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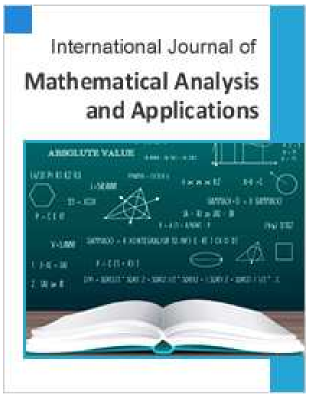

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KKT Conditions and Branch and Bound Methods on Pure Integer Nonlinear Programming

Amalia Umami*, Esther Nababan, Sawaluddin

Department of Mathematics, University of Sumatera Utara, Medan, Indonesia

Email address
umamiamalia@gmail.com (A. Umami), esther@usu.ac.id (E. Nababan),
sawal@usu.ac.id (Sawaluddin)

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Abstract
Optimization problems are not only formed into a linear programming but also nonlinear programming. In real life, often decision variables restricted on integer. Hence, came the nonlinear programming. One particular form of nonlinear programming is a convex quadratic programming which form the objective function is quadratic and convex and linear constraint functions. In this research designed a completion of a convex quadratic integer programming with Karush Kuhn Tucker conditions which then reduces the integer convex quadratic programming into a linear complementary problem. Then used a modified simplex method and *Branch and Bound* method to obtain the optimal and integer solution and fulfill all the constraints. The obtained solution by using KKT conditions is a global optimum solution due to the problem studied is convex. This method is effective in finding integer solution with result that is not too far from the initial solution to the problem which is quite simple.

1. Introduction

Many problems in economic, industrial, engineering and others can be expressed in the form of a mathematical model that is nonlinear programming. An optimization problem called nonlinear if the objective function and constraints have nonlinear form on one or both of them [9]. Nonlinear programs used in minimizing portfolio risk with a certain mean return [4]. Similarly to the paper industry, applications integer nonlinear programming model emerged as a nonlinear cutting stock problem [12]. Quadratic programming is a nonlinear program problem with a quadratic objective function and linear constraints. If the objective function is convex then the problem has a local minimum which is also a global minimum [17].

An optimization technique that can be used to search the optimum point of a constrained problem is Karush Kuhn Tucker (KKT) conditions. KKT conditions can be used to find the optimum solution of a function regardless the linearity [2]. If using KKT conditions in resolving convex problems, then the optimum KKT point is a global optimum. Convex quadratic program can be reduced to a linear complementarity problem using Karush Kuhn Tucker conditions as a condition in determining the optimum point of a problem [5].

Many optimization problems where the decision variables are restricted to integer variables. In the nonlinear optimization problem and all the variables are restricted in integers, then the problem stated as pure integer nonlinear programming [10]. Quadratic integer programming is part of nonlinear integer programming where the objective function is quadratic and the decision variables are restricted to integers [19].

Several methods have been used in solving problems of integer optimization. Branch and Bound is one of the methods commonly used to resolve the integer linear programming. There are two stages in this method, branching and bounding. At the branching stage, the problem is partitioned into several sub problems by adding constraints without changing the original integer solution set. At the bounding stage, the objective function value of an integer solution sub problem determined to be the bound value of the objective function of other sub problems [13]. Branch and bound method is then applied to the convex quadratic programming. YoonAnn [19] solve the problem in the binary quadratic programming in which the value of each decision variable is 0 or 1. Buchheim [3] uses the branch and bound method in solving a convex quadratic programming with the constraints that are also convex.

2. Pure Integer Convex Quadratic Programming Problems

In this paper, the issues discussed are the problems modeled into convex quadratic programming model and where all decision variables are restricted to integers and the inequality constraints form. The general form of pure integer quadratic programming is as follows:

$$Min: f(x) = \sum_{j=1}^{n} c_j x_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{m} d_{ij} x_j x_i$$

S.t: $\sum_{j=1}^{n} a_{ij} x_j \le b_i$

$x_j \ge 0$ *and integer*

where $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$. $f(x)$ is an objective function that is convex.

2.1. Convexity

2.1.1. Convex and Concave Functions

Convexity often used in the scope of nonlinear programming [1]. The set of convex can be seen as a collection of points with the convex function as the upper bound and concave function as lower bound.

A function $f(x)$ is called a convex function if for any two points x' and x'' where $x' < x''$,

$$f[\lambda x'' + (1-\lambda)x'] \le \lambda f(x'') + (1-\lambda) f(x')$$

for all values of λ that satisfies $0 < \lambda < 1$. That function is a strictly convex if $\le$ can be replaced by $<$.

Contrarily, for the concave function. $f(x)$ is convex if for every pair of points on the graph $f(x)$, the segment line connecting these two points on or above the graph $f(x)$ and contrarily with a concave function.

2.1.2. Convex and Concave Functions Several Variables

The concept of convex and concave functions of one variable can be generalized to more than one variables function. That way, when $f(x)$ to be $f(x_1, x_2, \dots, x_n)$ the definition still applied when x become $x_1, x_2, \dots, x_n$.

Line segment that connecting the two points $(x_1', x_2', \dots, x_m')$ and $(x_1'', x_2'', \dots, x_m'')$ is the sum of points $(x_1, x_2, \dots, x_m) = [\lambda x_1'' + (1-\lambda)x_1', \lambda x_2'' + (1-\lambda)x_2', \dots, \lambda x_m'' + (1-\lambda)x_m']$, where $0 \le \lambda \le 1$.

So the line segments in m-dimensional space is a direct generalization of line segments in two-dimensional space. $f(x_1, x_2, \dots, x_n)$ is a convex function if for every pair of points on the graph $f(x_1, x_2, \dots, x_n)$ line segment that connecting the two points are all above or right on the graph of the function $f(x_1, x_2, \dots, x_n)$. That function is a strictly convex if line segments are all above the graph except for the two end points. Contrarily with concave and strictly concave functions.

The second partial derivatives are used to test a function $f(x_1, x_2, \dots, x_n)$ convex or not. A function $f(x_1, x_2, \dots, x_n)$ is convex if and only if the matrix of second partial derivatives or Hessian matrix is semi-definite positive and the principal minor determinants are all nonnegative.

Hessian matrix is a matrix whose entries are the second partial derivative of a function. This matrix is used to test whether a function is a convex function or not. Suppose a function $f(x_1, x_2, \dots, x_n)$ with n variables $(x_1, x_2, \dots, x_n)$, then its Hessian matrix can be formed as follows.

$$H(X) = \left[\frac{\partial^2 f}{\partial_{x_i} \cdot \partial_{x_j}}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where $i = 1,2,\dots,m$ and $j = 1,2,\dots,n$

2.2. KKT Conditions of Quadratic Programming

From the general form of quadratic problems above, formed following Lagrange function

$$L = f(x) - \sum_{j=1}^{n} \lambda_i \left(a_{ij} x_j + S_i^2 - b_i\right) + \sum_{j=1}^{n} \mu_j\left(-x_j + r_j^2\right)$$

where S_i^2 and r_j^2 are slack variables that added to each constraint. λ_i are the value of the Lagrange multipliers for each constraint. μ_j is Lagrange multiplier values for every constraints where $x_j \ge 0$. The Karush Kuhn Tucker conditions obtained by differentiating partially Lagrange function for each decision variable and equating to 0.

$$\frac{\partial L}{\partial x_j} = c_j - \sum_{i=1}^{n} d_{ij} x_i - \sum_{i=1}^{n} \lambda_i a_{ij} + \mu_j = 0 \quad (2.1)$$

$$\frac{\partial L}{\partial \lambda_i} = \sum_{j=1}^{n} a_{ij} x_j + S_i^2 - b_i = 0 \quad (2.2)$$

$$\lambda_i S_i = 0 \quad (2.3)$$

$$\mu_j r_j = 0 \quad (2.4)$$

$$\frac{\partial L}{\partial \mu_i} = -x_j + r_j^2 = 0 \quad (2.5)$$

$$x_j, S_i, r_j, \lambda_i, \mu_j \geq 0 \tag{2.6}$$

The conditions $\lambda_i S_i = 0$ and $\mu_j x_j = 0$ called as complementary constraint conditions. $\lambda_i S_i = \mu_j x_j = 0$ implies that variables x_j, μ_j and S_i cannot be basis variables simultaneously because one of every decision variable must be 0.

2.3. Modified Simplex Methods

Two-phase simplex method has been modified to solve the problems of nonlinear programming. The steps for each iteration of the simplex method is as follows:

1. Change inequality constraints into the equations by adding slack variables S_i^2 and r_j^2 as described above.
2. Form Lagrange function and obtained The Karush Kuhn Tucker conditions.
3. Add artificial variables $A_j, j = 1,2,\ldots,n$ the Karush Kuhn Tucker conditions gained at step 3.
4. Form the objective function in linear programming form as follows

$$Min: Z = \sum_{j=1}^{n} A_j$$

1. Obtain a feasible solution to the problem

$$Min: Z = \sum_{j=1}^{n} A_j$$

Subject to:

$$\sum_{i=1}^{n} d_{ij} x_i + \sum_{i=1}^{n} \lambda_i a_{ij} - \mu_j + A_j = c_j$$

$$\sum_{j=1}^{n} a_{ij} x_j + S_i^2 = b_i$$

$$x_j, S_i, r_j, \lambda_i, \mu_j \geq 0$$

$$complementary\ constrains \begin{cases} \lambda_i S_i = 0 \\ \mu_j x_j = 0 \end{cases}$$

2. Solve with modified simplex method to obtain the optimal solution to the above problems. Modifications in the simplex method is the change in the selection of variables that will go to be the basis.

Limited Entering Rules: In the selection of entering basis variable, it is not allowed to include non basis variables that have a complementary form of basis variable. The option should be taken from other non basis variables agree with the usual criteria for the simplex method.

After forming the Lagrange function and obtain the KKT conditions in step 2, then the equations used as the new constraints of linear problem that exist in step 5. Due to the quadratic objective function, it can be ascertained the equations derived from the KKT conditions be linear equations. If the objective function is a polynomial of higher orde then it cannot be ascertained and require further research. The addition of artificial variables in step 3 and then making a linear objective function of the problem in step 5. The number of artificial variables added must be in accordance with the number of constraints derived from the KKT conditions in step 2. Thus forming a linear programming in step 5.

The purpose of this method certainly find an optimal solution to the linear programming that is minimizing the sum of artificial variables with the complementary constraints must be completed for each iteration.

2.4. Branch and Bound Algorithm

Branch and Bound Algorithm for pure integer quadratic programming is as follows:

1. Initialization

State the initial problem of pure integer convex quadratic programming as an element of P is a convex quadratic problem is not an integer, $P = L$. Set the upper bound of $\overline{Z} = +\infty$ and lower bound $\underline{Z} = -\infty$.

2. Termination Test

If $P = \emptyset$, stop the process. The process is terminated when the integer convex quadratic problem has been resolved and obtained for whole solutions are integers. Then x^* that produces the best value Z is an optimal solution.

3. Problem Selection and Relaxation

Select and remove the pure integer convex quadratic programming problem of P. Solve relaxation with modified simplex method described previously. Set Z^i as the optimal objective value of relaxation and x^i as a solution of the relaxation. If $\underline{Z} = -\infty$, then the set $\underline{Z} = Z^i$.

4. Pruning
   a. If $Z^i \geq \overline{Z}$, go to step 2.
   b. If $Z^i < \overline{Z}$ and x^i are feasible integres, then $Z^i = \overline{Z}$ and go to step 2.
   c. Otherwise go to step 5.
5. Branching

Let $\{S_{ij}\}_{j=1}^{j=k}$ be a feasible division of constraints S_i of the problem pure integer convex quadratic programming to the bound S_{ij}. Add the constraints to P and go to step 2.

Z^i is the lower bound of the optimum value of sub problem. The value of this sub problem is used to renew the Z^i. Problems in step 5 are called sub problems. Usually the division of the problem using the form of connective variables $x_i \leq a$ and $x_i \leq a + 1$ to variable x_i and a is integer. The constraints being resolved by the modified simplex method described earlier.

Branch and bound algorithm is a common method used to obtain a solution which is restricted to integer, both in linear and nonlinear program. However, this algorithm resulted in the `long process. This is because the search for solutions to be traced continuously until the result is entirely an integer. Then, it results in efficiency in turnaround time.

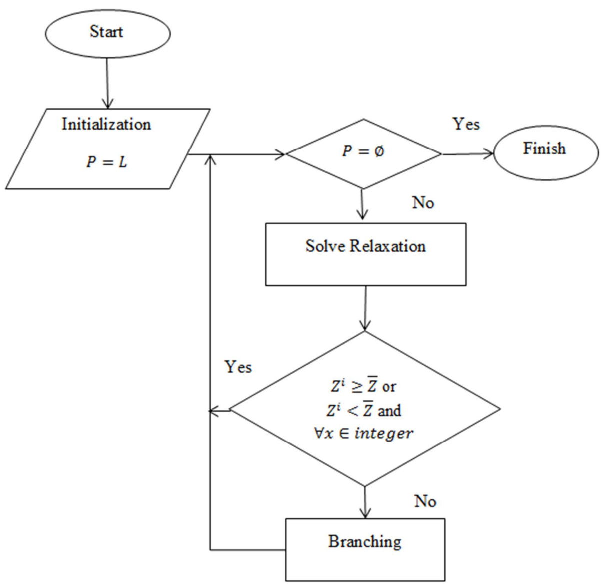

Figure 2.1. *Flowchart of Branch and Bound Algorithm.*

3. Result and Discussions

Example of the implementation on the methods on nonlinear problem is as follows,

Min: $(x_1-18)^2+(x_2-16)^2$

S.t: $2x_1+x_2\le 15$

$$x_1,x_2\ge 0\ and\ integer$$

Formulate the Lagrange function:

$$L=(x_1-18)^2+(x_2-16)^2+\lambda_1(2x_1+x_2+S_1^2-15)-\mu_1(-x_1+r_1^2)-\mu_2(-x_2+r_2^2)$$

and reduce the above problem into linear programming with Karush Kuhn Tucker condition is then solved with a modified simplex method

Min: $Z=A_1+A_2+A_3$

S.t: $2x_1+2\lambda_1-\mu_1+A_1=36$

$$2x_2+\lambda_1-\mu_2+A_2=32$$

$$2x_1+x_2+S_1+A_3=15$$

$$x_1,x_2\ge 0\ and\ integer$$

yield the following results

$x_1=\frac{32}{10}, x_2=\frac{43}{5}, \lambda_1=\frac{148}{10}$ with $Z=273\frac{4}{5}$. While the other decision variables are 0, ie $\mu_1=\mu_2=S_1=0$.

Table 3.1. *Modified Simplex.*

C	basic	0	0	0	0	0	0	1	1	1	B
		x_1	x_2	λ_1	μ_1	μ_2	S_1	A_1	A_2	A_3	
0	λ_1	0	0	1	$-\frac{2}{5}$	$\frac{1}{5}$	$-\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$-\frac{2}{5}$	$\frac{148}{10}$
0	x_2	0	1	0	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{43}{5}$
0	x_1	1	0	0	$-\frac{1}{10}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{32}{10}$
$Z_j - C_j$		0	0	0	0	0	0	-1	-1	-1	0

Feasible and optimal solution obtained has a non-integer values for the variables that are restricted integer, then the problem will be branched in one of variable is not an integer. Selection of variables to be branched greatly affect the efficiency of time and number of steps that must be solved.

3.1. Branching in x_1

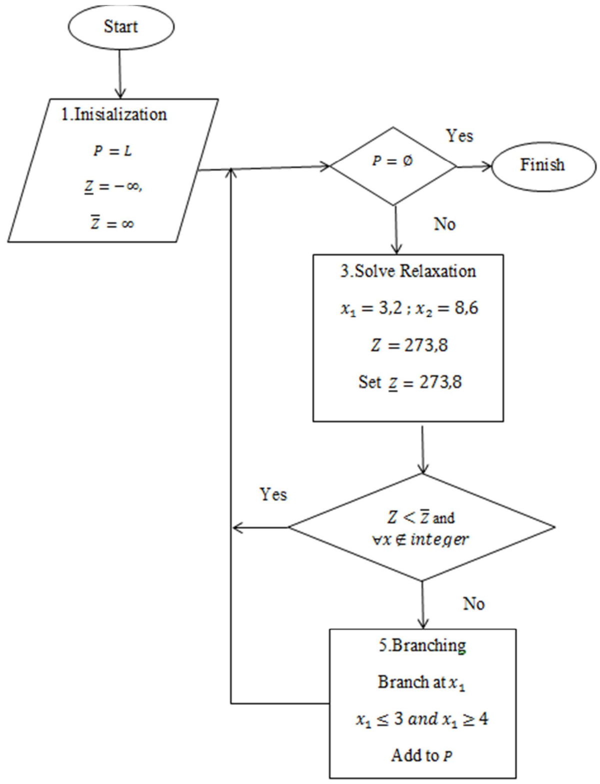

Figure 3.1. *Flowchart of branching in* x_1.

By selecting branching variable in x_1 , obtained new constraints from bounding process, that are sub problem 1 with addition constraint $x_1\le 3$ and sub problem 2 with constraint $x_1\ge 4$. With the same process obtained the result for sub problem 1 is $x_1=3, x_2=9, \lambda_1=14, \lambda_2=2$ with $Z=274$. While the other decision variables are 0, i.e $\mu_1=\mu_2=S_1=S_2=0$.

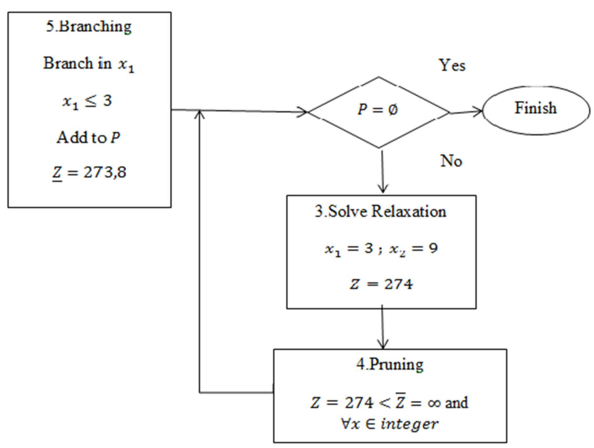

Figure 3.2. *Flowchart of Sub Problem 1.*

Then sub problem 2 obtained $x_1 = 4, x_2 = 7, \lambda_1 = 18, \lambda_2 = 8$ and $Z = 277$. While the other decision variables are 0, i.e $\mu_1 = \mu_2 = S_1 = S_2 = 0$.

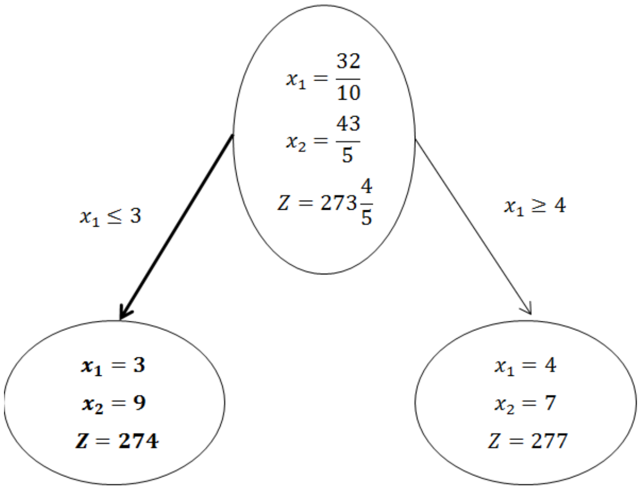

Figure 3.3. *Branch and Bound Algorithm Branching in* x_1.

From the value of Z from two sub problems above can be be concluded that $Z = 274$ is a minimum optimum solution satisfies two decision variables, that $x_1 = 3$ and $x_2 = 9$ are integer with $\lambda_1 = 14$ and $\lambda_2 = 2$. This means that change in the objective function is directly proportional to the change of the first constraint with Lagrange multiplier factor of 14. As to the second constraints is 2.

3.2. Branching in x_2

By selecting branching variable in x_2, obtained new constraints that are sub problem 1 with addition constraint $x_2 \leq 8$ and $x_2 \geq 9$ to sub problem 2. With the same process obtained the result for sub problem 1 is $x_1 = \frac{7}{2}, x_2 = 8, \lambda_1 = \frac{29}{2}, \lambda_2 = \frac{3}{2}$ *and* $Z = 274{,}2$. While the other decision variables are 0, i.e $\mu_1 = \mu_2 = S_1 = S_2 = 0$. Due to sub problem 1 gives a result real variable, then done the next branching $x_1 \leq 3$ and $x_1 \geq 4$. For the additional constraint $x_1 \leq 3$ obtained $x_1 = 3, x_2 = 8, S_1 = 1, \lambda_2 = 16, \lambda_3 = 30$ *and* $Z = 289$. While the other decision variables are 0, i.e $\mu_1 = \mu_2 = \lambda_1 = S_2 = S_3 = 0$. Then the additional constraint $x_1 \geq 4$ obtained $x_1 = 4, x_2 = 7, \lambda_1 = 10, S_2 = 1, \lambda_3 = 8$ *and* $Z = 277$. While the other decision variables are 0, i.e $\mu_1 = \mu_2 = S_1 = \lambda_2 = S_3 = 0$. Sub problem 2 gives a result $x_1 = 3, x_2 = 9, \lambda_1 = 15, \lambda_2 = 1$ *and* $Z = 274$. While the other decision variables are 0, i.e $\mu_1 = \mu_2 = S_1 = S_2 = 0$.

From these four nodes obtained integer solution for twice branching. However, for $x_2 \leq 9$ gives more minimum solution. Hence, the optimum solution for the problem is $x_1 = 3$; $x_2 = 9$ and $Z = 274$ with $\lambda_1 = 15$ and $\lambda_2 = 1$. This means that change in the objective function is directly proportional to the change in the first constraint with Lagrange multiplier factor of 15. As to the second constraint is 1.

From the solving process by branching in x_2 above provide more iterations than in x_1 at the beginning of the calculation. From all solving performed at each node, the optimal solution that satisfies all the constraints obtained in the second sub problem.

From the completion of the above process is obtained that the branch and bound method has resulted in the settlement process to be traced any node that might be to obtain the right optimal solution.

By selecting the smallest variable to branch provides shorter steps than the greater one. So the selection of variables to be branched greatly affect the efficiency of Branch and Bound algorithm.

KKT conditions provide a global optimal solution in a convex quadratic program. Combine KKT conditions for reducing the convex quadratic nonlinear program problem and then solved by a modified simplex method and branch and bound method for finding integer solution provide excellence solution are not far from the initial optimal solution. Hence it is a very effective to solve quite simple problems.

However, for the problems which contains more decision variables and constraints, it is not recommended. This is caused by the uncertainty of the number of iterations to be solved. Therefore, it would lead to inefficiency completion time. The use of software also requires quite large storage and the long turnaround time (CPU time).

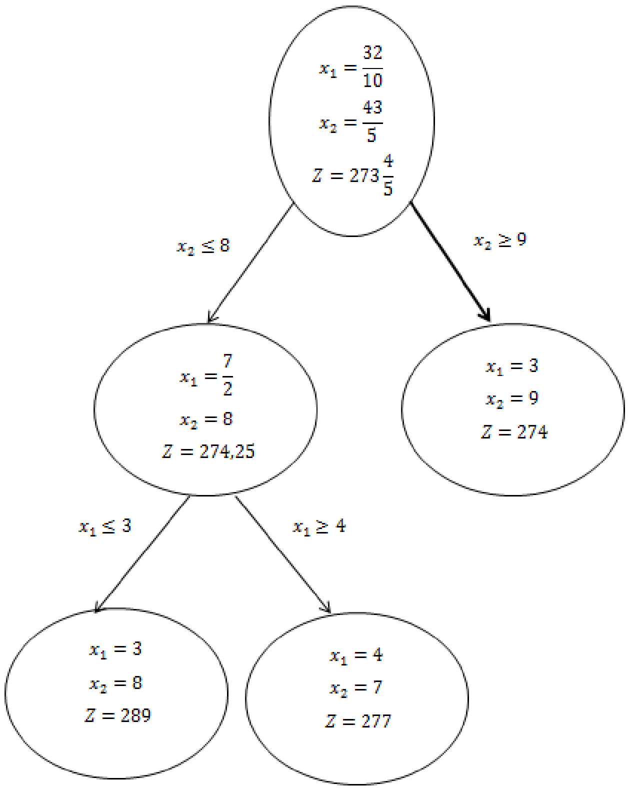

Figure 3.4. *Branch and Bound Algorithm Branching in* x_2.

4. Conclusion

In integer convex quadratic programming problem, KKT conditions reduce nonlinear problem to be linear then change them into linear constraints and form linear objective function with artificial variables then is solved with modified simplex method.

In modified simplex method attention is needed at the limited entering rules that satisfies complementary constraints. Branch and Bound method gives a role in forming new constraints by branching and bounding obtained from real optimum solution. Strategy in selecting branching variables at the initial completion in branch and bound method would lead to different completion steps and cause at the efficiency turnaround time of the completion. Hence, convexity test is required at the beginning of the process. Using KKT condition and branch and bound method in solving simple integer convex quadratic programming gives a result that is not far from the initial optimum calculation. KKT conditions provide a global optimum solution for convex problems. In this paper the problem been discussed is limited on integer convex quadratic programming with two decision variables. Hence, further research and the use of software such as Matlab are also required for more complicated problems such as convcave or polynomial problems. In addition, the method is expected to be applied directly on the real world problems.

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