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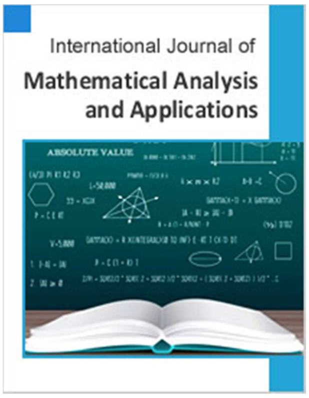

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Product of Extended Cesáro Operator and Composition Operator from B^α to $Q_k(p,q)$ Spaces

A. Kamal, T. I. Yassen

Department of Mathematics, Faculty of Science, Port Said University, Port Said, Egypt

Email address
taha_hmour@yahoo.com (T. I. Yassen)

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Abstract
In this paper, we study the properties boundedness and compactness of product of extended Cesáro operator and composition operator from the Bloch-type spaces B^α to $Q_K(p,q)$ spaces on the unit ball of C^n Moreover, necessary and sucient conditions are given for The product of extended Cesáro operator and composition operator from the Bloch-type spaces B^α to $Q_K(p,q)$ spaces to be bounded and compact.

1. Introduction

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ the open unit ball in $\mathbb{C}^n$, $H(\mathbb{B})$ denote the class of all analytic functions in $\mathbb{B}$. Let dA denote the Lebesegue measure on $\mathbb{B}$ normalized so that $A(\mathbb{B}) = 1$.

For $f \in H(\mathbb{B})$, let

$$f'(z) = \sum_{k=1}^{n} z_k \frac{\partial f}{\partial z_k}(z)$$

be the redial derivative of f.

Definition 1.1 *(see [20]) Let f be an analytic function in $\mathbb{B}$ and $0 < \alpha < \infty$*. The α-Bloch space $\mathbb{B}^\alpha$ is defined by

$$\mathbb{B}^\alpha = \{f \in H(\mathbb{B}) : \| f \|_{\mathbb{B}^\alpha} = \sup_{z \in \mathbb{B}} (1 - |z|^2)^\alpha | f'(z)| < \infty\},$$

the little α-Bloch space B_0^α is given as follows

$$\mathbb{B}_0^\alpha = \{f \in H(\mathbb{B}) : \| f \|_{\mathbb{B}_0^\alpha} = \lim_{|z| \to 1^-} (1 - |z|^2)^\alpha | f'(z)| = 0\}.$$

The spaces $\mathbb{B}^1$ and $\mathbb{B}_0^1$ are called the Bloch space and denoted by $\mathbb{B}$ and B_0 respectively (see [1]).

Let the Green's function of $\mathbb{B}$ be defined as $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$ is the Möbius transformation related to the point $a \in \mathbb{B}$.

Definition 1.2 *(see [17]) Let* $K:[0,\infty)\to[0,\infty)$ *be a right continuous and nondecreasing function. For* $0<p<\infty$ *and* $-2<q<\infty$. *The space* $Q_K(p,q)$ *is defined by*

$$Q_K(p,q)=\{f\in H(\mathrm{B}):\sup_{a\in\mathrm{B}}\int_{\mathrm{B}}|f'(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)<\infty\}.$$

If

$$\lim_{|a|\to1^-}\sup_{a\in\mathrm{B}}\int_{\mathrm{B}}|f'(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)=0,$$

then $f\in Q_{K,0}(p,q)$.

Wulan and Zhou in [18] mentioned the following properties of these spaces:

(a). For $p=2, q=0$, we obtain $Q_K(p,q)=Q_K$ (see [4, 11, 18]).

(b). For $p=2$, $q=0$, and $K(t)=t^p$, we obtain $Q_K(p,q)=Q_p$ (see [2]).

(c). For $K(t)=t^s$, then $Q_K(p,q)=F(p,q,s)$ (see [3, 19]).

A linear composition operator C_ϕ is defined by $C_\phi(f)=(f\circ\phi)$ for f in the set $H(\mathrm{B})$ of analytic functions on B. The study of composition operator C_ϕ acting on spaces of analytic functions has engaged many analysts for many years (see [3, 11] and others).

The problem of boundedness and compactness of C_ϕ has been studied in many Banach spaces of analytic functions and the study of such operators has recently attracted the most attention (see [9, 10, 16] and others).

Let $h\in H(\mathrm{B})$, the extended Cesáro operator T_h with symbol h is the operator on $H(\mathrm{B})$,

$$T_h f(z)=\int_0^1 f(tz)h'(tz)\frac{dt}{t},\qquad f\in H(\mathrm{B}), z\in\mathrm{B}\qquad (see\ [8]).$$

This operator is called generalized Cesáro operator, which has been studied in (see [5, 6, 7] and other).

Here, we consider the product of extended Cesáro operator T_h and of composition operator C_ϕ, which are defined by

$$T_hC_\varphi f(z)=\int_0^1 f(\varphi(tz))h'(tz)\frac{dt}{t},\quad f\in H(\mathrm{B}), z\in\mathrm{B}\quad (see\ [12]).$$

In this paper we characterize the boundedness and compactness of the product T_hC_ϕ of extended Cesáro operator and composition operator from Bloch-type space to $Q_K(p,q)$ spaces on the unit ball of C^n.

2. Auxiliary Results

In this section we state several results, which are used in the main result proofs.

Definition 2.1 The operator $T_hC_\phi:B^\alpha\to Q_K(p,q)$ is said to be bounded, if there is a positive constant C such that $\|T_hC_\phi f\|_{Q_K(p,q)}\le C\|f\|_{B^\alpha}$ for all $f\in B^\alpha$.

Definition 2.2 The operator $T_hC_\phi:B^\alpha\to Q_K(p,q)$ is said to be compact, if it maps any ball in B^α onto a pre-compact set in $Q_K(p,q)$.

The following lemma follows by standard arguments similar to those outlined in [16]. Hence we omit the proof.

Lemma 2.1 Assume ϕ is a analytic mapping from B into itself and let $0<p,\alpha<\infty, -2<q<\infty$, then $T_hC_\phi:B^\alpha\to Q_K(p,q)$ is compact if and only if for any bounded sequence $\{f_n\}_{n\in N}\in B^\alpha$ which converges to zero uniformly on compact subsets of B as $n\to\infty$ we have $\lim_{n\to\infty}\|T_hC_\phi f_n\|_{Q_K(p,q)}=0$.

Lemma 2.2 (see [13]) Let $f\in B^\alpha$. Then, for $z\in\mathrm{B}$, we have

$$|f(z)|\le C\begin{cases}\|f\|_{B^\alpha} & if\quad 0<\alpha<1,\\ \|f\|_{B^\alpha}\ln\dfrac{e}{1-|z|^2} & if\quad \alpha=1,\\ \dfrac{\|f\|_{B^\alpha}}{(1-|z|^2)^{\alpha-1}} & if\quad \alpha>1.\end{cases}$$

Lemma 2.3 (see [12]) Suppose that $f,h\in H(\mathrm{B})$. Then

$$[T_hC_\phi f(z)]'=f(\phi(z))h'(z).$$

The next lemma was obtained in (see [14]).

Lemma 2.4 If $\alpha>0, b>0$, then the elementary inequality holds,

$$(a+b)^p\le\begin{cases}a^p+b^p & for\quad 0<p<1,\\ 2^{b-1}(a^p+b^p) & for\quad p\ge1.\end{cases}$$

This lemma still holds for sum of finite number n, that is

$$(a_1+a_2+......+a_n)^p\le C(a_1^p+a_2^p+......+a_n^p),\quad (1)$$

where $a_1,a_2,.....a_n>0$, *and* $C>0$.

The next lemma was obtained in (see [15]).

Lemma 2.5 Assume $\alpha>1$. Then there exist $N=N(n)\in\mathrm{N}$ and functions $f_1,.......f_n\in\mathrm{B}^\alpha(\mathrm{B})$ such that

$$|f_1(z)|+......+|f_n(z)|\ge\frac{C}{(1-|z|^2)^{\alpha-1}},\qquad z\in\mathrm{B},\quad (2)$$

where C is a positive constant.

3. The Boundedness and Compactness of the Operator $T_hC_\varphi : \mathrm{B}^\alpha \to Q_K(p,q)$

3.1. The Case α>1

Theorem 3.1 Let $\alpha > 1$, $h \in H(\mathrm{B})$, ϕ is a analytic mapping from B into itself. Then $T_hC_\phi : \mathrm{B}^\alpha \to Q_K(p,q)$ is bounded if and only if

$$\Phi_1 := \sup_{a\in \mathrm{B}} \int_{\mathrm{B}} \frac{|h'(z)|^p (1-|z|^2)^q K(g(z,a))}{(1-|\phi(z)|^2)^{(\alpha-1)p}} dA(z) < \infty. \quad (3)$$

Proof: Assume first (3) is holds, and $f \in \mathrm{B}^\alpha$, by Lemma 2.2 and Lemma 2.3 we have

$$\begin{aligned}
\| T_hC_\phi f \|_{K,p,q}^p &= \sup_{a\in \mathrm{B}} \int_{\mathrm{B}} |(T_hC_\phi f)'(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) \\
&= \sup_{a\in \mathrm{B}} \int_{\mathrm{B}} |(f(\phi(z))h'(z))|^p (1-|z|^2)^q K(g(z,a)) dA(z) \\
&\le C \| f \|_{B^\alpha}^p \sup_{a\in \mathrm{B}} \int_{\mathrm{B}} \frac{|h'(z)|^p (1-|z|^2)^q K(g(z,a))}{(1-|\phi(z)|^2)^{(\alpha-1)p}} dA(z) \\
&\le C \| f \|_{B^\alpha}^p \Phi_1. \\
&< \infty.
\end{aligned}$$

It follows that $T_hC_\phi : \mathrm{B}^\alpha \to Q_K(p,q)$ is bounded.

For the other direction, we assume $T_hC_\phi : \mathrm{B}^\alpha \to Q_K(p,q)$ is bounded. Then using Lemma 2.4 and Lemma 2.5 we obtain

$$\begin{aligned}
&\{\| T_hC_\phi f_1 \|_{K,p,q}^p + \| T_hC_\phi f_2 \|_{K,p,q}^p\} \\
&= \{\sup_{a\in \mathrm{B}} \int_{\mathrm{B}} [|(T_hC_\phi f_1)'(z)|^p + |(T_hC_\phi f_2)'(z)|^p](1-|z|^2)^q K(g(z,a)) dA(z)\} \\
&\ge \{\sup_{a\in \mathrm{B}} \int_{\mathrm{B}} [|(T_hC_\phi f_1)'(z)| + |(T_hC_\phi f_2)'(z)|]^p (1-|z|^2)^q K(g(z,a)) dA(z)\} \\
&\ge \{\sup_{a\in \mathrm{B}} \int_{\mathrm{B}} [|f_1(\phi(z))| + |f_2(\phi(z))|]^p |h'(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z)\} \\
&\ge C\{\sup_{a\in \mathrm{B}} \int_{\mathrm{B}} \frac{|h'(z)|^p (1-|z|^2)^q K(g(z,a))}{(1-|\phi(z)|^2)^{(\alpha-1)p}} dA(z)\} \\
&\ge C\Phi_1.
\end{aligned}$$

Form this and the boundedness of T_hC_ϕ, it follows that (3) holds. The proof of this theorem is completed.

Theorem 3.2 Let $\alpha > 1$, $h \in H(\mathrm{B})$, ϕ is a analytic mapping from B into itself. Then $T_hC_\phi : \mathrm{B}^\alpha \to Q_K(p,q)$ is compact if and only if (3) holds.

Proof: Assume that $T_hC_\phi : \mathrm{B}^\alpha \to Q_K(p,q)$ is compact. Then it is bounded, then (3) holds from Theorem 3.1.

Conversely, assume that (3) holds. Then, form (3) we obtain

$$M = \sup_{a\in \mathrm{B}} \int_{\mathrm{B}} |h'(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) < \infty. \quad (4)$$

Since $\sup_{x\in[0,1)} (1-x^2)^{(\alpha-1)} > 0$.

Assum that $\{f_j\}_{j\in N}$ is bounded sequence in B^α, such that $f_j \to 0$ uniformly on the compact subsets of B as $j \to \infty$. Suppose that $\sup_{j\in N} \| f_j \|_{\mathrm{B}^\alpha} \le L$. It follows from (3) that for any $\varepsilon > 0$, there exist a constant $\delta \in (0,1)$, such that

$$\sup_{a\in \mathrm{B}} \int_{|\phi(z)|<\delta} \frac{|h'(z)|^p (1-|z|^2)^q K(g(z,a))}{(1-|\phi(z)|^2)^{(\alpha-1)p}} dA(z) < \varepsilon^p. \quad (5)$$

Let $M_1 = \{\omega \in \mathrm{B}, |\omega| \le \delta\}$, then M_1 is compact subset of B. Since $f_j \to 0$ uniformly on the compact subsets of B as $j \to \infty$. and $h \in Q_K(p,q)$, we have

$$\begin{aligned}
\| T_hC_\phi f_j \|_{K,p,q}^p &= \sup_{a\in \mathrm{B}} \int_{\mathrm{B}} |(T_hC_\phi f_j)'(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) \\
&= \sup_{a\in \mathrm{B}} \int_{|\phi(z)|\le\delta} |(f_j(\phi(z))h'(z))|^p (1-|z|^2)^q K(g(z,a)) dA(z) \\
&+ \sup_{a\in \mathrm{B}} \int_{|\phi(z)|>\delta} |(f_j(\phi(z))h'(z))|^p (1-|z|^2)^q K(g(z,a)) dA(z) \\
&= J_1 + J_2.
\end{aligned}$$

Since M_1 is compact and from (4) we have

$$J_1 := \sup_{a\in B}\int_{|\phi(z)|\le\delta} |(f_j(\phi(z))h'(z))|^p (1-|z|^2)^q K(g(z,a))dA(z)$$
$$\le \sup_{\omega\in M_1} |f_j(\omega)|^p \int_{|\phi(z)|\le\delta} |h'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z)$$
$$\le M \sup_{\omega\in M_1} |f_j(\omega)|^p \to 0, \quad J\to\infty. \quad (6)$$

On other hand, by Lemma 2.4 and from (5), we have

$$J_2 := \sup_{a\in B}\int_{|\phi(z)|>\delta} |(f_j(\phi(z))h'(z))|^p (1-|z|^2)^q K(g(z,a))dA(z)$$
$$\le C\|f_j\||_{B^\alpha} \sup_{a\in B}\int_{|\phi(z)|<\delta} \frac{|h'(z)|^p (1-|z|^2)^q K(g(z,a))}{(1-|\phi(z)|^2)^{(\alpha-1)p}} dA(z)$$
$$\le CM^p < \varepsilon^p. \ (7)$$

From (6), (7) and since ε is an arbitrary positive number, we get

$$\lim_{j\to\infty} \|T_hC_\phi f_j\|^p_{K,p,q} = 0. \quad (8)$$

Hence by (8) and Lemma 2.1 we get $T_hC_\phi : B^\alpha \to Q_K(p,q)$ is compact. This completes the proof of this theorem.

3.2. The Case 0<α<1

Theorem 3.3 Let $0<\alpha<1$, $h\in H(B)$, ϕ is a analytic mapping from B into itself. Then $T_hC_\phi : B^\alpha \to Q_K(p,q)$ is bounded if and only if $h\in Q_K(p,q)$. Moreover, if $T_hC_\phi : B^\alpha \to Q_K(p,q)$ is bounded. Then

$$\|T_hC_\phi f\|_{B^\alpha\to Q_K(p,q)} \approx |h|_{Q_K(p,q)} \quad (9)$$

Proof: Assume that $h\in Q_K(p,q)$. For any $f\in B^\alpha$, by Lemma 2.2 and Lemma 2.3 we have

$$\|T_hC_\phi f\|^p_{K,p,q} = \sup_{a\in B}\int_B |(T_hC_\phi f)'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z)$$
$$= \sup_{a\in B}\int_B |(f(\phi(z))h'(z))|^p (1-|z|^2)^q K(g(z,a))dA(z)$$
$$\le C\|f\|^p_{B^\alpha} \sup_{a\in B}\int_B |h'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z).$$

That is

$$\|T_hC_\phi f\|_{B^\alpha\to Q_K(p,q)} \le C\|h\|_{K,p,q}. \quad (10)$$

For the other direction, we assume $T_hC_\phi : B^\alpha \to Q_K(p,q)$ is bounded. By taking the function $f_0(z)=1\in B^\alpha$ and $\|f_0\|_{B^\alpha}=1$, then we obtain

$$\|T_hC_\phi\|^p_{B^\alpha\to Q_K(p,q)} = \|T_hC_\phi\|^p_{B^\alpha\to Q_K(p,q)} \|f_0\|^p_{B^\alpha}$$
$$\ge \|T_hC_\phi f_0\|^p_{Q_K(p,q)}$$
$$= \sup_{a\in B}\int_B |(T_hC_\phi f_0)'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z)$$
$$= \sup_{a\in B}\int_B |(f_0(\phi(z))h'(z))|^p (1-|z|^2)^q K(g(z,a))dA(z)$$
$$= \sup_{a\in B}\int_B |h'(z))|^p (1-|z|^2)^q K(g(z,a))dA(z)$$
$$= \|h\|^p_{K,p,q}.$$

That is

$$\|h\|_{K,p,q} \le \|T_hC_\phi f\|_{B^\alpha\to Q_K(p,q)}. \quad (11)$$

Thus from (10) and (11) we have the relation in (9). The proof of this theorem is completd.

Theorem 3.4 Let $0<\alpha<1$, $h\in H(B)$, ϕ is a analytic mapping from B into itself. Then $T_hC_\phi : B^\alpha \to Q_K(p,q)$ is compact if and only if $h\in Q_K(p,q)$.

Proof: The proof of this theorem is similar to that of Theorem 3.2.

3.3. The Case α=1

Theorem 3.5 Let $\alpha=1$, $h\in H(B)$, ϕ is a analytic mapping from B into itself. Then $T_hC_\phi : B^1 \to Q_K(p,q)$ is bounded (compact) if

$$\Phi_2 := \sup_{a\in B}\int_B |h'(z)|^p \left(ln\frac{e}{1-|\phi(z)|^2}\right)^p (1-|z|^2)^q K(g(z,a))dA(z) < \infty. \quad (12)$$

Proof: Assume that (12) holds. For any $f\in B^1$, by Lemma 2.2 and Lemma 2.3 we have

$$\|T_hC_\phi f\|^p_{K,p,q} = \sup_{a\in B}\int_B |(T_hC_\phi f)'(z)|^p (1-|z|^2)^q K(g(z,a))dA(z)$$
$$= \sup_{a\in B}\int_B |(f(\phi(z))h'(z))|^p (1-|z|^2)^q K(g(z,a))dA(z)$$
$$\le C\|f\|^p_{B^\alpha} \sup_{a\in B}\int_B |h'(z)|^p \left(ln\frac{e}{1-|\phi(z)|^2}\right)^p (1-|z|^2)^q K(g(z,a))dA(z).$$
$$\le C\|f\|^p_{B^1}\ \Phi_2 < \infty.$$

So $T_hC_\phi : B^\alpha \to Q_K(p,q)$ is bounded. The proof of compactness is similar to the corresponding part of Theorem 3.2.

4. Conclusion

We proved in many case depended on the value of $\alpha=1$ the boundedness and Compactness of product of extended Cesáro operator and composition operator from the Bloch-type spaces B^{α} to $Q_K(p,q)$ spaces on the unit ball still holds.

References

[1] J. Arazy, D. Fisher and J. Peetre, Möbius invariant function spaces, J. Reine Angew. Math. 363 (1986), 110-145.

[2] R. Aulaskari and P. Lappan, Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal, Complex Analysis and its applications, Pitman Research Notes in Math. 305, Longman Scientific and Technical Harlow, (1994), 136-146.

[3] A. El-Sayed Ahmed and M. A. Bakhit, Composition operators on some holomorphic Banach function spaces, Math. Scand. 104 (2) (2009), 275-295.

[4] M. Essén, H. Wulan, and J. Xiao, Several function-theoretic characterizations of Möbius invariant Q_K spaces, J. Funct. Anal. 230 (2006), 78-115.

[5] Z. S. Fang, Z. H. Zhou, Extended Cesàro operators from weighted Bloch spaces to Zygmund spaces in the unit ball, J. Math. Anal. Appl. 359 (2) (2009) 499-507.

[6] Z. S. Fang, Z. H. Zhou, Extended Cesàro operators on BMOA spaces in the unit ball, J. Math. Inequal. 4 (1) (2010) 27-36.

[7] D. Gu, Extended Cesàro operators from logarithmic-type spaces to Bloch-type spaces, Abstr. Appl. Anal. 2009, 9p, Article ID 246521.

[8] Z. J. Hu, Extended Cesàro operators on mixed norm spaces, Proc. Amer. Math. Soc. 131 (7) (2003) 2171-2179.

[9] A. Kamal, A. El-Sayed Ahmed and T. I. Yassen, Quasi-metric spaces and composition operators on $B^*_{\alpha,\log}$ and $F^*_{\log}(p;q;s)$ spaces, J. Comp. Theoretical Nanoscience. 12 (8) (2015), 1795-1801.

[10] A. Kamal, and T. I. Yassen, Some Properties of Composition Operator Acting Between General Hyperbolic Type Spaces, Int. J. Math. Anal. Appl. 2 (2) (2015), 17-26.

[11] S. Li and H. Wulan, Composition operators on Q_K spaces, J. Math. Anal. Appl. 327 (2007), 948-958.

[12] Yu-Xia Liang, Ze-Hua Zhou, Product of Extended Cesàro Operator and Composition Operator from Lipschitz Space to $F(p,q,s)$ Space on the Unit Ball. Abstr, Appl. Anal. 2011, 9 p, Article ID 152635.

[13] LÜ Xiao-fen, Weighted composition operators from $F(p,q,s)$ spaces to Bers-type spaces in the unit ball, Appl. Math. J. Chinese Univ., 24 (4) (2009), 462-472.

[14] J. Shi, Inequalities for the integral means of holomorphic function and their derivatives in the unit ball of C^n, Trans Amer Math Soc. 328 (1991), 619-637.

[15] S. Stevi `c , On operator P^g_φ from the logarithmic Bloch-type space to the mixed-norm space on the unit ball, Appl. Math. Comput. 215 (2010) 4248-4255.

[16] M. Tjani, Compact composition operators on Besov spaces, Trans. Amer. Math. Soc. 355 (2003), 4683-4698.

[17] H. Wulan and J. Zhou, Q_K type spaces of analytic functions. J. Funct. Spaces Appl. 4 (1) (2006), 73-84.

[18] H. Wulan and Y. Zhang, Hadamard products and Q_K spaces, J. Math. Anal. Appl, 337 (2) (2008), 1142-1150.

[19] R. Zhao, On a general family of function spaces, Ann. Acad. Sci. Fenn. Series A I. Mathematica. Dissertationes. 105. Helsinki: Suomalainen Tiedeakatemia, (1996), 1-56.

[20] R. Zhao, On α-Bloch functions and VMOA, Acta. Math. Sci, 3 (1996), 349 - 360.