}\beta_p{}^{(\alpha,\beta)}(b+n+s+1,-b-n)x^s \tag{41}$$

Proof. Making use of (8), gives

$$F_p{}^{(\alpha,\beta)}(a,b;\gamma;x)=\sum_{r=0}^{\infty}(a)_r\frac{\beta_p{}^{(\alpha,\beta)}(b+r,\gamma-b)}{\beta(b,\gamma-b)}\frac{x^r}{r!},$$

and accordingly,

$$\lim_{\gamma\to -n}(\Gamma(\gamma))^{-1}F_p{}^{(\alpha,\beta)}(a,b;\ \gamma;x)$$

$$= \lim_{\gamma\to -n}\frac{1}{\Gamma(\gamma)}\sum_{r=0}^{\infty}(a)_r\frac{\beta_p{}^{(\alpha,\beta)}(b+r,\gamma-b)}{\beta(b,\gamma-b)}\frac{x^r}{r!},$$

$$= \sum_{r=0}^{\infty}(a)_r\frac{\beta_p{}^{(\alpha,\beta)}(b+r,\gamma-b)}{\beta(b,\gamma-b)}\frac{x^r}{r!}\lim_{\gamma\to -n}\frac{1}{\Gamma(\gamma)},$$

and using (7), we obtain

$$= \sum_{r=0}^{\infty}(a)_r\int_0^1 t^{b+r-1}(1-t)^{\gamma-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\frac{x^r}{r!}dt$$

$$\times \lim_{\gamma\to -n}\frac{1}{\Gamma(b)\Gamma(\gamma-b)},$$

$$= \sum_{r=0}^{\infty}\frac{(a)_r}{\Gamma(b)}\int_0^1 t^{b+r-1}(1-t)^{-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\frac{x^r}{r!}$$

$$\times \lim_{\gamma\to -n}\frac{(1-t)^{\gamma}}{\Gamma(\gamma-b)}dt,$$

$$= \sum_{r=0}^{\infty}\frac{(a)_r}{\Gamma(b)}\int_0^1 t^{b+r-1}(1-t)^{-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\frac{x^r}{r!}$$

$$\times \frac{(1-t)^{-n}}{\Gamma(-n-b)}dt.$$

Now the substitution $r = n + s + 1$, yields

$$\sum_{s=0}^{\infty}\frac{(a)_{n+s+1}}{\Gamma(b)}\int_0^1 t^{b+n+s}(1-t)^{-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\frac{x^{n+s+1}}{(n+s+1)!}\times\frac{(1-t)^{-n}}{\Gamma(-n-b)}dt,$$

$$= \frac{x^{n+1}}{\Gamma(b)\Gamma(-n-b)}\sum_{s=0}^{\infty}(a)_{n+s+1}\int_0^1 t^{b+n+s}(1-t)^{-b-n-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\times\frac{x^s}{(n+s+1)!}dt, \tag{42}$$

Again, using (7), we obtain

$$= \frac{x^{n+1}}{\Gamma(b)\Gamma(-n-b)}\sum_{s=0}^{\infty}\frac{(a)_{n+s+1}}{(n+s+1)!}\beta_p{}^{(\alpha,\beta)}(b+n+s+1,-b-n)x^s,$$

$$= \frac{x^{n+1}}{\Gamma(a)\Gamma(b)\Gamma(-n-b)}\sum_{s=0}^{\infty}\frac{(a+n+s)!}{(n+s+1)!}\beta_p{}^{(\alpha,\beta)}(b+n+s+1,-b-n)x^s.$$

The particular expressions for the generalized Gauss hypergeometric function (GGHF) and confluent hypergeometric function (GCHF) may be obtained as special cases of formula (41). This is given in the following corollary

Corollary 11.

(i)

$$\lim_{\gamma\to -n}(\Gamma(\gamma))^{-1}\,{}_2F_1(a,b;\gamma;x)$$

$$= x^{n+1}\frac{(a)_{n+1}(b)_{n+1}}{(n+1)!}\,{}_2F_1\left(\begin{matrix}a+n+1,b+n+1\\ n+2\end{matrix};x\right), \tag{43}$$

$$= \frac{x^{n+1}}{\Gamma(a)\Gamma(b)}\sum_{s=0}^{\infty}\frac{(a+n+s)!(b+n+s)!}{(n+s+1)!}\frac{x^s}{s!}. \tag{44}$$

(ii)

$$\lim_{\gamma\to -n}(\Gamma(\gamma))^{-1}\varphi_p{}^{(\alpha,\beta)}(b;\gamma;x)$$

$$= \frac{x^{n+1}}{\Gamma(-n)} \sum_{s=0}^{\infty} \frac{1}{\Gamma(n+s+2)} \frac{\beta_p{}^{(\alpha,\beta)}(b+n+s+1,-b-n)}{\beta(b,-n-b)} \frac{x^s}{s!}. \quad (45)$$

4. Concluding Remarks

In this paper, we have expressed explicitly the nth derivative of Gauss hypergeometric and confluent hypergeometric functions in terms of the hypergeometric functions themselves. Some new integrals involving such functions are obtained. Many important results are also given. We hope to extend our results for special functions in the near future.

Acknowledgement

My most profound gratitude is to Professor Doha, E.H. for the comments he made on the first version of this paper.

References

[1] G.E. Andrews, R.Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge (1999).

[2] M.A. Chaudhry, A. Qadir, H.M. Srivastava, R.B. Paris, Extended hypergeometric and confluent hypergeometric functions, Appl. Math. Comput. 159, (2004) 589-602 .

[3] E. Özergin, M.A. Özaraslan, A. Altin, Extension of gamma, beta and hypergeometric functions, J. Comput. Appl. Math. 235, 16 (2011) 4601-4610 .

[4] M. Bozer, M.A.Özaraslan, Notes on Generalized Gamma, Beta and Hypergeometric functions. J. Computational Analysis and Application, Vol.15.No.7 (2013) 1194-1201 .

[5] M.A.Chaudhry, S.M.Zubair, Generalized incomplete gamma functions with applications. J.Comput.Appl.Math. 55(1994) 99-124.

[6] B.Fisher, A. Kilicma,Some results on the gamma function for negative integers, App. Math. Inf. Sci. 6, No.2 (2012) 173-176.

[7] M. A. Özaraslan, Some remarks on extended hypergeometric, extended confluent hypergeometric and extended appell's functions, J. of Comput. Anal. and Applic., 14, 6(2012) 1148-1153.

[8] M.A.Chaudhry, A. Qadir, M.Rafique, S.M. Zubair, Extension of Euler's beta function, J. Comput. Appl. Math. (1997) 78, 19-32.

[9] M.E. Ismail, Classical and Quantum Orthogonal Polynomials in one Variable, Cambridge University press (2005).

[10] E.H.Doha, S.I. El-Soubhy ,On the Legender coefficients of the moments of the general order derivative of an infinitely differentiable function, Inter. J. Comput. Math. 56, No.1 & 2, (1995)107-122.

[11] E.H. Doha, H.M. Ahmed, S.I. El-Soubhy, Explicit formulae for the coefficients of integrated expansions of Laguerre and Hermite polynomials and their integrals, Integral Transforms and special functions, 20, No.7 (2009) 491-503.