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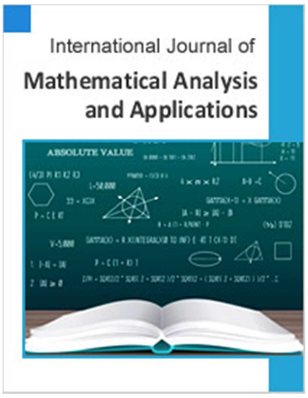

On the Generalization of Hypergeometric and Confluent Hypergeometric Functions and Their Applications for Finding the Derivatives of the Generalized Jacobi Polynomials

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Abstract
Recently, some generalizations of the generalized gamma, beta, Gauss hypergeometric and confluent hypergeometric functions have been introduced in the literature. Most of the special functions, such as Jacobi polynomials, can be expressed in terms of Gauss hypergeometric function (GHF) and confluent hypergeometric function (CHF). The main object of this paper is to express explicitly the derivatives of generalized Jacobi polynomials in terms of Jacobi polynomials themselves, by using generalized hypergeometric functions of any degree that have been differentiated an arbitrary numbers of times. The results for the special cases of generalized Ultraspherical polynomials and Chebyshev polynomials of the first, second, third and fourth kinds and Legendre polynomials are also given.

1. Introduction

Classical orthogonal polynomials are used extensively for the numerical solution of differential equations in spectral and pseudo spectral methods [3-14, 16-18]. If these polynomials are used as basis functions, then the rate of decay of the expansion coefficients is determined by the smoothness properties of the function being expanded and not by any special boundary conditions satisfied by the function itself. If the function of interest is infinitely differentiable, then the nth expansion coefficient will decrease faster than any finite power of $(\frac{1}{n})$ as n tends to infinity [9]. For spectral and pseudo spectral methods, explicit formulas for the basis functions of the derivatives in terms of the original functions are needed. Additionally, explicit expressions for the derivatives of the basis functions themselves are required [3-18].

A formula expressing the Chebyshev coefficients of a general order derivative of an infinitely differentiable function in terms of its Chebyshev coefficients is given in Karageorghis [11] and the corresponding formula for the Legendre coefficients has been obtained by Phillips [15].

In this recent work [3-8, 16, 17] Doha develops a class of spectral – Galerkin methods for the direct solution of higher order differential equations. One of particular interest is the Jacobi formula and its special cases. General formula for the coefficients of an expansion of Ultraspherical polynomials which has been differentiated an arbitrary number of times in terms of those in the original expansion is given in [3]. General formulae express the Legender coefficients of the general order derivative of an infinitely differentiable function in terms of its Legendre coefficients are given by Doha & El – Soubhy [7].

A formula expressing explicitly the derivatives of Jacobi polynomials of any degree in terms of Jacobi polynomials themselves is proved and the explicit formula, that expresses the Jacobi expansion coefficients of a general order derivative of an infinitely differentiable function in terms of the original Jacobi coefficients and the special cases are considered in Doha [4]. Doha et al. [16] give two analytical formulas expressing explicitly the derivatives of Chebyshev polynomials of third and fourth kinds of any degree in terms of Chebyshev polynomials of the third and fourth kinds themselves.

Chaudhry and Zubair [19] have introduced the extension of the gamma function, then Chaudhry et al. [20] considered the extension of Euler' s beta function. Afterwards, Chaudhry et al. [21] used $\beta_p(x,y)$ to extend the hypergeometric functions (and confluent hypergeometric functions) and gave the Euler type integral representation $F_p(a,b;c;z)$. They called these functions the extended Gauss hypergeometric function *(EGHF)* and the extended confluent hypergeometric function *(ECHF),* where $F_0(a,b;c;z) = {}_2F_1(a,b;c;z)$.

$\ddot{O}$zergin, $\ddot{O}$zaraslan and Altin [1] called the functions $\Gamma_p^{(\alpha,\beta)}(x,y), and\ \beta_p^{(\alpha,\beta)}(x,y)$ as the generalized Euler's gamma function *(GEGF)* and generalized Euler's beta function *(GEBF),* respectively. On the other hand using the generalizations of the Beta function, the generalized Gauss hypergeometric function *(GGHF)* and generalized confluent hypergeometric functions *(GCHF)* are discussed in [22].

Up to now, and to the best of our knowledge, many formulas corresponding to those mentioned previously are not known and cannot be found in the literature for the generalized Jacobi polynomials. This motivates our interest in such orthogonal polynomials.

The structure of this article is as follows. In Section 2, we give some properties of Jacobi polynomials and some properties of *(GGHF)* and generalized confluent hypergeometric function*(GCHF).* We re- derive the formulas given in [1] and Doha [4] to obtain a new explicit formula for generalized Jacobi polynomials; the formulas for differentiated Jacobi polynomials of any degree that have been differentiated an arbitrary number of times in terms of Jacobi polynomials themselves are given in a compact form of the *(GGHF)* and the generalized *(GCHF).* The special cases of generalized Ultraspherical polynomials are given; results for the generalized Chebyshev polynomials and Legendre polynomials are considered; further, the classical Ultraspherical polynomials and classical Chebyshev polynomials of the first, second, third and fourth kinds and classical Legendre polynomials are given in corollaries in Section 3. Finally, some concluding remarks are noted in section 4.

2. Generalized Gauss Hypergeometric Functions(GGHFs) and Jacobi Polynomials

2.1. Some Properties of the Generalized Gauss Hypergeometric Functions (GGHFs) and Jacobi Polynomials

In mathematics, there are several special functions that are of particular significance and are used in many applications [1-24]. In addition, some of these special functions find applications in such diverse areas as astrophysics, fluid dynamics and quantum physics [10, 18, 23, 24]. Examples of such well-known functions are the gamma, beta and hypergeometric functions [1-24]. Next, extensions of the gamma, beta, Gauss hypergeometric function *(GHF)* and confluent hypergeometric function *(CHF)* have been extensively studied in the recent past [19] by inserting a regularization factor $e^{\frac{-p}{t}}$.

The following extension of the gamma function is introduced by Chaudhry and Zubair [19]:

$$\Gamma_p(x) = \int_0^\infty t^{x-1}\exp\left(-t-\tfrac{p}{t}\right)dt, Re(p) > 0, \quad (1)$$

Chaudhry et al. [20] considered the extension of Euler's beta function in the following form

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\exp\left(\frac{-p}{t(1-t)}\right)dt,$$

$$Re(p) > 0, Re(x) > 0, Re(y) > 0.$$

It is clearly seen that

$$\Gamma_0(x) = \Gamma(x)\ and\ \beta_0(x,y) = \beta(x,y). \quad (2)$$

Afterwards, Chaudhry et al. [21] used $\beta_p(x,y)$ to define the extended hypergeometric.

Function *(EGHF)* and the extended confluent hypergeometric function *(ECHF)* as follows:

$$F_p(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n\frac{\beta_p(b+\mathrm{n},\mathrm{c}-\mathrm{b})}{\beta(\mathrm{b},\mathrm{c}-\mathrm{b})}\frac{(z)^{\mathrm{n}}}{\mathrm{n}!},$$

$$p \geq 0, Re(c) > Re(b) > 0, \quad (3)$$

$where\ (a)_n\ denotes\ the\ Pochhammer\ symbol\ defined\ by$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$$

$$= \begin{cases} 1, n = 0, a \in \mathbb{C} / \{0\} \\ a(a+1)(a+2)\dots(a+n-1), n \in \mathbb{N}, a \in \mathbb{C}. \end{cases}$$

This gave the Euler type integral representation

$$F_p(a,b;c;z) = \frac{1}{\beta(\mathrm{b},\mathrm{c}-\mathrm{b})}\int_0^1 \mathrm{t}^{\mathrm{b}-1}(1-\mathrm{t})^{\mathrm{c}-\mathrm{b}-1}(1-zt)^{-a}$$

$$\times\, exp\left(\frac{-p}{t(1-t)}\right)dt,$$

$p \geq 0, and\ |\arg(1-z)| < \pi < p;\ Re(c) > Re(b) > 0$ (4)

Note that

$$F_0(a,b;c;z) = {}_2F_1(a,b;c;z)$$

Additionally, the extended confluent hypergeometric functions *(ECHF)* is defined as

$$\phi_p(b;c;z) = \sum_{n=0}^{\infty} \frac{\beta_p(b+n, c-b)}{\beta(b,c-b)} \frac{(z)^n}{n!},$$

$$p \geq 0, Re(c) > Re(b) > 0, \quad (5)$$

The differentiation properties and the Mellin transforms of $F_p(a,b;c;z)$ were also considered in [21]. The transformation formulas, recurrence relations, summation and asymptotic formulas were also obtained in [21].

Very recently, some representations of these extended functions were reported in terms of a finite number of well-known higher transcendental functions [1, 22].

The following generalized Euler's gamma and generalized Euler's beta functions are defined in [1] as

$$\Gamma_p^{(\alpha,\beta)}(x) = \int_0^{\infty} t^{x-1}\, {}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right) dt,$$

$$Re(\alpha) > 0, Re(\beta) > 0, Re(p) > 0, Re(x) > 0, \quad (6)$$

$$\beta_p^{(\alpha,\beta)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt,$$

$$Re(p) > 0, Re(x) > 0, Re(y) > 0, Re(\alpha) > 0, Re(\beta) > 0, \quad (7)$$

Respectively.

It is obvious from (1) and (6), (2) and (7) that,

$$\Gamma_p^{(\alpha,\alpha)}(x) = \Gamma_p(x),\ \Gamma_0^{(\alpha,\alpha)}(x) = \Gamma(x),$$

$$\beta_p^{(\alpha,\alpha)}(x,y) = \beta_p(x,y) \text{ and}$$

$$\beta_0^{(\alpha,\beta)}(x,y) = \beta(x,y).$$

They called the functions $\Gamma_p^{(\alpha,\beta)}(x), and\ \beta_p^{(\alpha,\beta)}(x,y)$ generalized Euler's gamma function *(GEGF)* and generalized Euler's beta function *(GEBF)*, respectively. On the other hand, using the generalizations of (7) for the beta function, enable one to define the generalized Gauss hypergeometric function *(GGHF)* and generalized confluent hypergeometric function (*GCHF)* as,

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)} \frac{z^n}{n!}, \quad (8)$$

and

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \sum_{n=0}^{\infty} \frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)} \frac{z^n}{n!}, \quad (9)$$

Respectively.

The following integral representations were obtained in [1]:

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{\beta(b,c-b)} \times$$

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)(1-zt)^{-a}\, dt,$$

$Re(p) \geq 0, and\ |\arg(1-z)| < \pi < p;\ Re(c) > Re(b) > 0.$ (10)

and

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \frac{1}{\beta(b,c-b)} \times$$

$$\int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) dt,$$

$$p \geq 0, and\ Re(c) > Re(b) > 0. \quad (11)$$

It is to be noted here that

$$F_p^{(\alpha,\alpha)}(a,b;c;z) = F_p(a,b;c;z),$$

$$F_0^{(\alpha,\beta)}(a,b;c;z) = {}_2F_1(a,b;c;z).$$

Further,

$${}_1F_1^{(\alpha,\alpha;p)}(b;c;z) = {}_1F_1^{(p)}(b;c;z) = \phi_p(b;c;z),$$

$${}_1F_1^{(\alpha,\beta;0)}(b;c;z) = {}_1F_1(b;c;z).$$

The generalized hypergeometric function with p numerator q denominator parameters is defined by [23, 24],

$${}_pF_q(a_1,a_2,\dots,a_p;b_1,b_2,\dots,b_q;z) = {}_pF_q\left(\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, b_2, \dots, b_q \end{matrix}; z\right)$$

$$= \sum_{r=0}^{\infty} \frac{(a_1)_r(a_2)_r \dots (a_p)_r}{(b_1)_r(b_2)_r \dots (b_q)_r} \frac{z^r}{r!},$$

$$= \frac{\Gamma(b_1)\dots\Gamma(b_q)}{\Gamma(a_1)\dots\Gamma(a_p)} \sum_{r=0}^{\infty} \frac{\Gamma(a_1+r)\Gamma(a_2+r)\dots\Gamma(a_p+r)}{\Gamma(b_1+r)\Gamma(b_2+r)\dots\Gamma(b_q+r)} \frac{z^r}{r!},$$

$$where,\ z \in \mathbb{C}, p \leq q, a_i, b_j \in \mathbb{C}, b_j \neq 0,-1,-2,\dots,$$

$$i = 1,2,\dots,p, j = 1,2,\dots,q. \quad (12)$$

The nth derivative of $z^s F_p^{(\alpha,\beta)}(a,b;c;z)$ with respect to the variable z is given by El-Soubhy in [2].

2.2. Generalized Jacobi and Generalized Ultaspherecial Polynomials

The Jacobi polynomials and their special cases play important roles in approximation theory and its applications. The main objective of this section is to give the following generalization of Jacobi polynomials and generalization of its special cases.

Definition.

Using the definition of the generalization of the Gauss and confluent hypergeometric functions *GGHF* (8) and *GCHF* (9) to define the generalized Jacobi polynomials and their special cases, we have

$$(i)\ P_n^{(\alpha_1,\beta_1;\alpha,\beta;p)}(x)$$

$$= \frac{(\alpha_1+1)_n}{n!} F_p^{(\alpha,\beta)}\left(-n, n+\alpha_1+\beta_1+1; \alpha_1+1; \frac{1-x}{2}\right),$$

where $P_n^{(\alpha_1,\beta_1;\, \alpha,\, \beta;\, p)}(x)$ is called the generalized Jacobi polynomials.

$$(ii)\ C_n^{(\alpha,\beta;\gamma;p)}(x) = \frac{(2\gamma)_n}{n!} F_p^{(\alpha,\beta)}\left(-n, n+2\gamma; \gamma+\frac{1}{2}; \frac{1-x}{2}\right),$$

where $C_n^{(\alpha,\beta;\gamma;p)}(x)$ is called the generalized ultrspherecial polynomial.

$$(iii)\ T_n^{(\alpha,\beta;p)}(x) = F_p^{(\alpha,\beta)}\left(-n, n; \frac{1}{2}; \frac{1-x}{2}\right),$$

where $T_n^{(\alpha,\beta;p)}(x)$ is called the generalized Chebyshev polynomial of the first kind.

$$(iv)\ U_n^{(\alpha,\beta;p)}(x) = \sqrt{1-x^2}\, n \times F_p^{(\alpha,\beta)}\left(-n+1, n+1; \frac{3}{2}; \frac{1-x}{2}\right),$$

where $U_n^{(\alpha,\beta;p)}(x)$ is called the generalized Chebyshev polynomial of the second kind.

$$(v)\ P_n^{(\alpha,\beta;\,p)}(x) = F_p^{(\alpha,\beta)}\left(-n, n+1; 1; \frac{1-x}{2}\right),$$

where $P_n^{(\alpha,\beta;\,p)}(x)$ is called the generalized Legendre polynomial.

$$(vi)\ V_n^{(\alpha,\beta;\,p)}(x) = \frac{2^{2n} n!}{(2n)!}\left(\frac{1}{2}\right)_n F_p^{(\alpha,\beta)}\left(-n, n+1; \frac{1}{2}; \frac{1-x}{2}\right),$$

where $V_n^{(\alpha,\beta;\,p)}(x)$ is called the generalized Chebyshev polynomial of the third kind.

$$(vii)\ W_n^{(\alpha,\beta;\,p)}(x) = \frac{2^{2n} n!}{(2n)!}\left(\frac{3}{2}\right)_n F_p^{(\alpha,\beta)}\left(-n, n+1; \frac{3}{2}; \frac{1-x}{2}\right).$$

where $W_n^{(\alpha,\beta;\,p)}(x)$ is called the generalized Chebyshev polynomial of the fourth kind. (13)

It is well known that the classical Jacobi polynomials associated with the two real parameters ($\alpha_1 > -1, \beta_1 > -1$) are a sequence of polynomials $P_n^{(\alpha_1,\beta_1)}(x), (n = 0,1,2,\ldots)$, respectively of degree n, satisfying the orthogonality relation

$$\int_{-1}^{1}(1-x)^{\alpha_1}(1+x)^{\beta_1} P_m^{(\alpha_1,\beta_1)}(x) P_n^{(\alpha_1,\beta_1)}(x)\,dx$$

$$= \begin{cases} 0, & m \neq n, \\ \frac{2^{\alpha_1+\beta_1+1}\Gamma(n+\alpha_1+1)\Gamma(n+\beta_1+1)}{n!(2n+\alpha_1+\beta_1+1)\Gamma(n+\alpha_1+\beta_1+1)}, & m = n. \end{cases} \quad (14)$$

These polynomials are eigenfunctions of the following singular Sturm - Liouville equation

$$(1-x^2)\phi''(x) + [\beta_1 - \alpha_1 - (\alpha_1+\beta_1+2)x]\phi'(x) + n(n+\alpha_1+\beta_1+1)\phi(x) = 0.$$

For the present purposes it is convenient to standardize the classical Jacobi polynomials[3] so that

$$P_n^{(\alpha_1,\beta_1)}(1) = \frac{\Gamma(n+\alpha_1+1)}{n!\ \Gamma(\alpha_1+1)},$$

$$P_n^{(\alpha_1,\beta_1)}(-1) = (-1)^n \frac{\Gamma(n+\beta_1+1)}{n!\ \Gamma(\beta_1+1)}.$$

In this form the polynomials may be generated using the standard recurrence relation of Jacobi polynomials starting from $P_0^{(\alpha_1,\beta_1)}(x) = 1$, and $P_1^{(\alpha_1,\beta_1)}(x) = \frac{1}{2}[\alpha_1 - \beta_1 + (\lambda+1)x]$, or obtained from Rodrigue's formula

$$P_n^{(\alpha_1,\beta_1)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha_1}(1+x)^{-\beta_1} \times$$

$$D^n\left[(1-x)^{\alpha_1+n}(1+x)^{\beta_1+n}\right]$$

where $\lambda = \alpha_1 + \beta_1 + 1$, $D = \frac{d}{dx}$.

The classical Ultraspherical polynomials are Jacobi polynomials with $\alpha_1 = \beta_1$ and are thus a subclass of the Jacobi polynomials. It is convenient to weight the classical Ultraspherical polynomials

$$C_n^{(\gamma)}(x) = \frac{n!\Gamma\left(\gamma+\frac{1}{2}\right)}{\Gamma\left(n+\gamma+\frac{1}{2}\right)} P_n^{\left(\gamma-\frac{1}{2},\gamma-\frac{1}{2}\right)}(x), \text{ by taking}$$

$$C_n^{(\gamma)}(1) = 1, (n = 0,1,2,\ldots);$$

this is not the usual standardization, but has the desirable properties that the $C_n^{(0)}(x)$ are identical with the classical Chebyshev polynomials of the first kind $T_n(x)$, the $C_n^{\left(\frac{1}{2}\right)}(x)$ are the Legendre polynomials $P_n(x)$, and $C_n^{(1)}(x)$ are equal to $\frac{1}{n+1}U_n(x)$, where $U_n(x)$ are the classical Chebyshev polynomials of the second kind [4], and If $\alpha_1 = -\beta_1 = \pm\frac{1}{2}$, then this give the classical Chebyshev polynomials $V_n(x)$ *and* $W_n(x)$ *of the third and fourth kinds which are defined, respectively, by*

$$V_n(x) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(x),$$

$$W_n(x) = \frac{2^{2n}}{\binom{2n}{n}} P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(x),$$

$$\text{and } W_n(x) = (-1)^n V_n(-x).$$

It is to be noted here that the Chebyshev polynomials of the third and fourth kinds are not Ultraspherical polynomials as special cases, but they are only Jacobi polynomials [18].

3. Computation of K Times Derivatives of the Generalized Jacobi Polynomials

The main objective of this section is to prove the following theorem for the kth derivatives of Generalized Jacobi polynomials $P_n^{(\alpha_1,\beta_1;\,\alpha,\beta;p)}(x)$.

Theorem 1 For *GGHF*, we have the following differentiation formula:

$$\frac{d^k}{dz^k}\left\{F_p^{(\alpha,\beta)}(a,b;c;z)\right\} = \frac{(a)_k(b)_k}{(c)_k} F_p^{(\alpha,\beta)}(a+k, b+k; c+k; z). \quad (15)$$

(For proof, see [1]).

Theorem 2 For the derivatives of generalized Jacobi polynomials, we have

$$D^k P_n^{(\alpha_1,\beta_1;\,\alpha,\beta;p)}(x) = 2^{-k}\,(n+\alpha_1+\beta_1+1)_k\,P_{n-k}^{(\alpha_1+k,\beta_1+k;\,\alpha,\beta;p)}(x). \quad (16)$$

Proof.

The generalization of Jacobi polynomials (13) (i) and Theorem1 with the parameters,

$$a=-n, b=\ n+\alpha_1+\beta_1+1, c=\alpha_1+1, z=\frac{1-x}{2},$$

Immediately, yields,

$$\frac{d^k}{dx^k}\left\{F_p^{(\alpha,\beta)}\left(-n,n+\alpha_1+\beta_1+1;\alpha_1+1;\frac{1-x}{2}\right)\right\}=$$
$$(-\frac{1}{2})^k\ \frac{(-n)_k(n+\alpha_1+\beta_1+1)_k}{(\alpha_1+1)_k}F_p^{(\alpha,\beta)}\left(-n+k,n+\alpha_1+\beta_1+k+1;\alpha_1+k+1;\frac{1-x}{2}\right),$$

Multiplying by $\frac{(\alpha_1+1)_n}{n!}$, we get,

$$\frac{d^k}{dx^k}\left\{\frac{(\alpha_1+1)_n}{n!}F_p^{(\alpha,\beta)}\left(-n,n+\alpha_1+\beta_1+1;\alpha_1+1;\frac{1-x}{2}\right)\right\}=\frac{(\alpha_1+1)_n}{n!}(-1)^k(2)^{-k}\ \frac{(-n)_k(n+\alpha_1+\beta_1+1)_k}{(\alpha_1+1)_k}$$
$$\times F_p^{(\alpha,\beta)}\left(-n+k,n+\alpha_1+\beta_1+k+1;\alpha_1+k+1;\frac{1-x}{2}\right), \quad (17)$$

Now making use of generalization of Jacobi polynomials(13)(i), gives

$$D^k P_n^{(\alpha_1,\beta_1;\,\alpha,\beta;p)}(x) = \frac{(\alpha_1+1)_n}{n!}(-1)^k(2)^{-k}\ \frac{(-n)_k(n+\alpha_1+\beta_1+1)_k}{(\alpha_1+1)_k}\times$$
$$F_p^{(\alpha,\beta)}\left(-n+k,n+\alpha_1+\beta_1+k+1;\alpha_1+k+1;\frac{1-x}{2}\right),$$
$$=(-1)^k(2)^{-k}(n+\alpha_1+\beta_1+1)_k\,\frac{\Gamma(n+\alpha_1+1)}{n!\,\Gamma(\alpha_1+k+1)}\,\frac{\Gamma(-n+k)}{\Gamma(-n)}\times$$
$$F_p^{(\alpha,\beta)}\left(-n+k,n+\alpha_1+\beta_1+k+1;\alpha_1+k+1;\frac{1-x}{2}\right),$$

and after recalling the following facts,

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},\ \sin\pi(n-k)=\ (-1)^k\sin(n\pi),$$
$$-n\,\Gamma(-n)=\Gamma(1-n).$$

We have

$$D^k P_n^{(\alpha_1,\beta_1;\,\alpha,\beta;p)}(x)=(2)^{-k}(n+\alpha_1+\beta_1+1)_k\times$$
$$(-1)^k\frac{\pi}{\sin\pi(n-k)}\,\frac{\Gamma(n+\alpha_1+1)}{\Gamma(\alpha_1+k+1)}\,\frac{1}{n!\,\Gamma(-n)(-n+k)\Gamma(n-k)}$$
$$\times\ F_p^{(\alpha,\beta)}\left(-n+k,n+\alpha_1+\beta_1+k+1;\alpha_1+k+1;\frac{1-x}{2}\right),$$
$$=(2)^{-k}(n+\alpha_1+\beta_1+1)_k\times$$
$$\frac{\pi}{\sin n\pi}\,\frac{\Gamma(n+\alpha_1+1)}{\Gamma(\alpha_1+k+1)}\,\frac{1}{n!\,\Gamma(-n)(-n+k)\Gamma(n-k)}\times$$
$$F_p^{(\alpha,\beta)}\left(-n+k,n+\alpha_1+\beta_1+k+1;\alpha_1+k+1;\frac{1-x}{2}\right),$$
$$=(2)^{-k}(n+\alpha_1+\beta_1+1)_k\times$$
$$\frac{\Gamma(1-n)}{(n-k)\Gamma(n-k)}\,\frac{\Gamma(n+\alpha_1+1)}{\Gamma(\alpha_1+k+1)}\,\frac{1}{(-n)\Gamma(-n)}\times$$
$$F_p^{(\alpha,\beta)}\left(-n+k,n+\alpha_1+\beta_1+k+1;\alpha_1+k+1;\frac{1-x}{2}\right),$$
$$=(2)^{-k}(n+\alpha_1+\beta_1+1)_k\,\frac{1}{(n-k)!}\,\frac{\Gamma(n+\alpha_1+1)}{\Gamma(\alpha_1+k+1)}\times$$
$$F_p^{(\alpha,\beta)}\left(-n+k,n+\alpha_1+\beta_1+k+1;\alpha_1+k+1;\frac{1-x}{2}\right),$$
$$=2^{-k}(n+\alpha_1+\beta_1+1)_k\,\frac{\Gamma(n+\alpha_1+1)}{(n-k)!\,\Gamma(\alpha_1+k+1)}\times$$
$$F_p^{(\alpha,\beta)}\left(-n+k,n+\alpha_1+\beta_1+k+1;\alpha_1+k+1;\frac{1-x}{2}\right).$$

Finally, making use of the generalization of Jacobi polynomials (13)(i), immediately yields

$$=2^{-k}(n+\alpha_1+\beta_1+1)_k\,P_{n-k}^{(\alpha_1+k,\beta_1+k;\,\alpha,\beta;p)}(x), \quad (18)$$

and this completes the proof of Theorem 2.

Now, we need the following lemma

Lemma 1. Suppose

$$P_n^{(\gamma,\delta;\,\alpha,\beta;p)}(x)=\sum_{k=0}^{n}C_{nk}(\gamma,\delta,\alpha_1,\beta_1)\,P_k^{(\alpha_1,\beta_1;\,\alpha,\beta;p)}(x), \quad (19)$$

then

$$C_{nk}(\gamma,\delta,\alpha_1,\beta_1)=\frac{(n+\gamma+\delta+1)_k(k+\gamma+1)_{n-k}\Gamma(k+\alpha_1+\beta_1+1)}{(n-k)!\,\Gamma(2k+\alpha_1+\beta_1+1)}\times$$
$${}_3F_2\left(\begin{matrix}-n+k\,,n+k+\gamma+\delta+1\,,k+\alpha_1+1\\ k+\gamma+1\,,2k+\alpha_1+\beta_1+2\end{matrix};1\right). \quad (20)$$

Proof

Substitution of (20) into the R. H. S of (19) and making use of the generalization of Jacobi polynomials(13)(i), gives

$$\sum_{k=0}^{n}C_{nk}(\gamma,\delta,\alpha_1,\beta_1)\,P_k^{(\alpha_1,\beta_1;\,\alpha,\beta;p)}(x)$$
$$=\sum_{k=0}^{n}\frac{(n+\gamma+\delta+1)_k(k+\gamma+1)_{n-k}\Gamma(k+\alpha_1+\beta_1+1)}{(n-k)!\,\Gamma(2k+\alpha_1+\beta_1+1)}\times$$

$$ {}_3F_2\left(\begin{matrix}-n+k\,,n+k+\gamma+\delta+1\,,k+\alpha_1+1\\ k+\gamma+1\,,2k+\alpha_1+\beta_1+2\end{matrix};1\right) $$

$$ \times\frac{\Gamma(k+\alpha_1+1)}{k!\,\Gamma(\alpha_1+1)}F_p^{(\alpha,\beta)}\left(-k,k+\alpha_1+\beta_1+1;\alpha_1+1;\frac{1-x}{2}\right). $$

By making use of the *Pfaff – Saalschütz* theorem [10], when n is an even positive integer,

$$ {}_3F_2\left(\begin{matrix}-n\,,a\,,b\\ c\,,d\end{matrix};1\right)=\frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}, \tag{21} $$

and we have

$$ \sum_{k=0}^{n}C_{nk}(\gamma,\delta,\alpha_1,\beta_1)\,P_k^{(\alpha_1,\beta_1;\,\alpha,\beta;p)}(x) $$

$$ =\sum_{k=0}^{n}\frac{(n+\gamma+\delta+1)_k\Gamma(k+\alpha_1+\beta_1+1)}{(n-k)!\,\Gamma(2k+\alpha_1+\beta_1+1)}\times $$

$$ \frac{(-n-\delta)_{n-k}(\gamma-\alpha_1)_{n-k}}{(-n-\delta-k-\alpha_1-1)_{n-k}}\,\frac{\Gamma(k+\alpha_1+1)}{k!\,\Gamma(\alpha_1+1)}\times $$

$$ \sum_{r=0}^{\infty}\frac{\Gamma(-k+r)}{\Gamma(-k)}\frac{\beta_p{}^{(\alpha,\beta)}(k+\alpha_1+\beta_1+r+1,-k-\beta_1)}{\beta(k+\alpha_1+\beta_1+1,-k-\beta_1)}\frac{(\frac{1-x}{2})^r}{r!}, \tag{22} $$

Expanding (22) and collecting similar terms, we obtain

$$ \sum_{k=0}^{n}C_{nk}(\gamma,\delta,\alpha_1,\beta_1)\,P_k^{(\alpha_1,\beta_1;\,\alpha,\beta;p)}(x) $$

$$ =\frac{\Gamma(\gamma+n+1)}{n!\,\Gamma(\gamma+1)}\sum_{r=0}^{\infty}(-n)_r\frac{\beta_p{}^{(\alpha,\beta)}(n+\gamma+\delta+r+1,-n-\delta)}{\beta(n+\gamma+\delta+1,-n-\delta)} $$

$$ \times\frac{(\frac{1-x}{2})^r}{r!}, $$

$$ =\frac{\Gamma(\gamma+n+1)}{\Gamma(\gamma+1)n!}F_p{}^{(\alpha,\beta)}\left(-n,n+\gamma+\delta+1;\gamma+1;\frac{1-x}{2}\right) $$

$$ =\frac{(\gamma+1)_n}{n!}F_p{}^{(\alpha,\beta)}\left(-n,n+\gamma+\delta+1;\gamma+1;\frac{1-x}{2}\right) $$

$$ =P_n^{(\gamma,\delta;\,\alpha,\beta;p)}(x). $$

This completes the proof of Lemma 1.

Remark 1.

$$ D^kP_n^{(\alpha_1,\beta_1;\,\alpha,\beta;p)}(x)=2^{-k}(n+\alpha_1+\beta_1+1)_k\times $$

$$ \sum_{i=0}^{n-k}C_{n-k,i}(\alpha_1+k,\beta_1+k;\alpha_1,\beta_1)\,P_i^{(\alpha_1,\beta_1;\,\alpha,\beta;p)}(x). \tag{23} $$

Where

$$ C_{n-k,i}(\alpha_1+k,\beta_1+k;\alpha_1,\beta_1) $$

$$ =\frac{(n+k+\alpha_1+\beta_1+1)_i(k+i+\alpha_1+1)_{n-i-k}\Gamma(i+\alpha_1+\beta_1+1)}{(n-i-k)!\,\Gamma(2i+\alpha_1+\beta_1+1)}\times $$

$$ {}_3F_2\left(\begin{matrix}-n+k+i\,,n+k+i+\alpha_1+\beta_1+1\,,i+\alpha_1+1\\ k+i+\alpha_1+1\,,2i+\alpha_1+\beta_1+2\end{matrix};1\right), $$

$$ n\geq 0,k\geq 1. \tag{24} $$

It is worthy of note that the special cases may be obtained from formulas (16), (19), and (23) by taking p=0, which they are in complete agreement with those obtained by (Doha (2002)[4], formulas (17),(18), (19) and (15), respectively.

Next, we use the generalization of Gauss hypergeometric functions (8) and the generalization of Jacobi polynomials(13)(ii)
to obtain the kth derivative of generalized Ultrasphereceial polynomials, which are defined by,

$$ C_n{}^{(\alpha,\beta;\gamma;p)}(x) $$

$$ =\frac{(2\gamma)_n}{n!}\sum_{r=0}^{\infty}(-n)_r\frac{\beta_p{}^{(\alpha,\beta)}\left(n+2\gamma+r,\frac{1}{2}-\gamma-n\right)}{\beta\left(n+2\gamma,\frac{1}{2}-\gamma-n\right)}\frac{\left(\frac{1-x}{2}\right)^r}{r!}, $$

we call $C_n{}^{(\alpha,\beta;\gamma;p)}(x)$ the generalized Ultraspherical functions *(GUFs)* and observe that,

$$ C_n{}^{(\alpha,\alpha;\gamma;\,p)}(x)=C_n{}^{(\gamma;\,p)}(x) $$

$$ C_n{}^{(\alpha,\beta;0;p)}(x)=T_n{}^{(\alpha,\beta;p)}(x),C_n{}^{(\alpha,\beta;0;0)}(x)=T_n(x), $$

$$ C_n{}^{\left(\alpha,\beta;\frac{1}{2};p\right)}(x)=P_n{}^{(\alpha,\beta;p)}(x)\,,C_n{}^{\left(\alpha,\beta;\frac{1}{2};\,0\right)}(x)=P_n(x), $$

$$ C_n{}^{(\alpha,\beta;1;p)}(x)=\frac{1}{n+1}\,U_n{}^{(\alpha,\beta;p)}(x)\,, $$

$$ C_n{}^{(\alpha,\beta;1;\,0)}(x)=\frac{1}{n+1}U_n(x). \tag{25} $$

The particular expressions for the generalized ultraspherical polynomials may be derived as a special case of the main work. We give these as corollaries as follows:

Corollary 1.

If $\alpha_1=\beta_1$, and each is replaced by $\gamma-\frac{1}{2}$, then equation (23) will be as of the form

$$ D_x{}^kC_{k+2j}{}^{(\alpha,\,\beta;\gamma;\,p\,)}(x)=\frac{2^k(k+2j)!}{(k-1)!\,\Gamma(k+2j+2\gamma)} $$

$$ \sum_{i=0}^{j}\frac{(2i+\gamma)\Gamma(2i+2\gamma)\Gamma(k+j+i+\gamma)}{(2i)!\,(j-i)!\,\Gamma(j+i+\gamma+1)}(k+j-i-1)!\times $$

$$ C_{2i}{}^{(\alpha,\,\beta;\gamma;\,p\,)}(x), \tag{26} $$

and

$$ D_x{}^kC_{k+2j+1}{}^{(\alpha,\,\beta;\gamma;p\,)}(x)=\frac{2^k(k+2j+1)!}{(k-1)!\,\Gamma(k+2j+2\gamma+1)}\times $$

$$ \sum_{i=0}^{j}\frac{(2i+\gamma+1)\Gamma(2i+2\gamma+1)\Gamma(k+j+i+\gamma+1)}{(2i+1)!(j-i)!\Gamma(j+i+\gamma+2)}(k+j-i-1)!\times $$

$$ C_{2i+1}{}^{(\alpha,\,\beta;\gamma;p\,)}(x), \tag{27} $$

where

$$C_n^{(\alpha,\beta;\gamma;p)}(x) = \frac{(2\gamma)_n}{n!}\sum_{r=0}^{\infty}(-n)_r \frac{\beta_p^{(\alpha,\beta)}\left(n+2\gamma+r,\frac{1}{2}-\gamma-n\right)}{\beta\left(n+2\gamma,\frac{1}{2}-\gamma-n\right)} \frac{\left(\frac{1-x}{2}\right)^r}{r!}, \tag{28}$$

Proof.

Set $\alpha_1 = \beta_1 = \gamma - \frac{1}{2}$ in (23) then expanding the equation and collecting similar terms, we get the following two formulas that give explicitly the derivatives of generalized Ultraspherical polynomials of any degree and for any order in terms of generalized Ultraspherical polynomials in the forms:

$$D_x{}^k C_{k+2j}{}^{(\alpha,\beta;\gamma;p)}(x) = \frac{2^k(k+2j)!}{(k-1)!\,\Gamma(k+2j+2\gamma)} \times \sum_{i=0}^{j}\frac{(2i+\gamma)\Gamma(2i+2\gamma)\Gamma(k+j+i+\gamma)}{(2i)!\,(j-i)!\,\Gamma(j+i+\gamma+1)}(k+j-i-1)! \times \frac{(2\gamma)_{2i}}{(2i)!}\sum_{r=0}^{\infty}(-2i)_r \frac{\beta_p^{(\alpha,\beta)}\left(2i+2\gamma+r,\frac{1}{2}-\gamma-2i\right)}{\beta\left(2i+2\gamma,\frac{1}{2}-\gamma-2i\right)} \times \frac{\left(\frac{1-x}{2}\right)^r}{r!},$$

and

$$D_x{}^k C_{k+2j+1}{}^{(\alpha,\beta;\gamma;p)}(x) = \frac{2^k(k+2j+1)!}{(k-1)!\,\Gamma(k+2j+2\gamma+1)} \times \sum_{i=0}^{j}\frac{(2i+\gamma+1)\,\Gamma(2i+2\gamma+1)\,\Gamma(k+j+i+\gamma+1)}{(2i+1)!\,(j-i)!\,\Gamma(j+i+\gamma+2)}(k+j-i-1)! \times \frac{(2\gamma)_{2i+1}}{(2i+1)!}\sum_{r=0}^{\infty}(-2i-1)_r \frac{\beta_p^{(\alpha,\beta)}\left(2i+2\gamma+r+1,\frac{1}{2}-\gamma-2i-1\right)}{\beta\left(2i+2\gamma+1,\frac{1}{2}-\gamma-2i-1\right)} \times \frac{\left(\frac{1-x}{2}\right)^r}{r!},$$

Making use of (28), gives (26) and (27) directly.

Corollary 2.

If γ=0, (26) and (27) give derivative of generalized Chebyshev polynomials *of the first kind*,

$$D_x{}^k C_{k+2j}{}^{(\alpha,\beta;0;p)}(x) = \frac{2^k(k+2j)!}{(k-1)!\,\Gamma(k+2j)} \times \sum_{r=0}^{\infty}\sum_{i=0}^{j}\frac{\Gamma(k+j+i)}{(j-i)!\,\Gamma(j+i+1)}(k+j-i-1)! \times \frac{(-2i)_r}{(2i)!}\frac{\beta_p^{(\alpha,\beta)}\left(2i+r,\frac{1}{2}-2i\right)}{\beta\left(2i,\frac{1}{2}-2i\right)} \times \frac{\left(\frac{1-x}{2}\right)^r}{r!}, \tag{29}$$

$$D_x{}^k C_{k+2j+1}{}^{(\alpha,\beta;0;p)}(x) = \frac{2^k(k+2j+1)!}{(k-1)!\,\Gamma(k+2j+1)} \times \sum_{r=0}^{\infty}\sum_{i=0}^{j}\frac{\Gamma(k+j+i+1)}{(j-i)!\,\Gamma(j+i+2)}(k+j-i-1)! \times \frac{(-2i-1)_r}{(2i+1)!}\frac{\beta_p^{(\alpha,\beta)}\left(2i+r+1,-2i-\frac{1}{2}\right)}{\beta\left(2i+1,-2i-\frac{1}{2}\right)} \frac{\left(\frac{1-x}{2}\right)^r}{r!}, \tag{30}$$

and accordingly,

$$D_x{}^k C_{k+2j}{}^{(\alpha,\beta;0;p)}(x) = \frac{2^k(k+2j)}{(k-1)!} \sum_{i=0}^{j\,\prime}\frac{\Gamma(k+j+i)}{(j-i)!\,(j+i)!}(k+j-i-1)! \times \sum_{r=0}^{\infty}\frac{(-2i)_r}{(2i)!}\frac{\beta_p^{(\alpha,\beta)}\left(2i+r,\frac{1}{2}-2i\right)}{\beta\left(2i,\frac{1}{2}-2i\right)} \times \frac{\left(\frac{1-x}{2}\right)^r}{r!}, \tag{31}$$

and

$$D_x{}^k C_{k+2j+1}{}^{(\alpha,\beta;0;p)}(x) = \frac{2^k(k+2j+1)}{(k-1)!} \times \sum_{i=0}^{j}\frac{(k+j+i)!}{(j-i)!\,(j+i+1)!}(k+j-i-1)! \times \sum_{r=0}^{\infty}\frac{(-2i-1)_r}{(2i+1)!}\frac{\beta_p^{(\alpha,\beta)}\left(2i+r+1,-\frac{1}{2}-2i\right)}{\beta\left(2i+1,-\frac{1}{2}-2i\right)} \times \frac{\left(\frac{1-x}{2}\right)^r}{r!}, \tag{32}$$

In addition, using the generalization of Jacobi polynomials (13) (iii), we have

$$D_x{}^k T_{k+2j}{}^{(\alpha,\beta;p)}(x) = \frac{2^k(k+2j)}{(k-1)!} \times \sum_{i=0}^{j\,\prime}\frac{\Gamma(k+j+i)}{(j-i)!(j+i)!}(k+j-i-1)!\,T_{2i}{}^{(\alpha,\beta;p)}(x), \tag{33}$$

and

$$D_x{}^k T_{k+2j+1}{}^{(\alpha,\beta;p)}(x) = \frac{2^k(k+2j+1)}{(k-1)!} \times \sum_{i=0}^{j}\frac{(k+j+i)!}{(j-i)!(j+i+1)!}(k+j-i-1)!\,T_{2i+1}{}^{(\alpha,\beta;p)}(x). \tag{34}$$

Corollary 3.

Setting P=0 in (31) and (32), we get the derivatives of the classical Chebyshev polynomials of the first kind

$$D_x{}^k C_{k+2j}{}^{(\alpha,\beta;0;0)}(x) = \frac{2^k(k+2j)}{(k-1)!} \times \sum_{i=0}^{j\,\prime}\frac{\Gamma(k+j+i)}{(j-i)!(j+i)!}(k+j-i-1)!\,T_{2i}{}^{(\alpha,\beta;0)}(x), \tag{35}$$

and

$$D_x{}^k C_{k+2j+1}{}^{(\alpha,\beta;0;0)}(x) = \frac{2^k(k+2j+1)}{(k-1)!} \times \sum_{i=0}^{j}\frac{(k+j+i)!}{(j-i)!(j+i+1)!}(k+j-i-1)!\,T_{2i+1}{}^{(\alpha,\beta;0)}(x), \tag{36}$$

using the generalization of Jacobi polynomials(13)(iii), we obtain

$$D_x{}^k C_{k+2j}{}^{(\alpha,\beta;0;0)}(x) = \frac{2^k(k+2j)}{(k-1)!} \times \sum_{i=0}^{j\,\prime}\frac{\Gamma(k+j+i)}{(j-i)!(j+i)!}(k+j-i-1)!\frac{1}{(2i)!}{}_2F_1\left(-2i,2i;\frac{1}{2};\frac{1-x}{2}\right), \tag{37}$$

and

$$D_x{}^k C_{k+2j+1}{}^{(\alpha,\beta;0;0)}(x) = \frac{2^k(k+2j+1)}{(k-1)!} \times$$

$$\sum_{i=0}^{j} \frac{(k+j+i)!}{(j-i)!\,(j+i+1)!}\,(k+j-i-1)! \times$$

$$\frac{1}{(2i+1)!}\,{}_2F_1\left(-2i-1, 2i+1; \tfrac{1}{2}; \tfrac{1-x}{2}\right), \quad (38)$$

Making use of the generalization of Jacobi polynomials(13)(iii), then equations (37) and (38) may be written as,

$$D_x{}^k T_{k+2j}(x) = \frac{2^k(k+2j)}{(k-1)!} \times$$

$$\sum_{i=0}^{j\,\prime} \frac{\Gamma(k+j+i)}{(j-i)!(j+i)!}\,(k+j-i-1)!\; T_{2i}(x)\,, \quad (39)$$

$$D_x{}^k T_{k+2j+1}(x) = \frac{2^k(k+2j+1)}{(k-1)!} \times$$

$$\sum_{i=0}^{j} \frac{(k+j+i)!}{(j-i)!(j+i+1)!}\,(k+j-i-1)!\, T_{2i+1}(x)\,, \quad (40)$$

The two relations (39) and (40) are in complete agreement with those obtained in (Doha (1991) [3], P. 120, formulas (26) and (27), respectively).

($\sum{}'$ means that the first term is taking with factor $\frac{1}{2}$).

Corollary 4.

If γ= 1, (26) and (27) give derivatives of generalized Chebyshev polynomials of the second kind,

$$D_x{}^k C_{k+2j}{}^{(\alpha,\beta;1;p)}(x) = \frac{2^k}{(k-1)!}\sum_{i=0}^{j} \frac{(2i+1)^3(k+j-i-1)!(k+j+i)!}{(k+2j+1)(j+i+1)!} \times \sum_{r=0}^{\infty} \frac{(-2i)_r}{(j-i)!}\,\frac{\beta_p{}^{(\alpha,\beta)}\left(2i+r+2,-\frac{1}{2}-2i\right)}{\beta\left(2i+2,-\frac{1}{2}-2i\right)} \times \frac{\left(\frac{1-x}{2}\right)^r}{r!}, \quad (41)$$

and

$$D_x{}^k C_{k+2j+1}{}^{(\alpha,\beta;1;p)}(x) = \frac{2^{k+1}}{(k-1)!\,(k+2j+2)} \times \sum_{i=0}^{j} \frac{(k+j+i+1)!}{(j-i)!\,(j+i+2)!}(k+j-i-1)!$$

$$\times \sum_{r=0}^{\infty}(-2i-1)_r (i+1)^3\, \frac{\beta_p{}^{(\alpha,\beta)}\left(2i+r+3,-2i-\frac{3}{2}\right)}{\beta\left(2i+3,-2i-\frac{3}{2}\right)} \times \frac{\left(\frac{1-x}{2}\right)^r}{r!}, \quad (42)$$

and accordingly,

$$D_x{}^k C_{k+2j}{}^{(\alpha,\beta;1;p)}(x) = \frac{2^k}{(k-1)!}\sum_{i=0}^{j} \frac{(2i+1)(k+j-i-1)!(k+j+i)!}{(j-i)!(j+i+1)!}\,\frac{(2i+1)^2}{(k+2j+1)} \times \sum_{r=0}^{\infty} (-2i)_r\, \frac{\beta_p{}^{(\alpha,\beta)}\left(2i+r+2,-\frac{1}{2}-2i\right)}{\beta\left(2i+2,-\frac{1}{2}-2i\right)} \times \frac{\left(\frac{1-x}{2}\right)^r}{r!}, \quad (43)$$

and

$$D_x{}^k C_{k+2j+1}{}^{(\alpha,\beta;1;p)}(x) = \frac{2^{k+1}}{(k-1)!}\sum_{i=0}^{j} \frac{(i+1)(k+j-i-1)!(k+j+i+1)!}{(j-i)!(j+i+2)!}\,\frac{(2i+2)^2}{(k+2j+2)} \times \sum_{r=0}^{\infty} (-2i-1)_r\, \frac{\beta_p{}^{(\alpha,\beta)}\left(2i+r+3,-\frac{3}{2}-2i\right)}{\beta\left(2i+3,-\frac{3}{2}-2i\right)} \times \frac{\left(\frac{1-x}{2}\right)^r}{r!}, \quad (44)$$

Now, using the generalization of Jacobi polynomials (13)(iv), we obtain

$$D_x{}^k C_{k+2j}{}^{(\alpha,\beta;1;p)}(x) = \frac{2^k}{(k-1)!}\frac{1}{(k+2j+1)}\sum_{i=0}^{j} \frac{(2i+1)(k+j-i-1)!(k+j+i)!}{(j-i)!(j+i+1)!} \times U^{(\alpha,\beta;p)}{}_{2i}(x), \quad (45)$$

and

$$D_x{}^k C_{k+2j+1}{}^{(\alpha,\beta;1;p)}(x) = \frac{2^{k+1}}{(k-1)!}\frac{1}{(k+2j+2)}\sum_{i=0}^{j} \frac{(i+1)(k+j-i-1)!(k+j+i+1)!}{(j-i)!(j+i+2)!} \times U^{(\alpha,\beta;p)}{}_{2i+1}(x), \quad (46)$$

Corollary 5.

If we put, p=0, into (45) and (46), and again using the generalization of Jacobi polynomials(13)(iv), we get the derivatives of the classical Chebyshev polynomials of the second kind,

$$D_x{}^k\, U_{k+2j}(x) = \frac{2^k}{(k-1)!}\sum_{i=0}^{j} \frac{(2i+1)(k+j-i-1)!(k+j+i)!}{(j-i)!(j+i+1)!} \times U_{2i}(x), \quad (47)$$

and

$$D_x{}^k\, U_{k+2j+1}(x) = \frac{2^{k+1}}{(k-1)!}\sum_{i=0}^{j} \frac{(i+1)(k+j-i-1)!(k+j+i+1)!}{(j-i)!(j+i+2)!} \times U_{2i+1}(x), \quad (48)$$

Where $C_n{}^{(1)}(x) = \frac{U_n(x)}{n+1}$,

It is to be noted here that, (47) and (48) are in complete agreement with those obtained by Doha (1991) [3], formulas (28), and (29), respectively.

Corollary 6.

If $\gamma = \frac{1}{2}$, (26) and (27) give derivatives of generalized Legendre polynomials.

$$D_x{}^k C_{k+2j}{}^{\left(\alpha,\beta;\frac{1}{2};p\right)}(x) = \frac{2^{k-1}}{(k-1)!}\sum_{i=0}^{j}\frac{(4i+1)(k+j-i-1)!\Gamma\left(k+j+i+\frac{1}{2}\right)}{(j-i)!\Gamma\left(j+i+\frac{3}{2}\right)}\times\sum_{r=0}^{\infty}(-2i)_r\frac{\beta_p{}^{(\alpha,\beta)}(2i+r+1,-2i)}{\beta(2i+1,-2i)}\frac{\left(\frac{1-x}{2}\right)^r}{r!}, \tag{49}$$

and

$$D_x{}^k C_{k+2j+1}{}^{\left(\alpha,\beta;\frac{1}{2};p\right)}(x) = \frac{2^{k-1}}{(k-1)!}\times\sum_{i=0}^{j}\frac{(4i+3)\Gamma\left(k+j+i+\frac{3}{2}\right)}{(j-i)!\Gamma\left(j+i+\frac{5}{2}\right)}(k+j-i-1)!\times\sum_{r=0}^{\infty}(-2i-1)_r\frac{\beta_p{}^{(\alpha,\beta)}(2i+r+2,-2i-1)}{\beta(2i+2,-2i-1)}\frac{\left(\frac{1-x}{2}\right)^r}{r!}, \tag{50}$$

Substituting the generalization of Jacobi polynomials (13)(v) into (50) and using the Legendre duplication formulae

$$\Gamma(2m) = \frac{1}{\sqrt{2\pi}}2^{2m-\frac{1}{2}}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right),$$

we have directly,

$$D_x{}^k C_{k+2j}{}^{\left(\alpha,\beta;\frac{1}{2};p\right)}(x) = \frac{1}{2^{k-2}(k-1)!}\sum_{i=0}^{j}\frac{(4i+1)(k+j-i-1)!(2k+2j+2i-1)!}{(j-i)!(2j+2i+2)!}\times\frac{(j+i+1)!}{(k+j+i-1)!}P_{2i}{}^{(\alpha,\beta;p)}(x), \tag{51}$$

and

$$D_x{}^k C_{k+2j+1}{}^{\left(\alpha,\beta;\frac{1}{2};p\right)}(x) = \frac{1}{2^{k-2}(k-1)!}\times\sum_{i=0}^{j}\frac{(4i+3)(k+j-i-1)!}{(j-i)!(2j+2i+4)!}(2k+2j+2i+1)!\times\frac{(j+i+2)!}{(k+j+i)!}P_{2i+1}{}^{(\alpha,\beta;p)}(x), \tag{52}$$

Corollary 7.

Setting p=0, in (51) and (52), we get the derivatives of the classical Legendre polynomials,

$$D_x{}^k P_{k+2j}(x) = \frac{1}{2^{k-2}(k-1)!}\sum_{i=0}^{j}\frac{(4i+1)(k+j-i-1)!(2k+2j+2i-1)!}{(j-i)!(2j+2i+2)!}\times\frac{(j+i+1)!}{(k+j+i-1)!}P_{2i}(x), \tag{53}$$

and

$$D_x{}^k P_{k+2j+1}(x) = \frac{1}{2^{k-2}(k-1)!}\times\sum_{i=0}^{j}\frac{(4i+3)(k+j-i-1)!(2k+2j+2i+1)!}{(j-i)!(2j+2i+4)!}\frac{(j+i+2)!}{(k+j+i)!}\times P_{2i+1}(x), \tag{54}$$

(53) and (54) are in complete agreement with those obtained by Doha (1991) [3], page 120, formulas (30) and (31).

Finally, using the generalization of the Gauss hypergeometric functions (8) and the generalization of Jacobi polynomials (13) (vi) and (vii), we have the kth derivative of generalized Chebyshev polynomials of the third and fourth kinds, $V_n{}^{(\alpha,\beta;p)}(x)$ and $W_n{}^{(\alpha,\beta;p)}(x)$, respectively.

Corollary 8.

For the two nonsymmetric special cases, $\alpha_1 = -\frac{1}{2}, \beta_1 = \frac{1}{2}$, and $\alpha_1 = \frac{1}{2}, \beta_1 = -\frac{1}{2}$, we have

$$D_x{}^k V_n{}^{(\alpha,\beta;p)}(x) = \frac{2^{2n-k}n!}{(2n)!}\sum_{i=0}^{n-k}\frac{(2i)!}{(2^{2i})^2\, i!}\frac{1}{\Gamma\left(i+\frac{1}{2}\right)}\frac{\Gamma(n+k+i+1)}{(n-i-k)!}\frac{\Gamma\left(n+\frac{1}{2}\right)}{(k-1)!}\times\begin{cases}\dfrac{\left(\frac{n-i+k-2}{2}\right)!\,\Gamma\left(\frac{n-i-k+1}{2}\right)}{\left(\frac{n+i-k}{2}\right)!\,\Gamma\left(\frac{n+i+k+1}{2}\right)}, & (n-i-k)\ even,\\[2ex] \dfrac{\left(\frac{n-i+k-1}{2}\right)!\,\Gamma\left(\frac{n-i-k+2}{2}\right)}{\left(\frac{n+i-k+1}{2}\right)!\,\Gamma\left(\frac{n+i+k+2}{2}\right)}, & (n-i-k)\ odd,\end{cases}\times V_i{}^{(\alpha,\beta;p)}(x),$$

$$i,n,k\in\mathbb{Z}\geq 0, k\geq 1 and\ i\leq n-k \tag{55}$$

and

$$D_x{}^k W_n{}^{(\alpha,\beta;p)}(x) = \frac{2^{2n-k}n!}{(2n)!}\sum_{i=0}^{n-k}\frac{(2i)!}{2^{4i-1}\, i!}\frac{\Gamma(n+k+i+1)}{(n-i-k)!}\times\frac{1}{(2n+1)}\frac{\Gamma\left(n+\frac{3}{2}\right)}{(k-1)!\,\Gamma\left(i+\frac{1}{2}\right)}\times\begin{cases}\dfrac{\left(\frac{n-i+k-2}{2}\right)!\,\Gamma\left(\frac{n-i-k+1}{2}\right)}{\left(\frac{n+i-k}{2}\right)!\,\Gamma\left(\frac{n+i+k+1}{2}\right)}, & (n-i-k)\ even,\\[2ex] -\dfrac{\left(\frac{n-i+k-1}{2}\right)!\,\Gamma\left(\frac{n-i-k+2}{2}\right)}{\left(\frac{n+i-k+1}{2}\right)!\,\Gamma\left(\frac{n+i+k+2}{2}\right)}, & (n-i-k)\ odd,\end{cases}\times W_i{}^{(\alpha,\beta;p)}(x),$$

$$i,n,k\in\mathbb{Z}\geq 0, k\geq 1 and\ i\leq n-k \tag{56}$$

Proof.

Setting $\alpha_1 = -\frac{1}{2}$, $\beta_1 = \frac{1}{2}$ and $\alpha_1 = \frac{1}{2}$, $\beta_1 = -\frac{1}{2}$, respectively, in the generalization of Jacobi polynomials (13) (i) and making use of the generalization of Jacobi polynomials (13) (vi), (vii), give

$$V_n{}^{(\alpha,\beta;p)}(x) = \frac{2^{2n}n!}{(2n)!}\left(\frac{1}{2}\right)_n\times\sum_{r=0}^{\infty}(-n)_r\frac{\beta_p{}^{(\alpha,\beta)}\left(n+r+1,-n-\frac{1}{2}\right)}{\beta\left(n+1,-n-\frac{1}{2}\right)}\times\frac{\left(\frac{1-x}{2}\right)^r}{r!}, \tag{57}$$

and

$$W_n{}^{(\alpha,\beta;p)}(x) = \frac{2^{2n}n!}{(2n)!}\left(\frac{3}{2}\right)_n\times\sum_{r=0}^{\infty}(-n)_r\frac{\beta_p{}^{(\alpha,\beta)}\left(n+r+1,-n+\frac{1}{2}\right)}{\beta\left(n+1,-n+\frac{1}{2}\right)}\times\frac{\left(\frac{1-x}{2}\right)^r}{r!}, \tag{58}$$

Substitution of (57) and (58) into equations (23) and (24), yields the kth derivatives of generalized Chebyshev polynomials of third and fourth kinds (with $\alpha_1 = -\frac{1}{2}, \beta_1 = \frac{1}{2}$ and $\alpha_1 = \frac{1}{2}, \beta_1 = -\frac{1}{2}$), respectively, as,

$$D_x{}^k \frac{2^{2n} n!}{(2n)!} \left(\frac{1}{2}\right)_n \times$$

$$\sum_{r=0}^{\infty} (-n)_r \frac{\beta_p{}^{(\alpha,\beta)}\left(n+r+1, -n-\frac{1}{2}\right)}{\beta\left(n+1, -n-\frac{1}{2}\right)} \times \frac{(\frac{1-x}{2})^r}{r!} =$$

$$\frac{2^{2n-k} n!}{(2n)!} \Gamma(n+k+1) \sum_{i=0}^{n-k} \frac{1}{(i!)^2 2^{2i}} \frac{(n+k+1)_i}{(n-i-k)!} \left(k+i+\frac{1}{2}\right)_{n-i-k} \times$$

$${}_3F_2 \left(\begin{matrix} -n+k+i, n+k+i+1, i+\frac{1}{2} \\ k+i+\frac{1}{2}, 2i+2 \end{matrix} ; 1 \right) \times$$

$$\left(\frac{1}{2}\right)_i \sum_{r=0}^{\infty} (-i)_r \frac{\beta_p{}^{(\alpha,\beta)}\left(i+r+1, -i-\frac{1}{2}\right)}{\beta\left(i+1, -i-\frac{1}{2}\right)} \times \frac{(\frac{1-x}{2})^r}{r!},$$

$$i, n, k \in \mathbb{Z} \geq 0, k \geq 1 and\ i \leq n-k, \quad (59)$$

and

$$D_x{}^k \frac{2^{2n} n!}{(2n)!} \left(\frac{3}{2}\right)_n \times$$

$$\sum_{r=0}^{\infty} (-n)_r \frac{\beta_p{}^{(\alpha,\beta)}\left(n+r+1, -n+\frac{1}{2}\right)}{\beta\left(n+1, -n+\frac{1}{2}\right)} \times \frac{(\frac{1-x}{2})^r}{r!} =$$

$$\frac{2^{2n-k} n!}{(2n)!} \Gamma(n+k+1) \sum_{i=0}^{n-k} \frac{1}{(i!)^2 2^{2i}} \frac{(n+k+1)_i}{(n-i-k)!} \left(k+i+\frac{3}{2}\right)_{n-i-k} \times$$

$${}_3F_2 \left(\begin{matrix} -n+k+i, n+k+i+1, i+\frac{3}{2} \\ k+i+\frac{3}{2}, 2i+2 \end{matrix} ; 1 \right) \times$$

$$\left(\frac{3}{2}\right)_i \sum_{r=0}^{\infty} (-i)_r \frac{\beta_p{}^{(\alpha,\beta)}\left(i+r+1, -i+\frac{1}{2}\right)}{\beta\left(i+1, -i+\frac{1}{2}\right)} \times \frac{(\frac{1-x}{2})^r}{r!},$$

$$i, n, k \in \mathbb{Z} \geq 0, k \geq 1 and\ i \leq n-k \quad (60)$$

The following Lemma is needed to proceed with the proof of corollary 8.

Lemma 2.

For all i, n, k∈ Z≥0 and i ≤n-k, we have

$${}_3F_2 \left(\begin{matrix} -n+k+i, n+k+i+1, i+\frac{1}{2} \\ k+i+\frac{1}{2}, 2i+2 \end{matrix} ; 1 \right) =$$

$$\frac{(2i)!\Gamma\left(i+k+\frac{1}{2}\right)}{2^{2i}(k-1)!\Gamma\left(i+\frac{1}{2}\right)} \times \begin{cases} \frac{\left(\frac{n-i+k-2}{2}\right)!\,\Gamma\left(\frac{n-i-k+1}{2}\right)}{\left(\frac{n+i-k}{2}\right)!\,\Gamma\left(\frac{n+i+k+1}{2}\right)}, & (n-i-k)\ even \\ \frac{\left(\frac{n-i+k-1}{2}\right)!\,\Gamma\left(\frac{n-i-k+2}{2}\right)}{\left(\frac{n+i-k+1}{2}\right)!\,\Gamma\left(\frac{n+i+k+2}{2}\right)}, & (n-i-k)\ odd \end{cases} \quad (61)$$

and

$${}_3F_2 \left(\begin{matrix} -n+k+i, n+k+i+1, i+\frac{3}{2} \\ k+i+\frac{3}{2}, 2i+2 \end{matrix} ; 1 \right) =$$

$$\frac{(2i)!\Gamma\left(i+k+\frac{3}{2}\right)}{2^{2i-1}(k-1)!(2n+1)\Gamma\left(i+\frac{1}{2}\right)} \times \begin{cases} \frac{\left(\frac{n-i+k-2}{2}\right)!\,\Gamma\left(\frac{n-i-k+1}{2}\right)}{\left(\frac{n+i-k}{2}\right)!\,\Gamma\left(\frac{n+i+k+1}{2}\right)}, & (n-i-k)\ even \\ \frac{-\left(\frac{n-i+k-1}{2}\right)!\,\Gamma\left(\frac{n-i-k+2}{2}\right)}{\left(\frac{n+i-k+1}{2}\right)!\,\Gamma\left(\frac{n+i+k+2}{2}\right)}, & (n-i-k)\ odd \end{cases} \quad (62)$$

For proof see[16].

Application of Lemma 2, given in ((61) and (62)), by substituting into (59) and (60), respectively, and making use of the generalization of Jacobi polynomials((vi), (vii)) yield:

$$D_x{}^k V_n{}^{(\alpha,\beta;p)}(x) =$$

$$\frac{2^{2n-k} n!}{(2n)!} \Gamma(n+k+1) \sum_{i=0}^{n-k} \frac{(2i)!}{(2^{2i})^2\, i!} \frac{\Gamma\left(k+i+\frac{1}{2}\right)}{(k-1)!\,\Gamma\left(i+\frac{1}{2}\right)} \frac{(n+k+1)_i}{(n-i-k)!}$$

$$\times \left(k+i+\frac{1}{2}\right)_{n-i-k}$$

$$\times \begin{cases} \frac{\left(\frac{n-i+k-2}{2}\right)!\ \Gamma\left(\frac{n-i-k+1}{2}\right)}{\left(\frac{n+i-k}{2}\right)!\ \Gamma\left(\frac{n+i+k+1}{2}\right)}, & (n-i-k)\ even \\ \frac{\left(\frac{n-i+k-1}{2}\right)!\ \Gamma\left(\frac{n-i-k+2}{2}\right)}{\left(\frac{n+i-k+1}{2}\right)!\ \Gamma\left(\frac{n+i+k+2}{2}\right)}, & (n-i-k)\ odd \end{cases} \times V_i{}^{(\alpha,\beta;p)}(x),$$

$$i, n, k \in \mathbb{Z} \geq 0, k \geq 1 and\ i \leq n-k. \quad (63)$$

and

$$D_x{}^k W_n{}^{(\alpha,\beta;p)}(x) =$$

$$\frac{2^{2n-k} n!}{(2n)!} \Gamma(n+k+1) \times$$

$$\sum_{i=0}^{n-k} \frac{(2i)!}{2^{4i-1}\, i!} \frac{(n+k+1)_i}{(2n+1)(n-i-k)!} \frac{\Gamma\left(k+i+\frac{3}{2}\right)}{(k-1)!\,\Gamma\left(i+\frac{1}{2}\right)} \times$$

$$\left(k+i+\frac{3}{2}\right)_{n-i-k} \times$$

$$\begin{cases} \frac{\left(\frac{n-i+k-2}{2}\right)!\ \Gamma\left(\frac{n-i-k+1}{2}\right)}{\left(\frac{n+i-k}{2}\right)!\ \Gamma\left(\frac{n+i+k+1}{2}\right)}, & (n-i-k)\ even \\ \frac{-\left(\frac{n-i+k-1}{2}\right)!\ \Gamma\left(\frac{n-i-k+2}{2}\right)}{\left(\frac{n+i-k+1}{2}\right)!\ \Gamma\left(\frac{n+i+k+2}{2}\right)}, & (n-i-k)\ odd \end{cases}$$

$$\times W_i{}^{(\alpha,\beta;p)}(x),$$

$$i, n, k \in \mathbb{Z} \geq 0, k \geq 1 and\ i \leq n-k. \quad (64)$$

which also may be written as

$$D_x{}^k V_n{}^{(\alpha,\beta;p)}(x) = \frac{2^{2n-k}n!}{(2n)!}\sum_{i=0}^{n-k}\frac{(2i)!}{(2^{2i})^2\, i!}\,\frac{\Gamma(n+k+i+1)}{\Gamma\left(i+\frac{1}{2}\right)(n-i-k)!}\,\frac{\Gamma\left(n+\frac{1}{2}\right)}{(k-1)!}\times \begin{cases}\dfrac{\left(\frac{n-i+k-2}{2}\right)!\ \Gamma(\frac{n-i-k+1}{2})}{\left(\frac{n+i-k}{2}\right)!\ \Gamma\left(\frac{n+i+k+1}{2}\right)}, (n-i-k)\ even,\\ \dfrac{\left(\frac{n-i+k-1}{2}\right)!\ \Gamma(\frac{n-i-k+2}{2})}{\left(\frac{n+i-k+1}{2}\right)!\ \Gamma\left(\frac{n+i+k+2}{2}\right)}, (n-i-k)\ odd,\end{cases} \times V_i{}^{(\alpha,\beta;p)}(x),$$

$$i,n,k \in \mathbb{Z} \geq 0, k \geq 1 and\ i \leq n-k.$$

and

$$D_x{}^k W_n{}^{(\alpha,\beta;p)}(x) = \frac{2^{2n-k}n!}{(2n)!}\sum_{i=0}^{n-k}\frac{(2i)!}{2^{4i-1}\, i!}\,\frac{1}{(2n+1)}\,\frac{\Gamma(n+k+i+1)}{(n-i-k)!}\,\frac{\Gamma\left(n+\frac{3}{2}\right)}{(k-1)!\,\Gamma\left(i+\frac{1}{2}\right)}\times \begin{cases}\dfrac{\left(\frac{n-i+k-2}{2}\right)!\ \Gamma(\frac{n-i-k+1}{2})}{\left(\frac{n+i-k}{2}\right)!\ \Gamma\left(\frac{n+i+k+1}{2}\right)}, (n-i-k)\ even,\\ -\dfrac{\left(\frac{n-i+k-1}{2}\right)!\ \Gamma(\frac{n-i-k+2}{2})}{\left(\frac{n+i-k+1}{2}\right)!\ \Gamma\left(\frac{n+i+k+2}{2}\right)}, (n-i-k)\ odd,\end{cases} \times W_i{}^{(\alpha,\beta;p)}(x),$$

$$i,n,k \in \mathbb{Z} \geq 0, k \geq 1 and\ i \leq n-k.$$

Corollary 9.

Setting p=0, in (55) and (56), gives the derivatives of the classical Chebyshev polynomials of the third and fourth kinds,

$$D_x{}^k V_n(x) = \frac{2^{2n-k}n!}{(2n)!}\times \sum_{i=0}^{n-k}\frac{(2i)!}{(2^{2i})^2\, i!}\,\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(i+\frac{1}{2}\right)(k-1)!}\,\frac{\Gamma(n+k+i+1)}{(n-i-k)!}\times \begin{cases}\dfrac{\left(\frac{n-i+k-2}{2}\right)!\ \Gamma(\frac{n-i-k+1}{2})}{\left(\frac{n+i-k}{2}\right)!\ \Gamma\left(\frac{n+i+k+1}{2}\right)}, (n-i-k)\ even\\ \dfrac{\left(\frac{n-i+k-1}{2}\right)!\ \Gamma(\frac{n-i-k+2}{2})}{\left(\frac{n+i-k+1}{2}\right)!\ \Gamma\left(\frac{n+i+k+2}{2}\right)}, (n-i-k)\ odd\end{cases} \times V_i(x),$$

$$i,n,k \in \mathbb{Z} \geq 0, k \geq 1 and\ i \leq n-k. \quad (65)$$

and

$$D_x{}^k W_n(x) = \frac{2^{2n-k}n!}{(2n)!}\times \sum_{i=0}^{n-k}\frac{(2i)!}{2^{4i-1}\, i!}\,\frac{\Gamma(n+k+i+1)}{(n-i-k)!}\,\frac{\Gamma\left(n+\frac{3}{2}\right)}{(2n+1)(k-1)!\,\Gamma\left(i+\frac{1}{2}\right)}\times \begin{cases}\dfrac{\left(\frac{n-i+k-2}{2}\right)!\ \Gamma(\frac{n-i-k+1}{2})}{\left(\frac{n+i-k}{2}\right)!\ \Gamma\left(\frac{n+i+k+1}{2}\right)}, (n-i-k)\ even\\ -\dfrac{\left(\frac{n-i+k-1}{2}\right)!\ \Gamma(\frac{n-i-k+2}{2})}{\left(\frac{n+i-k+1}{2}\right)!\ \Gamma\left(\frac{n+i+k+2}{2}\right)}, (n-i-k)\ odd\end{cases} \times W_i(x),$$

$$i,n,k \in \mathbb{Z} \geq 0, k \geq 1 and\ i \leq n-k. \quad (66)$$

It is to be noted here that the two results (65)and (66) are more easily reached but are in complete agreement with those given in Doha et al. (2015) [16], pages 332-333, formulas (3.7) and (3.11).

4. Concluding Remarks

This paper gives formulas associated with the k times differentiation of Generalized Jacobi polynomials by using the generalizations of the beta function and generalized *(GGHF)* and *(GCHF)*. Moreover, we have used [1-24] and some properties of these orthogonal polynomials to obtain our new formulas, the corresponding extensions of several other familiar special functions are expected to be useful in studying the differentiation (or integration) [5, 6, 8, 17]) in the future.

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