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Solution to a Certain Non-Linear Black-Scholes Option Pricing Model Via the Riesz Representation Theorem

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Abstract

In this paper, a certain non-linear Black-Scholes equation which incorporates both transaction cost and volatile portfolio risk is obtained. A solution in Sobolev space via the Riesz representation theorem is proffered. Existence of the weak solution is established.

1. Introduction

Price in different markets tends to converge and this is due to the effect of arbitrage. The concept of arbitrage on some levels is analogous to the principle of “the invisible hand” of the market. Black and Scholes ([1], [2]) derived a singular equation that has moved the frontier of financial world to hitherto unimaginable length. In their derivation of this equation some basic assumptions were made: one such assumption is that there are no transaction costs in hedging. Since then there have been several and successful remodeling of the Black –Scholes equation that takes into account transaction costs. Accordingly, when transactions cost – directly proportional to trading- is incorporated in the Black-Scholes model the resulting hedging portfolio is prohibitively expensive. It is therefore acceptable that in the continuous-time model with transaction costs, there is no portfolio that can replicate the European call option transaction costs. To precede, the condition under which hedging can take place has to be relaxed such that the portfolio only dominates rather than replicates the value of the European call option at maturity. With this relaxation, there is always the trivial dominating hedging strategy of buying and holding one share of the stock on which the call is written. From arbitrage pricing theory, the price of an option should not be greater than the smallest initial capital that can support a dominating portfolio. Interesting results have evolved from this line of approach to pricing option without transaction cost, however, in the presence of constraints, in the presence of transaction costs, Soner et al [3] proved that the minimal hedging portfolio that dominates a European call option is the trivial one. In essence this suggests another way or technique to relaxing perfect hedging in models with transaction costs. Leland [4] used a relaxation with the effect that his model allowed transactions only at discrete times. By a formal δ - hedging argument, one can obtain a generalized option price that is equal to a Black- Scholes price but with an adjusted volatility of the form;

$$\sigma^2 = \hat{\sigma}^2\left(1 - LeSgn(\partial_S^2 V)\right),$$

where $\sigma > 0$ is a constant historical volatility, $Le = \sqrt{\frac{2}{\pi}}\frac{C}{\hat{\sigma}^2\sqrt{\Delta t}}$ is the Leland number and Δt is time lag.

Assuming that inventor's preferences are characterized by an exponential utility function, Barles and Soner[5] derived a nonlinear Black- Scholes equation with volatility $\sigma = \sigma(\partial_S^2 V, S, t)$ given by

$$\sigma^2 = \hat{\sigma}^2\left(1 + \varphi\left(a^2 e^{r(T-t)} S^2 \partial_S^2 V\right)\right),^2$$

where $\varphi(X) \approx \left(3/_2\right)^{2/_3} X^{1/_3}$ for close to the origin and $\hat{\sigma}^2$ is a constant.

Market models with transaction cost have been extensively dealt with (see for example Amster, et al [6], Avellanda and Paras [7]). A solution in Sobolev by stochastic iteration method for nonlinear Black-Scholes equation with transaction cost and volatile portfolio risk measure in Hilbert space had been obtained (see Osu and Olunkwa [8]). In a related paper, the solution of a nonlinear Black-Scholes equation with the Crank-Nicholson scheme had also been obtained (see Mawah [9] and the references therein). The objective of this paper is to further incorporate volatile portfolio risk and show that solution by RieszRepresentationTheorySobolev space subject to some boundary conditions is possible.

2. The Model

Transaction costs as well as the volatile portfolio risk depend on the time–lag between two consecutive transactions. Minimizing their sum yields the optimal length of the hedge interval –time lag (for numerical example, see references in [8]). This leads to a fully nonlinear parabolic PDE. If transaction costs are taken into account perfect replication of the contingent claim is no longer possible. Modeling the short rate $r = r(t)$ by a solution to a one factor stochastic differential equation,

$$dS = \mu(s,t)dt + \sigma(s,t)dw \qquad (2.1)$$

where $\mu(S,t)dt$ represent a trend or drift of the process and $\sigma(S,t)$ represents volatility part of the process, the risk adjusted Black-Scholes equation can be viewed as an equation with a variable volatility coefficient

$$\partial_t V + \frac{\sigma^2(s,t)}{2} S^2\left(1 - \mu(S\partial_S V)^{\frac{1}{3}}\right)\partial_S^2 V + rs\partial_S V - rV = 0, \qquad (2.2)$$

where $\sigma^2(s,t)$ depends on a solution $V = V(s,t)$ and $\mu = 3\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}}$, since

$$\hat{\sigma}^2(s,t) = \sigma^2(1 - \mu(S\partial_S^2 V(S,t))^{\frac{1}{3}}.$$

Incorporating both transaction costs and risk arising from a volatile portfolio into equation (2.2) we have the change in the value of portfolio to become.

$$\partial_t V + \frac{\hat{\sigma}^2(s,t)}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = (r_{TC} + r_{VP})S, \qquad (2.3)$$

where $r_{TC} = \frac{C|\Gamma|\hat{\sigma}S}{\sqrt{2\pi}}\frac{1}{\sqrt{\Delta t}}$ is the transaction costs measure, $r_{Vp} = \frac{1}{2}R\hat{\sigma}^4 S^2 \Gamma^2 \Delta t$ is the volatile portfolio risk measure and $\Gamma = \partial_S^2 V$. Minimizing the total risk with respect to the time lag Δt yields;

$$\min_{\Delta t}(r_{TC} + r_{VP}) = \frac{3}{2}\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{3}} \hat{\sigma}^2 |S\partial_S^2 V|^{\frac{4}{3}}.$$

For simplicity of solution and without loss of generality, we choose the minimized risk as

$$\{min_{\Delta t}(r_{TC} + r_{VP})\}^{\frac{3}{2}} = As^2 \partial_s^2 v, \qquad (2.4a)$$

with

$$A = \left(\frac{3}{2}\right)^{\frac{3}{2}}\left(\frac{C^2 R}{2\pi}\right)^{\frac{1}{2}} \hat{\sigma}^3. \qquad (2.4b)$$

They change in the value of the portfolio after minimizing the total risk with respect to time lag is given as

$$\partial_t V + \frac{\sigma^2(1-\mu(S\partial_S^2 v(s,t))^{\frac{1}{3}}}{2} S^2 \partial_S^2 V + rS\partial_S V - rV = As^2\partial_s^2 v, \qquad (2.5)$$

which can also be written as

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}(1` - \mu(S^2\partial_S^2 V(S,t))^{\frac{1}{3}} s^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = As^2\frac{\partial^2 V}{\partial S^2}.$$

Equation 2.5 is the Black-Scholes option pricing model that incorporates both transaction cost and volatile portfolio risk measure.

The left hand of equation (2.5) is the usually Black-Sholes formula. Setting

$$S = e^x, V(x,y) = u(e^x, t) and h(e^x) = g(x)$$

we have equation (2.5) becoming;

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}(1` - \mu(S^2\partial_S^2 U(S,t))^{\frac{1}{3}} s^2 \frac{\partial^2 u}{\partial x^2} + r\frac{\partial u}{\partial x} - ru = A\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right),$$

which implies;

$$\frac{\partial u}{\partial t} + \frac{\sigma^2}{2}(1` - \mu(S^2\partial_S^2 u(e^x,x))^{\frac{1}{3}}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right) + r\frac{\partial u}{\partial x} - ru = A\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right). \qquad (2.6)$$

Let $k = \frac{\sigma^2}{2}(1` - \mu(S^2\partial_S^2 u(e^x,x))^{\frac{1}{3}}$, then equation(2.6) reduces to

$$\frac{\partial u}{\partial t} + k\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right) + r\frac{\partial u}{\partial x} - ru = A\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right), \qquad (2.7)$$

or

$$\frac{\partial U}{\partial t} + k\frac{\partial^2 U}{\partial x^2} + (r-k)\frac{\partial U}{\partial x} - rU = A\left|\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial x}\right|, \qquad (2.8)$$

which is equivalent to that in (Mariamaetal[11]).

We further assume that there is no accumulated interest on the portfolio. Hence $r = 0$ and the new portfolio becomes

$$-\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + k\frac{\partial u}{\partial x} = -A\left|\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right|$$

or

$$-\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + k\frac{\partial u}{\partial x} = \bar{A}\left|\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right| \quad (2.9)$$

with the initial condition

$$U(x,0) = \max(1 - e^{-x}, 0)$$

$$-A = \bar{A}.$$

Our interest in this paper is to show that a solution of the equation

$$-\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} + k\frac{\partial u}{\partial x} = \bar{A}F\left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)\Omega \times (0,T) \quad (2.10)$$

$$u(x,0) = u_0(x) on x \in \Omega \quad (2.11)$$

Is possible using the RieszRepresentation theorem of Sobolev space.

3. The Sobolev Space

We are considering functions for which all the derivatives, in distribution sense belongs to L^2. Let $\Omega \subset \mathbb{R}^d$ and let u be a function of L^2 .It can be identified to a distribution on Ω as a function of $L^1_{loc}(\Omega)$, also denoted as u and we can define its derivative ∂u as distribution on Ω, since equation (2.10) is not an element of $L^2(\Omega)$. Hence, we introduce the Sobolev space.the easiest way to determine a function is in a Sobolev space is to explicitly compute the function's weak derivative. The study of Sobolev Spaces revolves around the concept of the weak derivative. Hence, we begin by setting up notation for this derivative and motivating its definition.

Definition 3.1: A multi-index α is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$, used to concisely denote the partial differential operator

$$D^\alpha(u) = \frac{d^\alpha}{dx_1^{\alpha_1} \ldots dx_n^{\alpha_n}}(u).$$

We define $|\alpha| = \alpha_1 + \cdots + \alpha_n$ to be the degree of α.

Definition 3.2: For two multi-indices, α, βwe define the following :

1) $\alpha \leq \beta if \alpha_i \leq \beta_i, for all\ 1 \leq i \leq n$
2) If $\alpha \leq \beta$,we define $\alpha - \beta = \gamma, where \gamma = (\alpha_i - \beta_i, \ldots, \alpha_n - \beta_n)$
3) $\alpha! = \alpha_1!\, \alpha_2! \ldots \alpha_n!$

Let Ω be a bounded open subset of $\mathbb{R}^n$. The weak derivative for some function $u \in C^1(\Omega)$ is defined by integrating against an arbitrary $\emptyset \in C_c^\infty(\Omega)$. Because our function u has continuous derivatives, we can integrate by parts to obtain the following:

$$\int_\Omega u\frac{\partial \emptyset}{\partial x_i}dx = u\emptyset \mathrm{I}_{\partial\Omega} - \int_\Omega \frac{\partial u}{\partial x_i}\emptyset dx = -\int_\Omega \frac{\partial u}{\partial x_i}\emptyset dx$$

($because\ \emptyset\ vanisheson\ the\ bound$ ary)

We see immediately that by repeated integration by parts we can generalize this result to a partial differential operator α of arbitrary degree, so long as we take $u \in C^{|\alpha|}(\Omega)$ and account for the parity dependence of the minus sign:

$$\int_\Omega uD^\alpha \emptyset dx = (-1)^{|\alpha|}\int_\Omega D^\alpha u\emptyset dx. \quad (3.1)$$

If $u \in C^{|\alpha|}(\Omega)$, the above formula is valid for every $\emptyset \in C_c^\infty(\Omega)$. The notion of the weak derivative asks if this formula is valid when u is not in $C^{|\alpha|}(\Omega)$. We insists that u be locally integrable function (that is, it be integrable on compact sets), because otherwise the left hand side of the above equality is meaningless. The right hand side of the equality poses an even bigger problem .How can we define $D^\alpha u$ if u is not in $C^{|\alpha|}(\Omega)$? We can no longer use the traditional analytic tools to derive the correct solution to this equation ,so instead we use this apparent ambiguity to define the weak derivative .In the definition ,$D^\alpha u$ is replaced by $v \in L^p(\Omega)$ and we say v satisfying the equality for every $\emptyset \in C_c^\infty(\Omega)$ is the weak derivative of u.

Definition3.3: Let $1 \leq p \leq \infty$ and k be a nonnegative integer.The sobolev Space $W^{k,p}(\Omega)$ consists of all function $u: \Omega \to \mathbb{R}$, $u \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)$ such that each weak derivative $D^\alpha u$ with $|\alpha| \leq k$ exists and belong to L^p.That is,

$$W^{k,p}(\Omega) = \left\{u \in \mathrm{L}^p_{\mathrm{loc}}(\Omega) \middle| \|D^\alpha u\|_{L^p(\Omega)} < \infty, \forall\ |\alpha| \leq k\right\} \quad (3.2)$$

Similarly, the space $\mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega)$ consists of the function u as above for which $D^\alpha u$ with $|\alpha| \leq k$ exists and belong to $\mathrm{L}^p_{\mathrm{loc}}(\mathrm{V})$, where V is an arbitrary compact subset of Ω.

Lemma 3.1: Given $u \in W^{k,p}(\Omega)$ and $\in C_c^\infty(\Omega)$, $uv \in W^{k,p}(\Omega)$.

Proof: Unfortunately, the easiest way to determine a function is in Sobolev space is to explicitly compute the function's weak derivative. To this end, we guess that the derivative will have the following form as a basis for induction on $|\alpha|$:

$$D^\alpha(uv) = \sum_{\beta \leq \alpha}\binom{\alpha}{\beta}D^\beta v D^{\alpha-\beta}u.$$

Fix $\emptyset \in C_c^\infty(\Omega)$ .Note that because $\emptyset\ and\ v$ are already differentiable ,their weak derivative necessarily obeys the product rule for classical derivatives .So if $|\alpha| = 1$,

$$\int_\Omega uv\, D^\alpha(\emptyset)dx = \int_\Omega uD^\alpha(\emptyset) - uD^\alpha(v)\emptyset\, dx$$

$$= -\int_\Omega \left(D^\alpha(u)v + uD^\alpha(v)\right)\emptyset\, dx,$$

where we have used integration by parts and the fact that $v\emptyset$ is still compactly supportd only on Ω to obtain the final equality. We know that $D^{\alpha}u$ exists (as $u \in W^{k,p}(\Omega)$ and $D^{\alpha}(v) exists\ (as\ v \in C_c^{\infty}(\Omega))$. Thus, the result holds in the base case .Assume $l < k$ and that the result holds for all $|\alpha| \leq l$and all functions $v \in C_c^{\infty}(\Omega). Choose$ a multiindex α with $|\alpha| \leq l+1$.Then, we can decompose $|\alpha|$ in terms of some other multiindicies β, γ satisfying $\alpha = \beta + \gamma$ with $|\beta| = l, |\gamma| = 1$. Then, for any $v, \emptyset \in C_c^{\infty}$,

$$\int_{\Omega} vu\, D^{\alpha}(\emptyset)dx = \int_{\Omega} uv\, D^{\beta}(D^{\gamma}\emptyset)dx = (-1)^{|\beta|}\int_{\Omega}\sum_{\sigma\leq\beta}\binom{\beta}{\sigma}D^{\sigma}vD^{\beta-\sigma}uD^{\gamma}\emptyset dx$$

$$= (-1)^{|\beta|+|\gamma|}\int_{\Omega}\sum_{\sigma\leq\gamma}\binom{\beta}{\sigma}D^{\gamma}\left(D^{\sigma}vD^{\beta-\sigma}u\right)\emptyset dx$$

$$= (-1)^{|\alpha|}\int_{\Omega}\sum_{\sigma\leq\alpha}\binom{\alpha}{\sigma}D^{\sigma}vD^{\alpha-\sigma}u\emptyset\, dx$$

which is the desire result.

In the following theorem, we characterize the dual of $W^{k,p}(\Omega)$, for all $1 \leq p < \infty$. Though this process is difficulty for many function space, clever application of the Riesz Representation Theorem and use of functional analysis make the following characterization quite direct.

Theorem 3.1: (Riesz's Representation Theorem in $W^{1,p}(\Omega)$) Let $1 \leq p < \infty$ and let $p' = \frac{p}{p-1}$. Then every $L \in \left(W^{1,p}(\Omega)\right)^*$ can be characterized in the following way: There exist $f_0, \ldots, f_N \in L^{p'}$ such that

$$L(u) = \int_{\Omega} f_0 u + \sum_{i=1}^{N}\frac{\partial u}{\partial x_i}f_i\, dx$$

for all $u \in W^{1,p}$ and

$$\|L\|_{(W^{1,p})^*} = \left(\sum_{i=0}^{\infty}\|f_i\|_{L^{p'}(\Omega)}^{p'}\right)^{\frac{1}{p'}}$$

Proof: Define a function

$$T: W^{1,p} \to L^p(\Omega, \mathbb{R}^{N+1})$$

$$u \mapsto \left(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right)$$

Because the first coordinate of the image is exactly the corresponding element of the domain,we see that this map is injective and that the pre-image of an open set will be open,so that the function is continuous. Also,

$$\|T(u)\|_{L^p(\Omega,\mathbb{R}^{N+1})} = \left(\int_{\mathbb{R}^{N+1}} |u|^p + \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^p dx\right)^{\frac{1}{p}}$$

For all $u \in W^{1,p}$, so T is norm preserving .We know that $W^{1,p}$ is a Banach space,so it must be closed.Hence,by the norm preserving property, $Y = T\left(W^{1,p}(\Omega)\right)$ must be closed in $L^p(\Omega, \mathbb{R}^{N+1})$. Given a linear functional $L \in W^{1,p}$, we now define a linear functional on Y.

$$L_1: Y \to \mathbb{R}$$

$$g = (g_o, \ldots g_N) \mapsto L\left(T^{-1}(g_o, \ldots g_N)\right)$$

Now T^{-1} is obviously linear, and L is linear by definition, so as a composition of linear maps L_1 is linear and because T^{-1} and L are bounded, their composition L_1 must be bounded and thus is continuous (a linear functional is bounded if and only if it is continous). Finally, because T preserves the norm, we have that:

$$\|L_1\|_{Y'} = \|L\|_{(W^{1,p})^*}$$

We are now free to apply the Hahn-Banach theorem and extend L_1 as a continous linear operator to:

$$\widetilde{L_1} = L^p(\Omega, \mathbb{R}^{N+1}) \to \mathbb{R}$$

Satisfying

$$\left\|\widetilde{L_1}\right\|_{L^p(\Omega,\mathbb{R}^{N+1})} = \|L_1\|_{Y'} = \|L\|_{(W^{1,p})^*}$$

We now apply the Riesz Representation Theorem in $L^p(\Omega, \mathbb{R}^{N+1})$ and conclude that there exist functions $f_0, \ldots, f_N \in L^p(\Omega)$ such that

$$L_1(g) = \int_{\Omega} f_0(x)g_0(x) + \sum_{i=1}^{N} f_i(x)g_i(x)dx$$

For all $g \in L^p(\Omega, \mathbb{R}^{N+1})$ and that

$$\|L\|_{\left(W^{1,p}(\Omega)\right)^*} = \left\|\widetilde{L_1}\right\|_{L^p(\Omega,\mathbb{R}^{N+1})} = \left(\sum_{i=0}^{N+1}\|f_i\|_{L^{p'}(\Omega)}^{p}\right)^{\frac{1}{p'}}$$

By identifying the above functions $g_o, \ldots, g_N$ with an $N+1-tuple$ of the form $(u, \frac{\partial u}{\partial x_i}, \ldots, \frac{\partial u}{\partial x_N})$,we obtain the desired result:

$$L(u) = \int f_0(x)u(x) + \sum_{i=1}^{N} f_i(x)\frac{\partial u}{\partial x_i}dx.$$

Theorem 3.2: (Lax-Miolgram) Let H be a Hilbert space and $B: H \times H \to \mathbb{R}$ be a bilinear mapping satisfying the following inequalities:

1) There exists $\alpha > 0$ such that $|B[u,v]| \le \alpha\|u\|\|v\|$ for all $u, v \in H$,
2) There exists $\beta > 0$ such that $\beta\|u\|^2 \le B[u,u]$ for all $u \in H$.

Then if $f: H \to \mathbb{R}$ is a bounded linear functional on H,there exists a unique element $u \in H$ such that

$$B[u,v] = \langle f, v\rangle$$

For all $v \in H$.

Note that we cannot directly apply the Riesz Representation Theorem because we do not know that our bilinear form is symmetric (ie it is possible that $B[u,v] \neq B[v,u]$).

Proof .By the assumed inequalities, we see that for a fixed element $u \in H$, the mapping $v \mapsto B[u,v]$ is a bounded linear functional on H .Here we can directly apply the Riesz Representation Theorem to fine an element $u \in H$ such that

$$B[u,v] = (w,v) forv \in H.$$

We define a mapping $A: H \to H$ by $u \mapsto w$, where w fits the above definition.This allows us to write

$$B[u,v] = (Au, v)(v \in H).$$

We claim that A is a bounded linear operator .It is easy to see that the operator is linear: take $\lambda \in \mathbb{R}, t, u, v \in H$. Then

$$(A(\lambda t + u), v) = B[\lambda t + u, v] = \lambda B[t,v] + B[u,v]$$

$$= (\lambda At, v) + (Au, v) = (\lambda At + Au, v),$$

Proving linearity. We can apply the first inequality required of bilinear form B to prove boundedness:

$$\|Au\|^2 = (Au, Au) = B[u, Au] \le \alpha\|u\|\|Au\|$$

So, $\|Au\| \le \alpha\|u\|$ for all $u \in H$, and so by definition A is bounded. We now apply the other assumed inequality to show that A is injective and has a closed range in H .This second criterion will be used later to show that our function is surjective .Directly applying the second assumed inequality,

$$\beta\|u\|^2 \le B[u,u] = (Au, u) \le \|Au\|\|u\|.$$

Hence, $\beta\|u\| \le \|Au\|$. If A sent some $u \neq 0\ to\ 0$, this inequality would be contradicted. This is enough to show injectivity .To show that the range is closed, choose some convergent sequence $\{Au_n\} \to w$. Now, $\beta\|u_n - u_m\| \le \|Au_n - Au_m\| \to 0$. So $\{u_n\}$ is a convergent sequence in H so $u_m \to u$ in H and thus by continuity, $\lim_{n\to\infty} Au_n = w$. Hence, the range of A is closed in H because $Au = w$.

We now show that the range of A, denoted $R(A)$ is in fact all of H and hence that our Map A is a bijection.We Know that the range of A is closed.Suppose that the range is not all of H . Then there exists a nonzero element $w \in H$ with $w \in R(A)^{\perp}$. But this implies that

$$\beta\|w\|^2 \le B[w,w] = (Aw, w) = 0,$$

which is a contradiction.

Again we can apply the Riesz Representation Theorem to find a $u \in H$ such that

$$\langle f, v\rangle = (w, v)$$

For all $v \in w$.But then because A is bijective, we can find $u \in H$ such that $Au = w$.this gives

$$B[u,v] = (Au, v) = (w,v) = \langle f, v\rangle.$$

This is exactly what we needed to show.All that is left is uniqueness and we prove this with a typical contradiction argument. Suppose there exist $t, u \in H$ satisfying the above equation. Then

$$B[u,v] = \langle f, v\rangle = B[t,v].$$

We conclude that $B[u-t, v] = 0$, for all $v \in H$. Setting $v = u - t$ we see that

$$\beta\|u-t\|^2 \le B[u-t, u-t] = 0.$$

We defined a weak solution of equation (2.10) in an analogous way to our formulation of weak derivatives.That is we integrate equation (2.10) against $v \in C_c^\infty(\Omega)$ and determine for which the function u the resulting integral equality holds.That is,we wish to fine a function u satisfying ,

$$\int_\Omega \left(\frac{\partial u}{\partial t} v + k\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - ku\frac{\partial u}{\partial x}\right) dx = -\bar{A}\int_\Omega F\left(u, \frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial x} dx\, forallv \in C_c^\infty(\Omega),$$

(where $u = \frac{\partial u}{\partial x}$)

By density of $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$, choosing $v \in H_0^1(\Omega)$ in the above process will yield the same result.

Definition 3.4: The bilinear form $B[.,]$ associated with a divergence form linear operator L is defined by

$$B[u,v] = \int_\Omega \left(\frac{\partial u}{\partial t} v + k\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - kv\frac{\partial u}{\partial x}\right) dx$$

Definition 3.5: We say that $u \in H_0^1(\Omega)$ is a weak solution of equation 2.10

$$-\frac{\partial u}{\partial t} - k\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + ku\frac{\partial u}{\partial x} = \bar{A}\frac{\partial}{\partial x}F\left(u, \frac{\partial u}{\partial x}\right)(x,t) \in \Omega \times (0,T)$$

$$u(x,0) = u_0(x)$$

If

$$B[u,v] = (f,v)$$

for all $v \in H_0^1(\Omega)$, where (.,.) denotes the inner product in $L^2(\Omega)$.

4. Main Result

Theorem 4.1; Let $u, v \in H_0^1(\Omega)$, then there exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

$(1)|B[u,v]| \leq \alpha\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}, and$,

$(2)\|u\|^2_{H_0^1(\Omega)} \leq B[u,u]$

Proof: The first inequality is given by direct computation, although we do need a bit of creativity

$$|B[u,v]| = \int_\Omega \left(\frac{\partial u}{\partial t}v + k\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - ku\frac{\partial v}{\partial x}\right)dx = -\bar{A}\int_\Omega F\left(u,\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial x}dx. \tag{4.1}$$

By (4.1) we have

$$\int_\Omega \frac{\partial u}{\partial t}vdx + \int_\Omega \left(k\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + ku\frac{\partial v}{\partial x}\right)dx = -\bar{A}\int_\Omega F\left(u,\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial x}dx.$$

Thus

$$\left|\int_\Omega \frac{\partial u}{\partial t}vdx\right| \leq \left|\int_\Omega \left(k\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + ku\frac{\partial v}{\partial x}\right)dx\right| + -\bar{A}\left|\int_\Omega F\left(u,\frac{\partial u}{\partial x}\right)\frac{\partial v}{\partial x}dx\right|.$$

By Holder inequality,

$$\left|\int_\Omega \frac{\partial u}{\partial t}v\,dx\right| \leq k\left(\int_\Omega \left|\frac{\partial u}{\partial x}\right|^2 dx\right)^{1/2}\left(\int_\Omega \left|\frac{\partial v}{\partial x}\right|^2 dx\right)^{1/2} + k\left(\int_\Omega |u|^2\,dx\right)^{1/2}\left(\int_\Omega \left|\frac{\partial v}{\partial x}\right|^2 dx\right)^{1/2} + \left(\int_\Omega \left|F\left(u,\frac{\partial u}{\partial x}\right)\right|^2 dx\right)^{1/2}\left(\int_\Omega \left|\frac{\partial v}{\partial x}\right|^2 dx\right)^{1/2}$$

Since $\|v\|_{H_0^1(\Omega)} \leq 1$, using the assumption

$$F(p,q) \leq |p| + |q|$$

And the Poincare's inequality we deduce

$$\left|\int_\Omega \frac{\partial u}{\partial t}vdx\right| \leq \alpha\|u(t)\|_{H_0^1(\Omega)}$$

where α is a constant.so

$$\left\|\frac{\partial u}{\partial t}(t)\right\|_{H^{-1}(\Omega)} \leq \alpha\|u(t)\|_{H_0^1(\Omega)}$$

Therefore,

$$\int_\Omega \left\|\frac{\partial u}{\partial t}(t)\right\|_{H^{-1}(\Omega)} dt \leq \alpha\int_\Omega \|u(t)\|_{H_0^1(\Omega)}\,dt = \alpha\|u(t)\|^2_{L^2(\Omega)}$$

Then this implies

$$\|u\|_{L^2(\Omega)} \leq \alpha\|u(t)\|_{L^2(\Omega)}$$

From this it is easy to see that

$$\beta\|u\|^2_{H_0^1} \leq B[u,u] + \gamma\|u(t)\|^2_{L^2(\Omega)}.$$

While this is a nice result, if$\gamma \neq 0$ we do not satisfy the hypothesis of the Lax- Milgram Theorem.In the following proof of the existence theorem, we will have to manufacture a new bilinear form which does satisfy the hypothesis.

Theorem 4.2 (First Existence Theorem for weak solutions): Let $Lu = -\frac{\partial u}{\partial t} - k\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + ku\frac{\partial u}{\partial x}$ and $f = \bar{A}\frac{\partial}{\partial x}F\left(u,\frac{\partial u}{\partial x}\right)$, there exists a number $\gamma \geq 0$ such that for each $\mu \geq \gamma$ and each function $f \in L^2(\Omega)$ there exists a unique weak solution $u \in H_0^1(\Omega)$ of the boundary –value problem:

$$\begin{cases} Lu + \mu u = f in\ \Omega \\ u = 0\ on \partial\Omega \end{cases}.$$

Proof: Choose γ to be the γ from the second inequality of the energy estimates theorem, and choose any $\mu \geq \gamma$. We defined the following bilinear form.

$$B_\mu = B[u,v] + \mu(u,v), foru, v \in H_0^1(\Omega),$$

Where $(.,.)$ denotes the inner product in $L^2(\Omega)$.We see that this bilinear form corresponds to the operator $L_\mu u = Lu + \mu u$ and that $B_\mu = [.,.]$ satisfies the hypothesis of the Lax-Milgram Theorem.Now fix $f \in L^2(\Omega)$ and set $\langle f, v\rangle = (f,v)_{L^2(\Omega)}$. This is a bounded linear functional on $L^2(\Omega)$ (because the inner product is continuous and linear), and hence is a bounded linear functional on $H_0^1(\Omega)$ as well. We apply the Lax-Milgram Theorem and fine a unique $u \in H_0^1(\Omega)$ satisfying

$$B_\mu = [u,v] = \langle f, v\rangle$$

For all $v \in H_0^1(\Omega)$. By definition u is a weak solution of equation 2.10.

5. Conclusion

The Sobolev spaces appear naturally in the solution of problem (2.10) in the sense that $C_c^\infty(\Omega)$ can be extended by continuity to $H_0^1(\Omega)$. We characterize the dual of $W^{k,p}(\Omega)$, for all $1 \leq p < \infty$. Though this process is difficult for many function space. The Riesz Representation Theorem in Sobolev space was established and we were able to find the weak solution of equation 2.10 using theRiesz representation theorem in Sobolev space.Existence of the weak solution is established.

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