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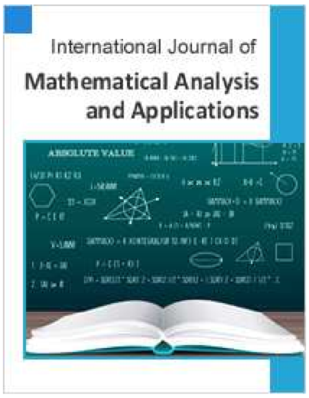

Exponentiated Nadarajah and Haghighi's exponential Distribution

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Abstract
In this paper, we introduce a new distribution called exponentiated Nadarajah and Haghighi's exponential distribution (ENHED). Some properties of this distribution will be discussed. The maximum likelihood estimation of the unknown parameters is discussed. Real data show that the new distribution can be used quite effectively to provide better than the Nadarajah and Haghighi's exponential distribution (NHED).

Keywords
Nadarajah and Haghighi's Exponential Distribution, Exponential Distribution, Exponentiated Distributions, Traditional Moments, Maximum Likelihood Estimation

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1. Introduction

Exponentiated distributions can be obtained by powering a positive real number λ to the cumulative distribution function (CDF). i.e, if we have CDF $F(x)$ of any random variable *X*, then the function

$$G(z) = \left[F(z)\right]^{\lambda}, (\lambda > 0), \tag{1}$$

is called an exponentiated distribution.

The probability density function (*pdf*) is given by

$$g(z) = \frac{dG(z)}{dz} = \lambda\left[F(z)\right]^{\lambda-1} f(z) \tag{2}$$

Many families of exponentiated distributions are proposed. The exponentiated Weibull (EW) was proposed in ref. [8]. The exponentiated exponential distribution was introduced in ref. [7]. Ref. [9] presented an exponentiated gumble distribution (EGD). The exponentiated Lomax distribution was proposed in ref. [1]. Ref. [5] introduce Exponentiated Transmuted Weibull Distribu-tion.

A random variable X is said to have a Nadarajah and Haghighi's exponential distribution (NHED) if its probability density function (*pdf*) is in the form:

$$f(x;\alpha,\beta) = \alpha\beta(1+\alpha x)^{\beta-1} e^{1-(1+\alpha x)^{\beta}}, \quad x > 0, \ (\alpha,\beta > 0). \tag{3}$$

The cumulative distribution function (CDF) and survival function (SF) are:

$$F(x;\alpha,\beta) = 1 - e^{1-(1+\alpha x)^{\beta}}, x > 0, \ (\alpha,\beta > 0), \tag{4}$$

and

$$\bar{F}(x;\alpha,\beta)=e^{1-(1+\alpha x)^{\beta}},\quad x>0,\ (\alpha,\beta>0). \qquad (5)$$

More details on this distribution and its applications can be found in ref. [10].

2. Exponentiated Nadarajah and Haghighi's Exponential Distribution

Using (1) and (4), we can define the CDF of ENHED as follows

$$G(x)=\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\lambda},\quad x\geq 0,\ (\alpha,\beta \text{ and } \lambda>0) \qquad (6)$$

The pdf of ENHED is

$$g(x)=\lambda\alpha\beta(1+\alpha x)^{\beta-1}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\lambda-1}e^{1-(1+\alpha x)^{\beta}},\quad x\geq 0,\ (\alpha,\beta \text{ and } \lambda>0) \qquad (7)$$

We can get the pdf for exponentiated exponential (EE), Nadarajah and Haghighi's exponential (NHE) and exponential (E) distributions by taking $\beta=1$, $\lambda=1$ and $\beta=\lambda=1$ respectively. The survival (reliability) function $S(x)$, the hazard rate function (HRF) $h_{\lambda}(x)$ and the reversed hazard rate function (RHRF) $h_{\lambda}^{*}(x)$ for ENHED are in the following forms:

$$S(x)=1-\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\lambda},\quad x\geq 0,\ (\alpha,\beta \text{ and } \lambda>0), \qquad (8)$$

$$h_{\lambda}(x)=\frac{\lambda\alpha\beta(1+\alpha x)^{\beta-1}\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\lambda-1}e^{1-(1+\alpha x)^{\beta}}}{1-\left(1-e^{1-(1+\alpha x)^{\beta}}\right)^{\lambda}},\quad x\geq 0,\ (\alpha,\beta \text{ and } \lambda>0) \qquad (9)$$

and

$$h_{\lambda}^{*}(x)=\frac{\lambda\alpha\beta(1+\alpha x)^{\beta-1}e^{1-(1+\alpha x)^{\beta}}}{1-e^{1-(1+\alpha x)^{\beta}}}=\lambda h^{*}(x),\quad x\geq 0,\ (\alpha,\beta \text{ and } \lambda>0), \qquad (10)$$

where $h^{*}(x)$ is the RHRF for the NHED.

The mode of ENHED is the root of the following equation:

$$\left[\beta\lambda(1+\alpha x)^{\beta}-(\beta-1)\right]e^{1-(1+\alpha x)^{\beta}}=\left[\beta(1+\alpha x)^{\beta}-(\beta-1)\right] \qquad (11)$$

The median of ENHED is

$$x_{0.5}=\alpha^{-1}\left\{\left[1-\ln\left(1-2^{\frac{-1}{\lambda}}\right)\right]^{\frac{1}{\beta}}-1\right\}. \qquad (12)$$

The formula to generate ENHED is

$$X_{ENHED}=\alpha^{-1}\left\{\left[1-\ln\left(1-U^{\frac{1}{\lambda}}\right)\right]^{\frac{1}{\beta}}-1\right\}, \qquad (13)$$

where U is uniformly distributed on $(0,\ 1)$.

In Figures 1, 2 and 3, we plot the density, cumulative and failure rate functions of the ENHED for selected parameter values, respectively.

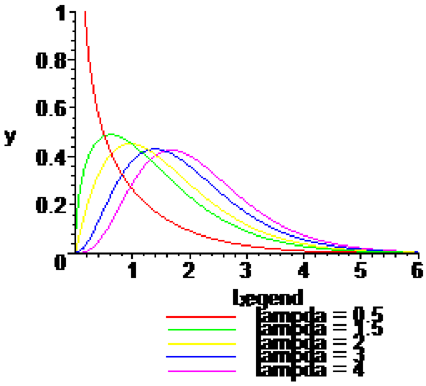

Figure 1. *Plot of the density function at* $\alpha = 0.5$ *and* $\beta = 1.5$ *with different values of* λ.

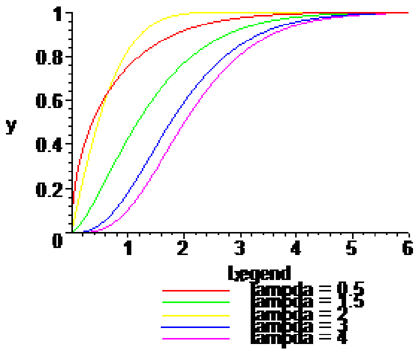

Figure 2. *Plot of the cumulative function at* $\alpha = 0.5$ *and* $\beta = 1.5$ *with different values of* λ.

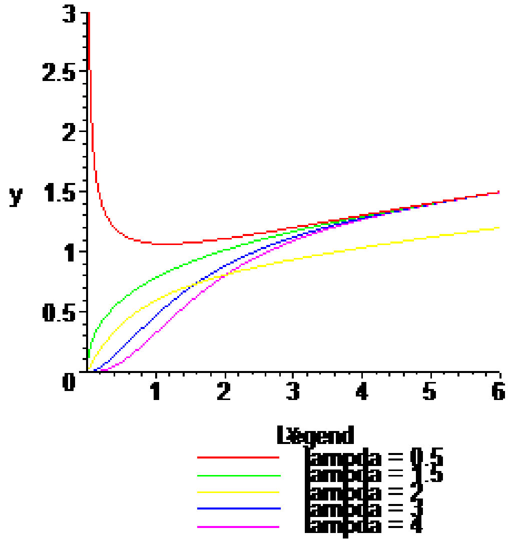

Figure 3. *Plot of the failure rate function at* $\alpha = 0.5$ *and* $\beta = 1.5$ *with different values of* λ.

3. Traditional Moments for ENHED

Here we obtain $E(1+\alpha X)^r$, and hence we can obtain the mean and the variance for ENHED as the following

$$E(1+\alpha X)^r = \lambda\alpha\beta\int_0^\infty (1+\alpha x)^{r+\beta-1}\left(1-e^{1-(1+\alpha x)^\beta}\right)^{\lambda-1} e^{1-(1+\alpha x)^\beta} dx$$

Using the transformation $(1+\alpha x)^\beta = y+1$, we get

$$E(1+\alpha X)^r = \lambda\int_0^\infty (y+1)^{\frac{r}{\beta}}\left(1-e^{-y}\right)^{\lambda-1} e^{-y} dy \;\; = \lambda\sum_{j=0}^\infty \binom{\lambda-1}{j}(-1)^j \int_0^\infty (y+1)^{\frac{r}{\beta}} e^{-(j+1)y} dy$$

$$= \lambda\sum_{j=0}^\infty \binom{\lambda-1}{j}(-1)^j (j+1)^{-\left(\frac{r+\beta}{\beta}\right)} e^{j+1}\Gamma\left(\frac{r+\beta}{\beta}, j+1\right) \tag{14}$$

To obtain the mean of ENHED, we set $r=1$ in (14) as follows

$$E(X) = \frac{\lambda}{\alpha}\sum_{j=0}^\infty \binom{\lambda-1}{j}(-1)^j (j+1)^{-\left(\frac{1+\beta}{\beta}\right)} e^{j+1}\Gamma\left(\frac{1+\beta}{\beta}, j+1\right) - \frac{1}{\alpha} \tag{15}$$

Similarly, we set $r=2$ in (14) as follows

$$E\left(1+2\alpha X+\alpha^2X^2\right)=\lambda\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j(j+1)^{-\left(\frac{2+\beta}{\beta}\right)}e^{j+1}\Gamma\left(\frac{2+\beta}{\beta},j+1\right)$$

$$1+2\alpha E(X)+\alpha^2E\left(X^2\right)=\lambda\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j(j+1)^{-\left(\frac{2+\beta}{\beta}\right)}e^{j+1}\Gamma\left(\frac{2+\beta}{\beta},j+1\right)$$

$$\Rightarrow E\left(X^2\right)=\frac{1}{\alpha^2}\left\{\lambda\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j(j+1)^{-\left(\frac{2+\beta}{\beta}\right)}e^{j+1}\Gamma\left(\frac{2+\beta}{\beta},j+1\right)\right.$$
$$\left.-2\lambda\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j(j+1)^{-\left(\frac{1+\beta}{\beta}\right)}e^{j+1}\Gamma\left(\frac{1+\beta}{\beta},j+1\right)-3\right\}$$

Then, the variance of ENHED is

$$Var(X)=\frac{1}{\alpha^2}\left\{\left[\lambda\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j(j+1)^{-\left(\frac{2+\beta}{\beta}\right)}e^{j+1}\Gamma\left(\frac{2+\beta}{\beta},j+1\right)\right.\right.$$
$$\left.-2\lambda\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j(j+1)^{-\left(\frac{1+\beta}{\beta}\right)}e^{j+1}\Gamma\left(\frac{1+\beta}{\beta},j+1\right)-3\right]$$
$$\left.-\left[\lambda\sum_{j=0}^{\infty}\binom{\lambda-1}{j}(-1)^j(j+1)^{-\left(\frac{1+\beta}{\beta}\right)}e^{j+1}\Gamma\left(\frac{1+\beta}{\beta},j+1\right)-1\right]^2\right\} \tag{16}$$

4. Maximum Likelihood Estimators (MLE)

In this section, we consider maximum likelihood estimators (MLE) of ENHED. Let $x_1,x_2,...,x_n$ be a random sample of size n from ENHED, then the log-likelihood function LL can be written as

$$LL\propto n\left[\ln(\lambda)+\ln(\alpha)+\ln(\beta)\right]+(\beta-1)\sum_{i=1}^{n}\ln(1+\alpha x_i)+(\lambda-1)\sum_{i=1}^{n}\ln\left(1-e^{1-(1+\alpha x_i)^\beta}\right)-\sum_{i=1}^{n}(1+\alpha x_i)^\beta$$

The normal equations become

$$\frac{\partial LL}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\ln\left(1-e^{1-(1+\alpha x_i)^\beta}\right) \tag{17}$$

$$\frac{\partial LL}{\partial\alpha}=\frac{n}{\alpha}+(\beta-1)\sum_{i=1}^{n}\frac{x_i}{1+\alpha x_i}+\beta(\lambda-1)\sum_{i=1}^{n}\frac{x_i(1+\alpha x_i)^{\beta-1}e^{1-(1+\alpha x_i)^\beta}}{1-e^{1-(1+\alpha x_i)^\beta}}-\beta\sum_{i=1}^{n}x_i(1+\alpha x_i)^{\beta-1} \tag{18}$$

$$\frac{\partial LL}{\partial\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln(1+\alpha x_i)+(\lambda-1)\sum_{i=1}^{n}\frac{(1+\alpha x_i)^\beta\ln(1+\alpha x_i)e^{1-(1+\alpha x_i)^\beta}}{1-e^{1-(1+\alpha x_i)^\beta}}-\sum_{i=1}^{n}(1+\alpha x_i)^\beta\ln(1+\alpha x_i) \tag{19}$$

The MLE of λ and α can be obtain by solving the equations (17), (18) and (19) using $\frac{\partial LL}{\partial\lambda}=0$, $\frac{\partial LL}{\partial\alpha}=0$ and $\frac{\partial LL}{\partial\beta}=0$.

5. Application of ENHED

In this section, we use a real data set to show that the ENHED can be a better model than one based on the generalized exponential distribution (GED), extended exponential distribution (EED), Nadarajah and Haghighi's exponential distribution (NHED) and exponential distribution

(ED). We consider a data set of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so we have complete data with the exact times of failure. For previous studies with the data sets see ref. [2], ref. [3], ref. [4] and ref. [6]. These data are:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

In order to compare the distributions, we consider some other criterion like K-S (Kolmogorow-Smirnov), −2LL, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected) and BIC (Bayesian information criterion) for the real data set. The best distribution corresponds to lower K-S, −2LL, AIC, AICC and BIC, where

$$KS = \max_{1\le i\le n}\left(F(x_i)-\frac{i-1}{n},\frac{i}{n}-F(x_i)\right),$$

$AIC = 2t - 2LL,\quad AICC = AIC + \dfrac{2t(t+1)}{n-t-1}$ and

$BIC = t\log(n) - 2LL$, with t is the number of parameters in the statistical model, n the sample size and LL is the maximized value of the likelihood function for the estimated model. Also, here for calculating the values of K-S we use the sample estimates of α, β and λ. Table 1 shows parameter MLE to each one of the two fitted distributions, Table 2 shows the values of K-S, −2LL, AIC, AICC and BIC values.

Table 1. *Estimated parameters of the ENHED, GED, EED, NHED and ED.*

Model	Parameters Estimates	- LL
ENHED	$\hat{\alpha}=2.279$, $\hat{\beta}=0.674$, $\hat{\lambda}=11.003$	45.487
GED	$\hat{\alpha}=0.703$, $\hat{\beta}=1.709$	122.244
EED	$\hat{\alpha}=0.954$, $\hat{\beta}=6.366$	121.650
NHED	$\hat{\alpha}=0.195$, $\hat{\beta}=2.007$	124.738
ED	$\hat{\alpha}=0.510$	127.114

Table 2. *Criteria for comparison.*

Model	K-S	$-2LL$	AIC	AICC	BIC
ENHED	0.08	90.97	96.97	97.31	103.97
GED	0.08	244.49	248.49	248.65	253.15
EED	0.09	243.30	247.30	247.47	251.96
NHED	0.13	249.48	253.48	253.64	258.14
ED	0.18	254.23	256.23	256.28	258.56

The values in Table 2 indicate that the ENHED leads to a better fit than both GED, EED, NHED and ED. We can see the same notes in the next figure.

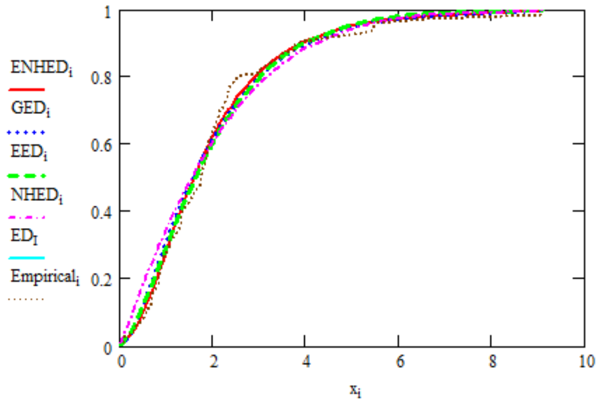

Figure 4. *Empirical, fitted ENHED, EED, NHED and ED CDF of the above data.*

6. Conclusion

In this article, we introduce a new generalization of the Nadarajah and Haghighi's exponential distribution called Exponentiated Nadarajah and Haghighi's exponential distribution and presented its theoretical properties. The estimation of parameters is approached by the method of maximum likelihood. An application of the Exponentiated Nadarajah and Haghighi's exponential distribution to real data show that the new distribution can be used quite effectively to provide better than the Nadarajah and Haghighi's exponential distribution.

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