The Behavior of Cauchy-Type Integral Near the Boundary of the Semicylindrical Domain

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Abstract

The purpose of this work is the elucidation of the behavior of Cauchy-type integrals near the boundary semicylindrical domain to characteristics celebrated jordanovic of closed curves (in the case when \(\theta(\delta)\sim\delta\), this class of curves is much wider class of piecewise-smooth class of curves for which the chord length relation to the pulling together arch are limited (K-curves), and also in it existence of cusps is allowed). The main characteristics for functions \(f \in \mathcal{C}_\Delta\) - the mixed and private modules of continuity which was proven continuously extendibility of \(n\) – multiple Cauchy-type integral to the border of the semicylindrical domain and the limit values of the types of Sokhoskiy’s formulas.

1. Introduction

Let \(y^k\) be a closed Jordan rectifiable curve (c.j.r.c.) with length \(l_k\) and diameter \(d_k\) in the complex planes of variables \(z_k (k = 1, n)\). A bounded domain \(D^+\) with the bound \(y^k\) we call an internal, the padding of \(D^+\) \(\cup y^k\) we call an external and denote by \(D^-\).

The contours \(y^1, y^2, ..., y^n\) defines in whole complex space of \(n\) variables \(Z^n\) various semicylindrical domains which are obtained by all possible combinations of characters in the topological multiplication

\[
D_1^+ \times D_2^+ \times ... \times D_n^+.
\]

Among them: one is of the type of \(D_1^+ \times D_2^+ \times ... \times D_n^+ (D_1^- \times D_2^- \times ... \times D_n^-)\), which we denote by \(D^+\) \((D^-)\); \(C_\Delta^n\) are domains of the type of \(D_1^+ \times D_2^+ \times ... \times D_{p-1}^+ \times D_p^- \times D_{p+1}^- \times ... \times D_n^-\) \((D_1^- \times D_2^- \times ... \times D_{p-1}^- \times D_p^+ \times D_{p+1}^+ \times ... \times D_n^+)\) which we denote by \(D^+_{\text{p}}\) \((D^-_{\text{p}})\); similarly, \(C_\Delta^n\) are domains of the type of

\[
D_1^+ \times D_2^+ \times ... \times D_{p-1}^+ \times D_{p}^- \times D_{p+1}^- \times ... \times D_{q-1}^- \times D_{q}^+ \times D_{q+1}^+ \times ... \times D_n^+\]

\[
(D_1^- \times D_2^- \times ... \times D_{p-1}^- \times D_p^+ \times D_{p+1}^+ \times ... \times D_{q-1}^+ \times D_{q}^- \times D_{q+1}^- \times ... \times D_n^-)
\]

which we denote by \(D^+_{\text{pq}}\) \((D^-_{\text{pq}})\) and etc.

Borders of all these semicylindrical domains have the common part, namely, \(\Delta = y^1 \times y^2 \times ... \times y^n\) which is called the core.

If the function \(\Phi(z):= \Phi(z_1, z_2, ..., z_n)\) is defined in \(D^+\) and for an arbitrary \(t := (t_1, t_2, ..., t_n) \in \Delta\) there exists

\[
\int_{C_\Delta^n} \Phi(z)\, dz = 0.
\]
\[ \Phi^+(t) := \lim_{D^+ \ni z \to t} \Phi(z), \]

we say \( \Phi(z) \) is continuously extended up to the bound of \( D^+ \). Analogically, define continuously extendibility up to core of domains \( D^-, D^+_p(D^+_p), D^+_p(D^-_p), \text{etc.} \). The corresponding limit values of the function \( \Phi(z) \) is denoted by \( \Phi^-(t), \Phi^+_p(t), (\Phi^+_p(t)), \Phi^+_{pq}(t) (\Phi^+_{pq}(t)) \) etc., respectively.

If \( \Phi(z) \) is continuously extended up to core from the domain \( D^\pm \) then we say that the function \( \Phi(z) \) is continuously extendible up to core. We say that a function \( \Phi(z) \) is continuously extended to the given boundary point of semicylindrical domain, if the function \( \Phi(z) \) tends to a given boundary point along any path, while remaining at all times in this semicylindrical domain. The corresponding limits we call boundary values \( \Phi(z) \) in this domain, and we denote them as well as the boundary values \( \Phi(z) \) on the core \( \Delta \), with the replacement \( t \) core of \( \Delta \) corresponding boundary point. It is easily seen that if the function \( \Phi(z) \) is continuously extended to the core \( \Delta \) from every semicylindrical domain which boundaries have a common core of \( \Delta \), then it will continuously be extended to any boundary point of each of these semicylindrical domains.

Let us consider \( n \) - multiple integral of Cauchy-type

\[
\Phi(z_1, z_2, ..., z_n) = \frac{1}{(2\pi)^n} \int_{\Delta} \frac{f(t_1, t_2, ..., t_n)}{\prod_{p=q+1}^{n} (t_k-z_k)} \, dt_1 \, dt_2 \, ... \, dt_n,
\]

where \( f(t_1, t_2, ..., t_n) \in C_\Delta, C_\Delta^- \) is the space of continuous functions on \( \Delta \).

In the paper [1] was studied the behavior of integral (1) for smooth contours and functions of Holder's class and in papers [2], [3] and [4] (for \( n=2 \)) under some assumptions on the curves \( \gamma^k \) and the function \( f(t) \) the continuity up to core of the integral was investigated. In [5] and [6] (for \( n=2 \)) the investigation of the integral (1) was extended to a case of summable density. The papers [7], [8], [9] and [12] are focused on study the integral near a bicylindrical fields and contain results the behavior of integral Martinelli-Bochner, which (1) turns into a Cauchy-type integral for \( n = 1 \).

In the current work the behavior of \( n \) - multiple integral (1) on the border of semicylindrical domain in terms of the continuity modulus and \( \theta(\delta) \) characteristic curve \( \gamma^k \) (it was first given in [7], and then generalized in [10], [15-17]). The paper is organized as follows: in the next section are presented some results and notations which will be used in the formulation of the main theorems. In Section 3 we give our main results and their proofs.

2. Preliminaries

For the brevity of the writing we introduce the following notations as in [1]

\[ \tau = (\tau_1, \tau_2, ..., \tau_n), \quad t = (t_1, t_2, ..., t_n), \quad z = (z_1, z_2, ..., z_n), \]

\[ \tau_{tp} = (t_1, t_2, ..., t_{p-1}, t_p, t_{p+1}, ..., t_n), \]

\[ \tau_{tpq} = (\tau_1, \tau_2, ..., \tau_{p-1}, \tau_p, \tau_{p+1}, ..., \tau_{q-1}, \tau_q, \tau_{q+1}, ..., \tau_n). \]

Then be these notations (1) takes form

\[ \Phi(z) = \frac{1}{(2\pi)^n} \int_{\Delta} \frac{f(t)}{\Psi(t-z)} \, dt. \]

Let us denote by \( \Delta f(t; t) \) the following

\[ \Delta f(t; t) = f(t) - \sum_{p=1}^{n} f(t_{tp}) + \sum_{p=1}^{n} \sum_{q=1}^{n} f(t_{tpq}) - \cdots + \]
It is easy to verify that holds the identity
\[
G = G + e \Delta G + e \Delta 2 G + \cdots + e \Delta n G
\]
holds. Using this and (1) we have
\[
\Phi = 1 \frac{2ab}{c \Delta G} \sum_{p=1}^{n} \frac{\Delta f (t_p)}{\Delta [p]},
\]
\[
+ \sum_{p=1}^{n} \sum_{q=1}^{n} \Delta f (t_{pq}; t) + \sum_{p=1}^{n} \sum_{q=p+1}^{n} \Delta f (t_{pq}; t) + \cdots + \Delta f (t_{pq}; t) + f (t)
\]
holds. Using this and (1) we have
\[
\Phi(z) = \frac{1}{(2\pi i)^2} \int_{\gamma} \frac{\Delta f (t; z)}{\tau - z} d\tau + \sum_{p=1}^{n} \frac{\Delta f (t_p; t)}{\Delta [p]} d\tau [p] + \cdots + \Delta f (t_{pq}; t) + \cdots + \Delta f (t_{pq}; t) + f (t)
\]
where \( \Delta [p] = \sum_{q=1}^{p-1} \Delta f (t_{pq}; t) + \sum_{q=p+1}^{n} \Delta f (t_{pq}; t) + \cdots + \Delta f (t_{pq}; t) + f (t) \),
\[
\Phi^n (z) + \psi^n (z) + \psi^{n-2} (z_{pq}) + \cdots + \psi^2 (t_{pq}) + \psi^1 (t_p) + \nu (t) f (t),
\]
which will be further used.

Let \( \gamma^k \) be a closed rectifiable Jordan curve (c.r.j.c.) \( t_k = t(s_k), (0 \leq s_k \leq l_k) \), \( l_k \) be the length of the curve and \( \gamma^k \) be an equation of the curve in arc coordinates \( k = 1, n \). Let us denote
\[
\theta (t_k; \delta) = mes \{ t \in \gamma^k \mid |t_k - t| \leq \delta, \delta \in (0, \delta_k) \},
\]
\[
d_k = sup \{ t_k - t_k \}, \theta_k (\delta) = sup \theta (t_k; \delta), k = 1, n.
\]
\[
\omega^k (\delta) = \omega^k (\delta_1; \delta_2, \ldots; \delta_n) = \delta_1 \delta_2 \cdots \delta_n \sup_{\xi_1, \xi_2, \ldots, \xi_n \in \delta} \omega^N (f; \xi_1, \xi_2, \ldots, \xi_n) = \delta \sup_{\xi_1, \xi_2, \ldots, \xi_n \in \delta} \omega^N (f; \xi_1, \xi_2, \ldots, \xi_n) = \delta \sup_{\xi \in \delta} \omega^N (f; \xi),
\]
where
\[
\delta_i > 0, i = 1, n, \omega^N (f; \delta) = \sup_{|t_1 - t_2| \delta \in (0, \delta_k), \delta \in (0, \delta_k)} |f (t_1; t_2)| = \sup_{|t_1 - t_2| \delta \in (0, \delta_k)} |f (t_1; t_2)|
\]
2). private continuity modules
\[
\omega_f^N(\delta) := \delta_{[p]} \sup_{\xi_{[p]} \in \delta_{[p]}} \frac{\omega_f(f; \xi_{[p]})}{\xi_{[p]}}
\]

where

\[
\omega_f^N(\delta) = \sup_{t_p} \sup_{|t_p - t| \leq |\delta_{[p]}|} |Df(t_p; t)|, \quad p = 1, n
\]

\[
\omega_f^N(\delta_{[p]}) := \delta_{[p]} \sup_{\xi_{[p]} \in \delta_{[p]}} \frac{\omega_f(f; \xi_{[p]})}{\xi_{[p]}}
\]

where

\[
\omega_f^N(p; f; \delta_{[p]}) := \sup_{t_p, t_q} \sup_{|t_p - t_q| \leq |\delta_{[p]}|} \frac{|Df(t_p; t_q)|}{\xi_{[p]}},
\]

\[
\omega_f(\delta) = \delta \sup_{\xi \geq \delta} \frac{\omega_f(f; \xi)}{\xi},
\]

\[
\omega_f^N(\delta_{[p]}) := \delta_{[p]} \sup_{\xi_{[p]} \in \delta_{[p]}} \frac{\omega_f(f; \xi_{[p]})}{\xi_{[p]}}
\]

Let us denote by \(\Phi^+(0, d)\) a multiple nonnegative monotone increasing function \(\varphi(\delta)\) on \((0, d)\) such that \(\lim_{\delta \to 0} \varphi(\delta) = 0\), and \(\varphi(\delta)\) monoton decreases. By

\[
\Phi_{[p]}^{+1(0, d)} := \Phi(0, d)
\]

denote a set of functions \(\omega(\delta, \delta_2, \ldots, \delta_n) := \omega(\delta)\) defined on \((0, d)\) and lying in \(\Phi^+\) on each argument, i.e., \(\omega(\delta) \in \Phi^+(0, d)\) by \(\delta_{[p]}\) at fixed \(\delta_{[i]}; i = 1, n; i \neq p\). It is clear that

\[
\omega_f(\delta) \in \Phi(0, d).
\]

**Lemma 1** ([7], [10]).

1. Let \(g(\xi)\) be a nonincreasing function on \((0, d)\).

Then the following

\[
\int_{\epsilon}(\xi) \Psi^n(t) d\xi = \int_{\epsilon}(\xi) g(\xi) d\theta(\xi)
\]

holds for arbitrary \(\epsilon, \epsilon' \in (0, d), \epsilon < \epsilon'\).

2. Let \(g(\xi)\), a nonincreasing function on \((0, d)\) and \(\mu_k(\delta)\) satisfy the conditions

\[
|\Psi_{p}^{n-1}(\xi_d) - \Psi_{p}^{n-1}(t)| = \frac{1}{(2\pi)^n \int_{|\xi|} \Delta f(t; \pi) d\tau(\pi) - \int_{|\tau|} \Delta f(t; \pi) d\tau(\pi)} \leq C_{n-1} Z(\omega_f^{N_p}, |z - t|_{|p|}, \xi_{|p|}) + \sum_{q < p} Z(\omega_f^{N_q}, |z - t|_{|p|}, \xi_{|p|})
\]

\[
\mu_k(\delta) > 0, \mu_k(\delta) \uparrow, \mu_k(\delta) \to 0, k = 1, 2, \mu_1(\delta) \leq \mu_2(\delta), \text{then}
\]

\[
\int_{0}^{d} g(y) d\mu_1(y) \leq \int_{0}^{d} g(y) d\mu_2(y);
\]

3. Let \(g(\xi)\) be nonincreasing function on \((0, d)\). Then

\[
\int_{0}^{d} g(y) d\mu_k(y) \leq \int_{0}^{d} g(y) d\mu_1(y).
\]

Let us emphasize

\[
f_{\epsilon_0}(\sigma, \xi) := \int_{\xi_0} \omega(\sigma) \xi d\theta(\xi) \leq \infty,
\]

\[
\int_{0}^{d} \omega(\sigma) \xi d\theta(\xi) \leq \infty, \quad \infty, \quad \cdots
\]

\[
\int_{0}^{d} \omega(\sigma) \xi d\theta(\xi) \leq \infty.
\]

**3. Main Results**

**Theorem 3.1**. Let \(y^k\) be a closed rectifiable Jordan curve and \(f_{\epsilon_0}(\sigma, \xi)\). Then for arbitrary \(z = (z_1, z_2, \ldots, z_n) \in \Delta, t = (t_1, t_2, \ldots, t_n) \in \Delta, |z_k - t_k| \leq d_k\) the following estimate holds

\[
|\Psi^n(z) - \Psi^n(t)| = \frac{1}{(2\pi)^n \int_{\Delta} \Delta f(t; \pi) d\tau(\pi)} \leq C_{n-1} Z(\omega_f^{N_p}, |z - t|_{|p|}, \xi_{|p|}) + \sum_{p < q} Z(\omega_f^{N_q} |z - t|_{|p|}, d_{q, p, \theta}) + \sum_{p < q} Z(\omega_f^{N_q}, |z - t|_{|p|}, d_{q, p, \theta}) + \cdots + \sum_{p < q} Z(\omega_f^{N_q}, |z - t|_{|p|}, d_{q, p, \theta}) + \sum_{p < q} Z(\omega_f^{N_q}, |z - t|_{|p|}, d_{q, p, \theta}) + \cdots + \sum_{p < q} Z(\omega_f^{N_q}, |z - t|_{|p|}, d_{q, p, \theta}).
\]
\[ + \sum_{q=1}^{n} \sum_{r=1}^{n} Z \left( \omega_{j}^{N \mid q}, |z - t|_{pqr}, d, d_r, \theta_{[p]} \right) + \ldots + \\
+ \sum_{q<r; \quad q,r \neq p}^{n} \sum_{r=1}^{n} Z \left( \omega_{j}^{N \mid q}, |z_q - z_r|, |z_r - t_r|, d_{[q]}, d_r, \theta_{[p]} \right) + \sum_{q=1}^{n} Z \left( \omega_{j}^{N \mid q}, |z_q - t_q|, d_{[pq]}, d_{[q]}, \theta_{[p]} \right), \tag{4} \]

\[ |\Psi^2_{pq}(t_{2pq}) - \Psi^2_{pq}(t)| = \left| \frac{1}{(2\pi i)^{2}} \int_{\gamma} \frac{\Delta f \left( t_{2pq} \right)}{2\pi i} dt \right| = C_{2} \left[ Z \left( \omega_{pq}^{p \mid q}, |z_{pq} - t_p|, |z_q - t_q|, \theta_p, \theta_q \right) \right] + \\
+ Z \left( \omega_{pq}^{p \mid q}, |z_p - t_{pq}|, d_p, \theta_p, \theta_q \right) + Z \left( \omega_{pq}^{p \mid q}, |z_q - t_{pq}|, d_q, \theta_p, \theta_q \right), \tag{5} \]

\[ |\Psi^1_p(t_{2p}) - \Psi^1_p(t)| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{\Delta f \left( t_{2pq} \right)}{2\pi i} dt \right| = C_{2} \left[ Z \left( \omega_{pq}^{p \mid q}, |z_p - t_p|, \theta_p \right) \right], \tag{6} \]

where

\[ Z \left( \omega_{pq}^{p \mid q}, \delta_1, \delta_2, \ldots, \delta_n, \theta_1, \theta_2, \ldots, \theta_n \right) = Z \left( \omega_{pq}^{p \mid q}, \delta, \theta \right) = \int_{0}^{1} \frac{\omega_{pq}^{p \mid q}(\xi)}{\xi} d\theta(\xi) + \sum_{p=1}^{n} \delta_{[p]} \int_{0}^{1} \frac{\omega_{pq}^{p \mid q}(\xi)}{\xi} d\theta(\xi) + + \sum_{p=1}^{n} \sum_{q=1}^{n} \delta_{[pq]} \int_{0}^{1} \int_{0}^{1} \frac{\omega_{pq}^{p \mid q}(\xi)}{\xi^{2}} d\theta(\xi) + \ldots + \\
+ \sum_{p=1}^{n} \sum_{q=1}^{n} \delta_{[pq]} \int_{0}^{1} \int_{0}^{1} \frac{\omega_{pq}^{p \mid q}(\xi)}{\xi^{2}} d\theta(\xi) + \sum_{p=1}^{n} \delta_{[p]} \int_{0}^{1} \int_{0}^{1} \frac{\omega_{pq}^{p \mid q}(\xi)}{\xi^{2}} d\theta(\xi) + \sum_{q=1}^{n} \delta_{[p]} \int_{0}^{1} \int_{0}^{1} \frac{\omega_{pq}^{p \mid q}(\xi)}{\xi^{2}} d\theta(\xi) + \sum_{\delta} \frac{\omega_{pq}^{p \mid q}(\xi)}{\xi^{2} \delta^{2}} d\theta(\xi), \tag{7} \]

\[ C_{2} (i = 1, n) - \text{constant.} \]

Proof. Let us denote by \( t_{2p} \) an arbitrary point of the border \( \gamma \), such that \( |z_p - t_p| = \inf_{z_p \in \gamma} |z_p - t_p| = \rho(z_p, \gamma) \). Then the identity takes the place

\[ \Delta f \left( \tau, t \right) = \Delta f \left( t_{2p}, t \right) - \sum_{p=1}^{n} \Delta f \left( t_{2; t_{pq}} \right) + \sum_{p=1}^{n} \sum_{q=1}^{n} \Delta f \left( t_{2; t_{pq}} \right) + \ldots + \\
+ (-1)^{n-1} \sum_{p=1}^{n} \sum_{q=1}^{n} \Delta f \left( t_{2; t_{pq}} \right) \tag{7} \]

the validity of which is easily shown by direct calculations of items. It is easy to see also that

\[ \Delta f \left( \tau, t \right) = \Delta f \left( t_{2; t} \right) \tag{8} \]

Let us consider the difference

\[ \Psi^n(z) - \Psi^n(t) = \left( \frac{1}{(2\pi i)^{n}} \right) \left( \int \frac{\Delta f(t_{2; t})}{t - z} d\tau - \int \frac{\Delta f(t_{2; t})}{t - \tau} d\tau \right) \tag{9} \]

Using (8) and (7) in (9) we obtain
\[\Psi^n(z) - \Psi^n(t) = \frac{1}{(2\pi i)^n} \left( \int_{\Delta} \frac{\Delta f(\tau; t_z)}{\tau - z} \, d\tau - \int_{\Delta} \frac{\Delta f(\tau; t_z)}{\tau - t_z} \, d\tau \right) + 
\]

\[+ (-1)^{n-1} \frac{\chi_p(z_p)}{(2\pi i)^{n-1}} \sum_{p=1}^{n} \int_{\Delta_{p[p]}} \frac{\Delta f(t_z; \tau_p)}{\tau - z} \, d\tau_p + 
\]

\[+ (-1)^{n-2} \frac{\chi_p(z_p) \chi_q(z_q)}{(2\pi i)^{n-2}} \sum_{p=1}^{n} \sum_{q<q=1}^{n} \int_{\Delta_{p[q]}} \frac{\Delta f(t_z; \tau_{pq})}{\tau - z} \, d\tau_{pq} + \n\]

\[\ldots + \sum_{p=1}^{n} \sum_{p<q}^{n} \frac{\chi(z_p) \chi(z_q)}{(2\pi i)^2} \int_{\gamma_p} \int_{\gamma_q} \frac{\Delta f(t_z; \tau_{pq})}{\tau - z} \, d\tau_p \, d\tau_q - 
\]

\[- \sum_{p=1}^{n} \frac{\chi(z_p)}{2\pi i} \int_{\gamma_p} \frac{\Delta f(t_z; \tau_{tp})}{\tau - t_p} \, d\tau_p + \Delta f(t_z; t) \chi(z) + \frac{1}{(2\pi i)^m} \left( \int_{\Delta} \frac{\delta f(\tau_z; t)}{\tau - z} \, d\tau - \int_{\Delta} \frac{\delta f(\tau_z; t)}{\tau - t_z} \, d\tau \right). \tag{10} \]

Let us first denote items on the right hand side of equality (10) by \(J_n, J_{n-1}, \ldots, J_1, J_0\), \(n, m\), respectively, and we estimate each of them separately.

Before proceed to assess \(J_n\), remind the is an identity

\[\frac{1}{\tau - z} - \frac{1}{\tau - t_z} = \prod_{k=1}^{n} (\tau_k - z_k) - \prod_{k=1}^{n} (\tau_k - t_z) = 
\]

\[\frac{1}{\prod_{k=1}^{n} (\tau_k - z_k)} \left( \prod_{r=1}^{n} \frac{z_k - t_z}{t_k - z_k} + \sum_{k=1}^{n} \prod_{p=k, r}^{n} \frac{z_p - z_k}{t_p - z_k} + \ldots + \sum_{k=1}^{n} \prod_{r=1}^{n} \frac{z_k - t_z}{t_k - t_z} \right) \]

\[\text{Let } \varepsilon_k \text{ be an arbitrary number from } (0, d_k) \quad (k = 1, n), \\
\]

\[y^k_{\varepsilon_k}(t_{z_k}) = \{ \tau \in y^k | |t_{z_k} - \tau| < \varepsilon_k \}. \]

The integral (12) represent table a type of the sum of two integrals of \(J'_n\) and \(J''_n\), taken, on \(\Delta_z\) and \(\Delta \Delta_z\), respectively, where

\[\Delta_z = y^1_{\varepsilon_1}(t_{z_1}) \times y^2_{\varepsilon_1}(t_{z_2}) \times \ldots \times y^n_{\varepsilon_n}(t_{z_n}) , \]

\[\Delta \Delta_z = \sum_{p=1}^{n} D_{p}^{(n-1)} + \sum_{p<q}^{n} D_{pq}^{(n-2)} + \ldots + \sum_{p<q}^{n} D_{pq}^{(2)} + \sum_{p=1}^{n} D_{p}^{(1)} + D_{\varepsilon} , \]

\[D_{p}^{(n-1)} = y^1_{\varepsilon_1}(t_{z_1}) \times y^p_{\varepsilon_p-1}(t_{z_{p-1}}) \times y^p_{\varepsilon_p}(t_{z_p}) \times y^{p+1}_{\varepsilon_{p+1}}(t_{z_{p+1}}) \times \ldots \times y^n_{\varepsilon_n}(t_{z_{n}}) , \]

\[D_{pq}^{(n-1)} = y^q_{\varepsilon_q-1}(t_{z_{q-1}}) \times y^q_{\varepsilon_q}(t_{z_q}) \times y^{q+1}_{\varepsilon_{q+1}}(t_{z_{q+1}}) \times \ldots \times y^n_{\varepsilon_n}(t_{z_{n}}) , \]

\[D_{pq}^{(2)} = y^1_{\varepsilon_1}(t_{z_1}) \times y^{p-1}_{\varepsilon_{p-1}}(t_{z_{p-1}}) \times y^p_{\varepsilon_p}(t_{z_p}) \times y^{p+1}_{\varepsilon_{p+1}}(t_{z_{p+1}}) \times \ldots \times y^n_{\varepsilon_n}(t_{z_{n}}) , \]

\[D_{p}^{(1)} = y^1_{\varepsilon_1}(t_{z_1}) \times y^{p-1}_{\varepsilon_{p-1}}(t_{z_{p-1}}) \times y^p_{\varepsilon_p}(t_{z_p}) \times y^{p+1}_{\varepsilon_{p+1}}(t_{z_{p+1}}) \times \ldots \times y^n_{\varepsilon_n}(t_{z_{n}}) , \]

\[D_{\varepsilon} = y^1_{\varepsilon_1}(t_{z_1}) \times y^p_{\varepsilon_p-1}(t_{z_{p-1}}) \times y^p_{\varepsilon_p}(t_{z_p}) \times y^{p+1}_{\varepsilon_{p+1}}(t_{z_{p+1}}) \times \ldots \times y^n_{\varepsilon_n}(t_{z_{n}}) . \]
\[
\begin{align*}
&\times y^{p+1}y^{p+1}(t_{p+1}) \times \ldots \times y^{q-1}y^{q-1}(t_{q-1}) \times y^p(t_q) \\
&\quad \times y^{p+1}y^{p+1}(t_{p+1}) \times \ldots \times y^n(t_n), \\
D_p^{(1)} &= y^1y^2(t_1) \times y^2y^2(t_2) \times \ldots \times y^{p-1}y^{p-1}(t_{p-1}) \times y^p(t_p) \\
&\quad \times y^{p+1}y^{p+1}(t_{p+1}) \times \ldots \times y^n(t_n), \\
D_e &= y^1y^2(t_1) \times y^2y^2(t_2) \times \ldots \times y^n(t_n).
\end{align*}
\]

Let us denote by

\[Q_p^{-1}(z, t), Q_p^{-2}(z, t), \ldots, Q_p^n(z, t)\]

the integrals taken piecemeal cores: \(D_p^{(n-1)}, D_p^{(n-2)}, \ldots, D_p^{(2)}, D_p^{(1)}, D_e\), respectively. As for every

\[\tau_k \in y^k|z_k - t_k| \leq M_k < 1, \quad k = 1, n,\]

we have

\[|J_n| \leq \frac{1}{(2\pi)^n} \int_{\Delta_1} \cdots \int_{\Delta_n} \frac{\omega^N(|\tau_k - t_k|)|d\tau|}{|\tau - t_k|} \leq M \int_{\gamma^1} \cdots \int_{\gamma^n} \frac{\omega^N(|\tau_n - t_{n-1}|)}{|\tau - t_n|} |d\tau_n|,
\]

where

\[M = \prod_{k=1}^n M_k + \sum_{k=1}^n M_p + \sum_{k=1}^n M_m + \sum_{k=1}^n M_k M_m + \sum_{k=1}^n M_k.
\]

Consequently applying 1), 2), 3) of Lemma 2.1 and choosing \(\varepsilon_k = |z_k - t_k|, k = 1, n\), we obtain

\[|J_n| \leq M \int_{\gamma^1} \cdots \int_{\gamma^n} \frac{\omega^N(\prod_{k=1}^n |\tau_k - t_k|, \xi_n)}{\prod_{k=1}^n |\xi_n|} d\theta| \xi_n| \leq M \int_{\gamma^1} \cdots \int_{\gamma^n} \frac{|d\tau|}{|\tau - t_k|} \epsilon_k
\]

\[x \int_{0}^{\epsilon_k} \omega^N(\prod_{k=1}^{n-1} |\tau_k - t_k|, \xi_n) d\theta| \xi_n| \leq M \int_{0}^{\epsilon_k} \omega^N(\xi_n) d\theta| \xi_n| \equiv M \int_{0}^{\epsilon_k} \omega^N(\xi_n) d\theta| \xi_n|.\]

Thus,

\[|J_n| \leq M \int_{0}^{\epsilon_k} \omega^N(\xi_n) d\theta| \xi_n|.\]
Now we estimate $I''_n$. For this aim we first estimate integral $Q^{n-1}_{p} \left(z, t_z \right)$:

Now we estimate the integral $Q^{n-1}_{p} \left(z, t_z \right)$ as follows

\[
\left| Q^{n-1}_{p} \left(z, t_z \right) \right| \leq \frac{1}{(2\pi i)^n} \int_{|\tau - \tau_p|} \left( \prod_{k=p} |\tau_k - \tau_k| \right) + 
\sum_{k=1}^{n} \prod_{i=1}^{k=p} \left( \frac{z_i - t_{z_k}}{\tau_i - z_i} \right) + 
\sum_{k=p}^{n} \sum_{i=1}^{k=m} \prod_{j=1}^{k=p} \left( \frac{z_i - t_{z_k}}{\tau_i - z_i} \right) + \ldots + 
\sum_{k=m}^{n} \prod_{m=1}^{k=\infty} \left( \frac{z_i - t_{z_k}}{\tau_i - z_i} \right) + 1) + 
\sum_{k=p}^{n} \prod_{i=1}^{k=m} \left( \frac{z_i - t_{z_k}}{\tau_i - z_i} \right) + 
\sum_{k=m}^{n} \prod_{m=1}^{k=\infty} \left( \frac{z_i - t_{z_k}}{\tau_i - z_i} \right) + \ldots + 
\sum_{k=1}^{n} \prod_{k=m}^{n} \left( \frac{z_i - t_{z_k}}{\tau_i - z_i} \right) + \frac{\omega_0^{N} \left( \prod_{k=1}^{n} |\tau_k - t_{z_k}| \right)}{\prod_{\infty}^{k=1} |\tau_k - t_{z_k}|} d\tau .
\]

From this and (13) we obtain the following

\[
\left| Q^{n-1}_{p} \left(z, t_z \right) \right| \leq (M_{|p|} + 1) |z_p - t_p| x 
\times \int_{0}^{\infty} |z_p - t_p| d\theta(\xi) + \int_{0}^{\infty} \omega_0^{N}(\xi) \left[ \frac{1}{\xi_{|p|}^{\xi_{p}}} d\theta(\xi) \right] \left( \frac{\omega_0^{N}(\xi)}{\xi_{|p|}^{\xi_{p}}} d\theta(\xi) \right) 
\]

for every $\tau_k \in \gamma^k$ such $|\tau_k - t_{z_k}| \leq 2$. Taking into account that $|z_k - t_{z_k}| \leq |z_k - t_k|$ and $\epsilon_k = |z_k - t_k|, k = 1, n$, from the last estimate we have

\[
\left| Q^{n-1}_{p} \left(z, t_z \right) \right| \leq (M_{|p|} + 1) |z_p - t_p| x 
\times \int_{0}^{\epsilon_{|p|}} |z_p - t_p| d\theta(\xi) + \int_{0}^{\epsilon_{|p|}} \omega_0^{N}(\xi) \left[ \frac{1}{\xi_{|p|}^{\xi_{p}}} d\theta(\xi) \right] \left( \frac{\omega_0^{N}(\xi)}{\xi_{|p|}^{\xi_{p}}} d\theta(\xi) \right) 
\]

Taking similar transforms which were used in the estimate of (15), we have for the integral $Q^{n-2}_{p,q} \left(z, t_z \right)$ the following:

\[
\left| Q^{n-2}_{p,q} \left(z, t_z \right) \right| \leq (M_{|p|} + 1) |z_p - t_p| |z_q - t_q| x 
\times \int_{0}^{\epsilon_{|p|}} \int_{0}^{\epsilon_{|q|}} d\theta(\xi) + \int_{0}^{\epsilon_{|p|}} d\theta(\xi) + \int_{0}^{\epsilon_{|q|}} d\theta(\xi) + \int_{0}^{\epsilon_{|p|}} d\theta(\xi) 
\times \left( M_p + M_q + M_p + M_q + 1 \right) x
\]
\[
\begin{align*}
&= x \left[ \int_{|z-t|_{pq}}^{d_{pq}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) + \sum_{r,s=p,q}^{n} \int_{|z-t|_{pq}}^{d_{pq}} d_{r} \int_{|z_{r}-t_{r}|}^{d_{r}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) + \sum_{r<s}^{n} \sum_{r,s=p,q}^{n} \int_{|z-t|_{pq}}^{d_{pq}} d_{r} d_{s} \int_{|z_{r}-t_{r}|}^{d_{r}} \int_{|z_{s}-t_{s}|}^{d_{s}} x \int_{|z_{r}-t_{r}|}^{d_{r}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) + \sum_{r=1}^{n} \int_{|z_{r}-t_{r}|}^{d_{r}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) + \int_{0}^{d_{pq}} \int_{0}^{d_{pq}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) \right] + \sum_{r=1}^{n} \int_{0}^{d_{p}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) + \sum_{q=1}^{n} \int_{0}^{d_{q}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) + \sum_{q=1}^{n} \sum_{r=1}^{n} \int_{0}^{d_{pq}} d_{r} d_{q} \int_{|z_{r}-t_{r}|}^{d_{r}} \int_{|z_{q}-t_{q}|}^{d_{q}} x \int_{|z_{r}-t_{r}|}^{d_{r}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) + \sum_{q<r}^{n} \sum_{r=1}^{n} \sum_{q=1}^{n} \int_{|z_{q}-t_{q}|}^{d_{q}} \int_{|z_{r}-t_{r}|}^{d_{r}} \int_{|z_{r}-t_{r}|}^{d_{r}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) + \sum_{r=1}^{n} \int_{0}^{d_{p}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) + \sum_{r=1}^{n} \int_{0}^{d_{pq}} d_{r} \int_{|z_{r}-t_{r}|}^{d_{r}} \int_{|z_{r}-t_{r}|}^{d_{r}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi) + \int_{0}^{d_{pq}} \int_{0}^{d_{pq}} \frac{\omega_{f}^{N}(\xi)}{\xi_{pq}^{\xi_{p}^{2}}} d\theta(\xi). \end{align*}
\]
\[ |Q_\varepsilon(z; t_\varepsilon)| \leq (|z - t|) \int_{|z-t|}^d |\frac{\omega^N(\xi)}{\xi^2}| d\theta(\xi) + \]

\[ + \sum_{p=1}^n \sum_{q=1}^n \frac{d_{pq}}{|z-t| |z_q - t_q|} \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{\xi^2} d\theta(\xi) + \]

\[ + \sum_{p=1}^n \sum_{q=1}^n \frac{d_{pq}}{|z_{pq} - t_{pq}|} \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{\xi^2} d\theta(\xi) + \]

\[ + \sum_{p=1}^n \sum_{q=1}^n \frac{d_{pq}}{|z_{pq} - t_{pq}|} \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{\xi^2} d\theta(\xi) + \]

\[ + \sum_{p=1}^n \frac{d_{pq}}{|z_{pq} - t_{pq}|} \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{\xi^2} d\theta(\xi). \tag{19} \]

Summarizing the obtained estimates in (15) – (19), we get an estimate for \( J_{n-1} \).

Now we estimate \( J_{n-1} \). As \( |t_{z_k} - t_k| \leq 2|z_k - t_k| \), \( |t_k - t_{z_k}| \leq 2|t_k - t_k| \), \( k = \overline{1,n} \), applying items 1) and 2) of Lemma 2.1 we have

\[ |J_{n-1}| \leq \left( \frac{1}{2\pi \cdot I_0} \right)^{n-1} \sum_{p=1}^n \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{\xi^2} d\theta(\xi) |d\xi| \leq \]

\[ \leq \frac{1}{\pi^{n-1}} \sum_{p=1}^n \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{\xi^2} d\theta(\xi) |d\xi|. \]

Owing to a lack of growth \( \frac{\omega^N(\xi)}{\xi^2} \) on \( \delta_p \) (\( p = \overline{1,n} \)),

\[ \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{\xi} d\theta(\xi) \leq \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{|z_{pq} - t_{pq}|} d\theta(\xi) = \]

\[ = \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{|z_{pq} - t_{pq}|} d\theta(\xi). \]

Therefore,

\[ |J_{n-1}| \leq C_{n-1} \sum_{p=1}^n \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{\xi} d\theta(\xi). \tag{20} \]

Similarly estimating the rest integrals we have

\[ |J_{n-2}| \leq C_{n-2} \sum_{p=1}^n \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{\xi} d\theta(\xi). \tag{21} \]

\[ |J_1| \leq C_1 \sum_{p=1}^n \int_0^{|z_{pq} - t_{pq}|} \frac{\omega^N(\xi)}{\xi} d\theta(\xi). \tag{22} \]
\[ \| \varphi \| \leq C_1 \sum_{p=1}^n \left| \int_{r=p-t_p}^{z_p} \omega_f^{N_p}(\xi) \frac{d\theta(\xi)}{\xi} \right. \]

(23)

For get an estimate for the integral \( i_n \) we apply Theorem 2 in [18]. Summarizing all estimates for integrals \( f_n, j_n \) and \( i_n \), and estimates (14), (15) – (19) and (20) – (23) we finally obtain the required estimate (3).

To estimate the following difference

\[ \Psi_{p-1}^{n-1}(z_{t_p}) - \Psi_{p-1}^{n-1}(t) = \frac{1}{(2\pi i)^{n-1}} \left( \int_{\Delta_p} \frac{\Delta f(t_{t_p}; t)}{(t-z)^{111} \Delta_r \Delta_p} d\tau \right) - \int_{\Delta_p} \frac{\Delta f(t_{t_p}; t)}{(t-t)^{111} \Delta_r \Delta_p} d\tau \]

we use the following identity

\[ \Delta f(t_{t_p}; t) = \Delta f(t_{t_p}; t) - \sum_{q=1}^{n} \Delta f(t_{t_p}; t_{r_q}) + \sum_{q<r}^{n} \Delta f(t_{t_p}; t_{r_q r_q}) + \cdots + (-1)^{n-3} \sum_{q<r}^{n} \Delta f(t_{t_p}; t_{r_q r_q}) + \]

\[ \sum_{q,r,s}^{n} \Delta f(t_{t_p}; t_{r_q r_q}) \]

the validity of which follows from (7). Then \( \Psi_{p-1}^{n-1}(z_{t_p}) - \Psi_{p-1}^{n-1}(t) \) is represented in a form of a difference (10) and is estimated also as estimates for \( \Psi, \Psi, \ldots, \Psi, \Psi \) and \( i_n \) the distinction consists only among integrals. Therefore, we obtain

\[ \left| \Psi_{p-1}^{n-1}(z_{t_p}) - \Psi_{p-1}^{n-1}(t) \right| \leq C_{n-1}(Z \left( \omega_f^{N_p}, [z-t]_{111}, \theta_{111} \right)) + \]

\[ + \sum_{q=1}^{n} Z \left( \omega_f^{N_p}, [z-t]_{111}, d_q, \theta_{111} \right) + \sum_{q<r}^{n} Z \left( \omega_f^{N_p}, [z-t]_{111}, d_q, d_r, \theta_{111} \right) + \cdots + \]

\[ + \sum_{q=r}^{n} Z \left( \omega_f^{N_p}, [z-t]_{111}, Q_{111} \right) + \]

\[ + \sum_{q<r}^{n} Z \left( \omega_f^{N_p}, [z-t]_{111}, Q_{111} \right) \]

Similarly, we have

\[ \left| \Psi_{pq-2}^{n-2}(z_{t_p q}) - \Psi_{pq-2}^{n-2}(t) \right| \leq C_{n-2}(Z \left( \omega_f^{N_p q}, [z-t]_{111}, Q_{111} \right)) + \]

\[ + \sum_{r=1}^{n} Z \left( \omega_f^{N_p q}, [z-t]_{111}, Q_{111} \right) + \]

\[ + \sum_{r<m}^{n} Z \left( \omega_f^{N_p q}, [z-t]_{111}, Q_{111} \right) + \cdots + \]

\[ + \sum_{r<m}^{n} Z \left( \omega_f^{N_p q}, [z-t]_{111}, Q_{111} \right) + \]
By continuing this process we show estimates for the differences

\[ \Psi^2_{pq}(z_{tpq}) - \Psi^2_{pq}(t), \Psi^1_{pq}(z_{tpq}) - \Psi^1_{pq}(t) \]

have the forms:

\[
\left| \Psi^2_{pq}(z_{tpq}) - \Psi^2_{pq}(t) \right| \leq C_2(Z(\omega_f^p; |z_p - t_p|, |z_q - t_q|, Q_p, Q_q) + \\
+ Z(\omega_f^p; |z_p - t_p|, d_q, \theta_p, \theta_q) + Z(\omega_f^p; |z_q - t_q|, d_p, \theta_p, \theta_q)),
\]

\[
\left| \Psi^1_{pq}(z_{tpq}) - \Psi^1_{pq}(t) \right| \leq C_3(Z(\omega_f^p; |z_p - t_p|, \theta_p).
\]

These prove the theorem.

From the theorem immediately follow the following equalities

\[ \Psi^1_p(t) = \Psi^{p,\pm}_p(t), \Psi^3_p(t) = \Psi^{p,\pm_2}_p(t), ..., \Psi^n_p(t) = \Psi^{p,\pm_\ldots\pm}_p(t). \]

**Theorem 3.2.** If \( f \in J_\varrho(\Delta, \theta) \), then for every \( z \notin \Delta \), and \( t \in \Delta \) such \( |z_k - t_k| < d_k, k = 1, n \), the following estimate

\[
|\Phi(z) - \left( \frac{1}{(2\pi i)^n} \int_{\Delta} \Delta f(\tau; t) \right| t - t \right) d\tau + \sum_{p=1}^{n} \left( \frac{\Delta f(\tau_{tpq}; t)}{(2\pi i)^{n-1}} \right) d\tau_{tpq} + \cdots \]

\[
+ \sum_{p<q}^{n} \sum_{q=1}^{n} \left( \frac{\Delta f(\tau_{tpq}; t)}{(2\pi i)^2} \right) d\tau_{tpq} d\tau_{tq} + \cdots + \]

\[
+ \sum_{p=1}^{n} \frac{\Delta f(\tau_{tpq}; t)}{2\pi i} d\tau_{tpq} + f(t)z(t) \right| \leq \]

\[
\leq c \left( Z(\omega_f^p; |z - t|, Q) + \sum_{p=1}^{n} Z(\omega_f^p; |z - t|, d_p, \theta) + \right.
\]

\[
+ \sum_{p=1}^{n} \sum_{q=1}^{n} Z(\omega_f^p; |z - t|, d_p, d_q, \theta) + \cdots + \]

\[
+ \sum_{p<q}^{n} \sum_{q=1}^{n} Z(\omega_f^p; |z - t|, d_p, d_q, \theta) + \right) + (Z(\omega_f^p; |z - t|, Q) + \right)
\]
\[
+ \sum_{q=1}^{n} Z(\omega_j^{N_p}; |x - t|_{pq}, d_q, \theta_{[p]}) + \\
+ \sum_{q=1}^{n} \sum_{r=1}^{n} Z(\omega_j^{N_p}; |x - t|_{pqr}, d_q, d_r, \theta_{[p]}) + \cdots + \\
+ \sum_{q<r,q,r=1}^{n} Z(\omega_j^{N_p}; d_{[pqr]}, |x_q - t_q|, |x_r - t_r|, \theta_{[p]}) + \\
+ \sum_{q=1}^{n} Z(\omega_j^{p}; d_{[pq]}, |x_q - t_q|, \theta_{[p]}) \cdots + \\
+(Z(\omega_j^{p}; |x_p - t_p|, |x_q - t_q|, d_{p}, \theta_{p}, \theta_{q}) + Z(\omega_j^{p}; |x_p - t_p|, d_{p}, \theta_{p}, \theta_{q}) + \\
+ Z(\omega_j^{p}; |x_q - t_q|, d_{p}, \theta_{p}, \theta_{q}) + Z(\omega_j^{p}; |x_q - t_q|, \theta_{p}))
\]
holds.

The proof follows from Theorem 3.1.

**Theorem 3.3.** Let \( y^k \ (k=1, n) - c.j.t.c., f \in f_0(\Delta, \theta) \). Then function \( \Phi(z) \) continuously extendable on a core \( \Delta \) from each of \( 2^n \) semicylindrical domains for which the core is common.

By Theorems 3.1 and 3.2 and taking into account (2), we get that the function \( \Phi(z) \) is continuously extended to the cores \( \Delta \) and for the limiting values of the function \( \Phi(z) \) equitable Sokhotskii’s formulas:

\[
\Phi\,^+ (t) = f(t) + \frac{1}{(2\pi i)^n} \int_\Delta \frac{\Delta f(t; t)}{t - t} dt \quad + \cdots + \sum_{p=1}^{n} \sum_{q<p}^{n} \frac{1}{(2\pi i)^{n-1}} \int_\Delta \frac{\Delta f(t_{pq}; t)}{(t - t)_{[pq]} - t_p} dt_{pq} \\
+ \cdots + \sum_{p=1}^{n} \sum_{q<p}^{n} \frac{1}{(2\pi i)^{n-1}} \int_\Delta \frac{\Delta f(t_{pq}; t)}{(t - t)_{[pq]} - t_p} dt_{pq} + \cdots \\
\Phi\,^- (t) = \frac{1}{(2\pi i)^n} \int_\Delta \frac{\Delta f(t; t)}{t - t} dt + \cdots + \sum_{k=1}^{\frac{n}{2}} \sum_{k<p}^{n} \frac{1}{(2\pi i)^{n-1}} \int_\Delta \frac{\Delta f(t_{pq}; t)}{(t - t)_{[pq]} - t_p} dt_{pq} \\
+ \cdots + \sum_{k=1}^{\frac{n}{2}} \sum_{k<p}^{n} \frac{1}{(2\pi i)^{n-1}} \int_\Delta \frac{\Delta f(t_{pq}; t)}{(t - t)_{[pq]} - t_p} dt_{pq} + \cdots \\
\Phi\,^+_{pq} = \frac{1}{(2\pi i)^n} \int_\Delta \frac{\Delta f(t; t)}{t - t} dt + \sum_{k=1}^{\frac{n}{2}} \frac{1}{(2\pi i)^{n-1}} \int_\Delta \frac{\Delta f(t_{pq}; t)}{(t - t)_{[pq]} - t_p} dt_{pq} + \cdots \\
+ \sum_{k=1}^{\frac{n}{2}} \frac{1}{(2\pi i)^{n-1}} \int_\Delta \frac{\Delta f(t_{pq}; t)}{(t - t)_{[pq]} - t_p} dt_{pq} + \cdots.
\]
+ \sum_{k=1}^{n} \sum_{r=1}^{n} \frac{1}{(2\pi i)^{n-2}} \int_{\Delta_{[k]}} \frac{\Delta f(t_{kr}; t)}{(r-t)_{[kr]}} \, dt_{kr} + \ldots +

+ \sum_{k=1}^{n} \sum_{r=1}^{n} \frac{1}{(2\pi i)^{2}} \int_{\gamma_p} \int_{\gamma_q} \int_{\gamma_k} \frac{f(t_{pqk}; t)}{(r_p - r_q)(r_p - r_q)(r_k - r_k)(r_r - r_r)} \, dt_{pq} \, dt_{k} \, dt_{r} + \ldots +

\Phi^{+pq}_{+pq} (t) = \frac{1}{(2\pi i)^{n}} \int_{\Delta} \frac{\phi(t, t)}{r-t} \, dt + \sum_{k=p, q}^{n} \frac{1}{(2\pi i)^{n-1}} \int_{\Delta_{[k]}} \frac{\Delta f(t_{k}; t)}{(r-t)_{[k]}} \, dt_{k} + \ldots +

\Phi^{-pq}_{+pq} (t) = \frac{1}{(2\pi i)^{n}} \int_{\Delta} \frac{\phi(t, t)}{r-t} \, dt + \sum_{k=p, q}^{n} \frac{1}{(2\pi i)^{n-1}} \int_{\Delta_{[k]}} \frac{\Delta f(t_{k}; t)}{(r-t)_{[k]}} \, dt_{k} + \ldots +

\Phi^{-pq}_{+pq} (t) = \frac{1}{(2\pi i)^{n}} \int_{\Delta} \frac{\phi(t, t)}{r-t} \, dt + \sum_{k=p, q}^{n} \frac{1}{(2\pi i)^{n-1}} \int_{\Delta_{[k]}} \frac{\Delta f(t_{k}; t)}{(r-t)_{[k]}} \, dt_{k} + \ldots +

(24)
\[ \Phi^-(t) = \frac{1}{(2\pi i)^n} \int \frac{\Delta f(t_\tau)}{\tau-t} \, dt. \]

In particular, for the case \( n = 1 \), the Sokhotskii’s formulas (24) take the forms
\[
\left\{ \begin{array}{l}
\Phi^+(t) = \frac{f(t)}{2} + \frac{1}{2\pi i} \int_{t_1}^t \frac{f(t_\tau)}{t_\tau-t} \, dt, \\
\Phi^-\tau(t) = -\frac{f(t)}{2} + \frac{1}{2\pi i} \int_t^t \frac{f(t_\tau)}{t_\tau-t} \, dt,
\end{array} \right. \quad n = 1;
\]
and for the case \( n = 2 \):
\[
\Phi^{++}(t_1, t_2) = \frac{1}{(2\pi i)^2} \int_{t_1}^{t_2} \int_{t_2}^{t_1} \frac{\Delta f(t_1, t_2, t_3)}{(t_1-t_2)(t_2-t_3)} \, dt_1 \, dt_2 +
\frac{1}{2\pi i} \int_{t_1}^{t_2} \int_{t_2}^{t_1} \frac{f(t_1, t_2)-f(t_2, t_1)}{t_2-t_1} \, dt_1 \, dt_2.
\]
\[
\Phi^{+-}(t_1, t_2) = \frac{1}{(2\pi i)^2} \int_{t_1}^{t_2} \int_{t_2}^{t_1} \frac{\Delta f(t_1, t_2, t_3)}{(t_1-t_2)(t_2-t_3)} \, dt_1 \, dt_2 +
\frac{1}{2\pi i} \int_{t_1}^{t_2} \int_{t_2}^{t_1} \frac{f(t_1, t_2)-f(t_2, t_1)}{t_1-t_2} \, dt_1 \, dt_2;
\]
\[
\Phi^{-+}(t_1, t_2) = \frac{1}{(2\pi i)^2} \int_{t_1}^{t_2} \int_{t_2}^{t_1} \frac{\Delta f(t_1, t_2, t_3)}{(t_1-t_2)(t_2-t_3)} \, dt_1 \, dt_2 +
\frac{1}{2\pi i} \int_{t_1}^{t_2} \int_{t_2}^{t_1} \frac{f(t_1, t_2)-f(t_2, t_1)}{t_2-t_1} \, dt_1 \, dt_2;
\]
\[
\Phi^{--}(t_1, t_2) = \frac{1}{(2\pi i)^2} \int_{t_1}^{t_2} \int_{t_2}^{t_1} \frac{\Delta f(t_1, t_2, t_3)}{(t_1-t_2)(t_2-t_3)} \, dt_1 \, dt_2.
\]

Above we have shown the behavior of Cauchy-type integrals on the core \( \Delta \) of the border. Now we explain the behavior of the integrals on the whole boundary of the semicylindrical domain. To reduce the entries in detail, consider the case \( n = 3 \) for
\[ D_1^+ \times D_2^- \times D_3^+. \]

The boundary of this domain consists of the sets: \( \gamma^1 \times D_2^- \times D_3^+; D_1^+ \times \gamma^2 \times D_3^+; D_1^+ \times D_2^- \times D_3^+; \gamma^1 \times \gamma^2 \times D_3^+; \gamma^1 \times D_2^- \times \gamma^3 ; D_1^+ \times \gamma^2 \times \gamma^3 \) and \( \Delta = \gamma^1 \times \gamma^2 \times \gamma^3 \).

The integral (1) for \( n = 3 \) we write as follows
\[
\Phi(z) = \frac{1}{2\pi i} \int_{\gamma^1} \frac{f(t) \delta \tau}{\tau-z} \, dt, \quad p = 1, 2, 3, \quad \tau \neq z_p,
\]
\[
\Phi(z) = \frac{1}{(2\pi i)^2} \int_{\Delta^1} \left[ \frac{1}{2\pi i} \int_{\gamma^1} \frac{f(t) \delta \tau}{(\tau-z) \delta \tau} \, dt \right] \frac{d\tau_p}{\tau^p-z_p}, \quad p = 1, 2, 3.
\]

Let us consider integral (30) as integral of Cauchy-type of a complex variable with a core
\[ \Phi_{p=3} \left( \gamma^1 \times D_2^- \times D_3^+; D_1^+ \times \gamma^2 \times D_3^+; D_1^+ \times D_2^- \times D_3^+; \gamma^1 \times \gamma^2 \times D_3^+; \gamma^1 \times D_2^- \times \gamma^3 ; D_1^+ \times \gamma^2 \times \gamma^3 \right), \]

depending on the parameters \( z_p, k = 1, 2, 3, k \neq p \) and applying to them Sokhotskii’s formulas (26) of the variable \( z_p \) and we obtain
\[
\Phi^{++}(z_p) = \frac{(-1)^{p-1}}{(2\pi i)^2} \int_{\gamma^1} \left[ \frac{1}{2\pi i} \int_{\gamma^1} \frac{f(t_\tau) [d \tau_p]}{(t_\tau-z) \delta \tau} \right] \frac{d\tau_p}{\tau^p-z_p},
\]
\[
\Phi^{+-}(z_p) = \frac{1}{(2\pi i)^2} \int_{\gamma^1} \left[ \frac{1}{2\pi i} \int_{\gamma^1} \frac{f(t_\tau) [d \tau_p]}{(t_\tau-z) \delta \tau} \right] \frac{d\tau_p}{\tau^p-z_p},
\]
\[
\Phi^{-+}(z_p) = \frac{1}{(2\pi i)^2} \int_{\gamma^1} \left[ \frac{1}{2\pi i} \int_{\gamma^1} \frac{f(t_\tau) [d \tau_p]}{(t_\tau-z) \delta \tau} \right] \frac{d\tau_p}{\tau^p-z_p}.
\]
At \( p = 1, 2, 3 \) the formula (32) gives boundary values of integral (1) in points of boundary sets: \( \gamma ^1 \times D_2^- \times D_3^+; D_1^+ \times \gamma ^2 \times D_3^+; D_1^+ \times D_2^- \times \gamma ^3 \).

Considering integrals (31) as integral of Cauchy-type of two complex variables of \( z_k, k = 1, 2, 3, k \neq p \) with the core
\[ \Phi_{p=3} \left( z_p \right) = \int_{\gamma^1} \frac{f(t_\tau)}{(\tau-z_p)}, \]

depending on the parameter \( z_p \) and applying to them Sokhotskii’s formulas (26), (27) and (28) on the corresponding variables and obtain...
\[
\Phi^{++}(z_{t_2}) = -\frac{1}{4} \frac{1}{2\pi i} \int_{D_3} f(\tau_{t_4}) d\tau_3 + \frac{1}{2} \frac{1}{(2\pi i)^2} \int_{\Delta_{[2]}} f(\tau_{t_2}) d\tau_2 + \frac{1}{2} \frac{1}{(2\pi i)^3} \int_{\Delta_{[3]}} f(\tau_{t_3}) d\tau_3 + \frac{1}{(2\pi i)^4} \int_{\Delta_{[4]}} f(\tau_{t_4}) d\tau_4 + \frac{1}{(2\pi i)^5} \int_{\Delta_{[5]}} f(\tau_{t_5}) d\tau_5 + \frac{1}{(2\pi i)^6} \int_{\Delta_{[6]}} f(\tau_{t_6}) d\tau_6
\]

Formulas (33), (34) and (35) give values of integral (1) at points of sets \(y^1 \times y^2 \times D^+_1 \times D^+_2 \times y^3 \times D^+_3 \times y^4 \times y^5 \times y^6 \). All shifts integrals at a conclusion of formulas (32)-(35) are admissible as only shifts of special integrals with routine. All shifts integrals at a conclusion of formulas (32)-(35) are extended to the entire of border of the semicylindrical domain and the limiting values of the function \(\Phi(z)\) are formulas such as Sokhotski’s formulas for the case of the core (24).

Thus the following theorem is proven.

**Theorem 3.4.** Let \(y^k (k = \overline{1, n}) - r.i.c.c., f \in I_0(\Delta, \theta).\) Then the function \(\Phi(z)\) defined as (1) is continuously extended to the entire of border of the semicylindrical domain and the limiting values of the function \(\Phi(z)\) are formulas such as Sokhotski’s formulas for the case of the core (24).

### References


[18] Gaziev A., Bubnov E. A. About the special Cauchy integrals with a continuous density for functions of several variables.// DEP. in Uzniinti.08.07.1985. P. 47.