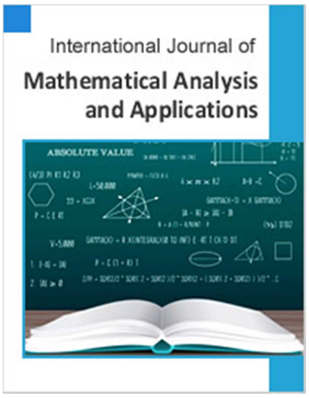
International Journal of Mathematical Analysis and Applications
2016; 3(2): 17-25
http://www.aascit.org/journal/ijmaa
ISSN: 2375-3927

American Association for Science and Technology



Exact Proof of the Riemann Hypothesis

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Citation
Fayez Fok Al Adeh. Exact Proof of the Riemann Hypothesis. *International Journal of Mathematical Analysis and Applications.* Vol. 3, No. 2, 2016, pp. 17-25.

Abstract
I have already discovered a simple proof of the Riemann Hypothesis. The hypothesis states that the nontrivial zeros of the Riemann zeta function have real part equal to 0.5. I assume that any such zero is s =a+ bi. I use integral calculus in the first part of the proof. In the second part I employ variational calculus. Through equations (50) to (59) I consider (a) as a fixed exponent, and verify that a =0.5. From equation (60) onward I view (a) as a parameter (a <0.5) and arrive at a contradiction. At the end of the proof (from equation (73)) and through the assumption that (a) is a parameter, I verify again that a = 0.5.

Keywords
Definite Integrel,
Indefinite Integral,
Variational Calculus

Received: June 24, 2016
Accepted: July 4, 2016
Published: July 29, 2016

1. Introduction

The Riemann zeta function is the function of the complex variable s = a + bi (i = $\sqrt{-1}$), defined in the half plane a >1 by the absolute convergent series

$$\zeta(s)=\sum_{1}^{\infty}\frac{1}{n^{s}} \tag{1}$$

and in the whole complex plane by analytic continuation.

The function $\zeta(s)$ has zeros at the negative even integers -2, -4, … and one refers to them as the trivial zeros. The Riemann hypothesis states that the nontrivial zeros of $\zeta(s)$ have real part equal to 0.5.

2. First Part of the Proof of the Hypothesis

We begin with the equation

$$\zeta(s)=0 \tag{2}$$

And with

$$s= a+bi \tag{3}$$

$$\zeta(a+bi)=0 \tag{4}$$

It is known that the nontrivial zeros of $\zeta(s)$ are all complex. Their real parts lie between zero and one.

If 0 < a < 1 then

$$\zeta(s) = s\int_0^\infty \frac{[x]-x}{x^{s+1}}\,dx \quad (0 < a < 1) \tag{5}$$

[x] is the integer function
Hence

$$\int_0^\infty \frac{[x]-x}{x^{s+1}}\,dx = 0 \tag{6}$$

Therefore

$$\int_0^\infty ([x]-x)x^{-1-a-bi}dx = 0 \tag{7}$$

$$\int_0^\infty ([x]-x)x^{-1-a}x^{-bi}dx = 0 \tag{8}$$

$$\int_0^\infty x^{-1-a}([x]-x)(\cos(b\log x) - i\sin(b\log x))dx = 0 \tag{9}$$

Separating the real and imaginary parts we get

$$\int_0^\infty x^{-1-a}([x]-x)\cos(b\log x)dx = 0 \tag{10}$$

$$\int_0^\infty x^{-1-a}([x]-x)\sin(b\log x)dx = 0 \tag{11}$$

According to the functional equation, if $\zeta(s)$ =0 then $\zeta(1-s)$ =0. Hence we get besides equation (11)

$$\int_0^\infty x^{-2+a}([x]-x)\sin(b\log x)dx = 0 \tag{12}$$

In equation (11) replace the dummy variable x by the dummy variable y

$$\int_0^\infty y^{-1-a}([y]-y)\sin(b\log y)dy = 0 \tag{13}$$

We form the product of the integrals (12)and (13). This is justified by the fact that both integrals (12) and (13) are absolutely convergent .As to integral (12) we notice that

$$\int_0^\infty x^{-2+a}([x]-x)\sin(b\log x)dx \le \int_0^\infty \left| x^{-2+a}([x]-x)\sin(b\log x) \right| dx$$

$$\le \int_0^\infty x^{-2+a}((x))dx$$

(where ((z)) is the fractional part of z, $0 \le ((z)) < 1$)

$= \lim(t \to 0) \int_0^{1-t} x^{-1+a}\,dx + \lim(t \to 0) \int_{1+t}^\infty x^{-2+a}((x))dx$

(t is avery small positive number) (since ((x)) =x whenever $0 \le x < 1$)

$= \frac{1}{a} + \lim(t \to 0) \int_{1+t}^\infty x^{-2+a}((x))dx$

$< \frac{1}{a} + \lim(t \to 0) \int_{1+t}^\infty x^{-2+a}dx = \frac{1}{a} + \frac{1}{a-1}$

And as to integral (13) $\int_0^\infty y^{-1-a}([y]-y)\sin(b\log y)dy$

$\le \int_0^\infty \left| y^{-1-a}([y]-y)\sin(b\log y) \right| dy$

$\le \int_0^\infty y^{-1-a}((y))dy$

$= \lim\,(t\to 0) \int_0^{1-t} y^{-a}\,dy + \lim(t\to 0) \int_{1+t}^{\infty} y^{-1-a}((y))dy$

(t is avery small positive number) (since ((y)) =y whenever $0 \le y<1$)

$= \frac{1}{1-a} + \lim(t\to 0) \int_{1+t}^{\infty} y^{-1-a}((y))dy$

$< \frac{1}{1-a} + \int_{1+t}^{\infty} y^{-1-a}dy = \frac{1}{1-a} + \frac{1}{a}$

Since the limits of integration do not involve x or y, the product can be expressed as the double integral

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y)\sin(b\log y)\sin(b\log x)dxdy = 0 \tag{14}$$

Thus

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y)(\cos(b\log y + b\log x) - \cos(b\log y - b\log x))dxdy = 0 \tag{15}$$

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y)(\cos(b\log xy) - \cos(b\log \frac{y}{x}))dxdy = 0 \tag{16}$$

That is

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y)\cos(b\log xy)dxdy =$$

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y)\cos(b\log \frac{y}{x})dxdy \tag{17}$$

Consider the integral on the right-hand side of equation (17)

$$\int_0^{\infty}\int_0^{\infty} x^{-2+a} y^{-1-a}([x]-x)([y]-y)\cos(b\log \frac{y}{x})dxdy \tag{18}$$

In this integral make the substitution x = $\frac{1}{z}$ dx = $\frac{-dz}{z^2}$

The integral becomes

$$\int_0^{\infty}\int_{\infty}^{0} z^{2-a} y^{-1-a}([\frac{1}{z}]-\frac{1}{z})([y]-y)\cos(b\log zy)\frac{-dz}{z^2}dy \tag{19}$$

That is

$$-\int_0^{\infty}\int_{\infty}^{0} z^{-a} y^{-1-a}([\frac{1}{z}]-\frac{1}{z})([y]-y)\cos(b\log zy)dzdy \tag{20}$$

This is equivalent to

$$\int_0^{\infty}\int_0^{\infty} z^{-a} y^{-1-a}([\frac{1}{z}]-\frac{1}{z})([y]-y)\cos(b\log zy)dzdy \tag{21}$$

If we replace the dummy variable z by the dummy variable x, the integral takes the form

$$\int_0^\infty\int_0^\infty x^{-a}y^{-1-a}([\frac{1}{x}]-\frac{1}{x})([y]-y)\cos(b\log xy)dxdy \tag{22}$$

Rewrite this integral in the equivalent form

$$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})([y]-y)\cos(b\log xy)dxdy \tag{23}$$

Thus equation 17 becomes

$$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}([x]-x)([y]-y)\cos(b\log xy)dxdy =$$

$$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})([y]-y)\cos(b\log xy)dxdy \tag{24}$$

Write the last equation in the form

$$\int_0^\infty\int_0^\infty x^{-2+a}y^{-1-a}([y]-y)\cos(b\log xy)\{$$

$$(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})\text{-([x]-x)\}dxdy=0} \tag{25}$$

Let p >0 be an arbitrary small positive number.We consider the following regions in the x –y plane.

The region of integration I = $[0,\infty)\times[0,\infty)$ (26)

The large region I1 = $[p,\infty)\times[p,\infty)$ (27)

The narrow strip I 2 = $[p,\infty)\times[0,p]$ (28)

The narrow strip I 3 = $[0,p]\times[0,\infty)$ (29)

Note that

$$I = I1 \bigcup I2 \bigcup I3 \tag{30}$$

Denote the integrand in the left hand side of equation (25) by

$$F(x,y) = x^{-2+a}y^{-1-a}([y]-y)\cos(b\log xy)\{$$

$$(x^{2-2a}[\frac{1}{x}]-\frac{x^{2-2a}}{x})\text{-([x]-x)\}} \tag{31}$$

Let us find the limit of F (x,y) as x $\longrightarrow\infty$ and y $\longrightarrow\infty$. This limit is given by

$$\text{Lim } x^{-a}y^{-1-a}[-((y))]\cos(b\log xy)[-((\frac{1}{x})) + ((x))\, x^{2a-2}] \tag{32}$$

((z)) is the fractional part of the number z,$0\leq$ ((z)) < 1

The above limit vanishes, since all the functions [- ((y))], cos (blog xy), - $((\frac{1}{x}))$, and ((x)) remain bounded as x $\longrightarrow\infty$ and y $\longrightarrow\infty$

Note that the function F (x,y) is defined and bounded in the region I 1. We can prove that the integral

$\iint_{I1}$ F(x, y) dx dy is bounded as follows (33)

$$\iint_{I1} F(x,y)\,dx\,dy = \iint_{I1} x^{-a}y^{-1-a}[-((y))]\cos(b\log xy)[-((\frac{1}{x})) + ((x)) = x^{2a-2}]\,dx\,dy$$

$$\leq \left|\iint_{I1} x^{-a}y^{-1-a}[-((y))]\cos(b\log xy)[-((\frac{1}{x})) + ((x))\, x^{2a-2}]\,dx\,dy\right|$$

$$= \left| \int_p^\infty \left(\int_p^\infty x^{-a} \cos(b\log xy) \left[-\left(\left(\frac{1}{x}\right)\right) + ((x)) \, x^{2a-2} \right] dx \right) y^{-1-a} \left[-((y)) \right] dy \right|$$

$$\le \int_p^\infty \left| \left(\int_p^\infty x^{-a} \cos(b\log xy) \left[-\left(\left(\frac{1}{x}\right)\right) + ((x)) \, x^{2a-2} \right] dx \right) \right| \left| y^{-1-a} \left[-((y)) \right] \right| dy$$

$$\le \int_p^\infty \left(\int_p^\infty x^{-a} \left| \cos(b\log xy) \right| \left| \left[-\left(\left(\frac{1}{x}\right)\right) + ((x)) \, x^{2a-2} \right] \right| dx \right) \left| y^{-1-a} \left[-((y)) \right] \right| dy$$

$$< \int_P^\infty x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x)) \, x^{2a-2} \right] dx \int_P^\infty y^{-1-a}$$

$$= \frac{1}{ap^a} \int_P^\infty x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x)) \, x^{2a-2} \right] dx$$

$$= \frac{1}{ap^a} \left\{ \lim(t \to 0) \int_P^{1-t} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x)) \, x^{2a-2} \right] dx + \lim(t \to 0) \int_{1+t}^\infty x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x)) \, x^{2a-2} \right] dx \right\}$$

where t is avery small arbitrary positive. number.Since the integral

$\lim(t \to 0) \int_P^{1-t} x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x)) \, x^{2a-2} \right] dx$ is bounded, it remains to show that $\lim(t \to 0)$

$\int_{1+t}^\infty x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x)) \, x^{2a-2} \right] dx$ is bounded.

Since x >1,then $\left(\left(\frac{1}{x}\right)\right) = \frac{1}{x}$ and we have $\lim(t \to 0)$

$$\int_{1+t}^\infty x^{-a} \left[\left(\left(\frac{1}{x}\right)\right) + ((x)) \, x^{2a-2} \right] dx$$

$$= \lim(t \to 0) \int_{1+t}^\infty x^{-a} \left[\frac{1}{x} + ((x)) \, x^{2a-2} \right] dx$$

$$= \lim(t \to 0) \int_{1+t}^\infty \left[x^{-a-1} + ((x)) \, x^{a-2} \right] dx$$

$$< \lim(t \to 0) \int_{1+t}^\infty \left[x^{-a-1} + x^{a-2} \right] dx = \frac{1}{a(1-a)} \tag{34}$$

Hence the boundedness of the integral $\iint_{I1} F(x,y)\,dx\,dy$ is proved.

Consider the region

$$I4 = I2 \bigcup I3 \tag{35}$$

We know that

$$0 = \iint_{I} F(x,y)\,dx\,dy = \iint_{I1} F(x,y)\,dx\,dy + \iint_{I4} F(x,y)\,dx\,dy \tag{36}$$

and that

$$\iint_{I1} F(x,y)\,dx\,dy \text{ is bounded} \tag{37}$$

From which we deduce that the integral

$$\iint_{I4} F(x,y)\,dx\,dy \text{ is bounded} \tag{38}$$

Remember that

$$\iint_{I4} F(x,y)\,dx\,dy = \iint_{I2} F(x,y)\,dx\,dy + \iint_{I3} F(x,y)\,dx\,dy \tag{39}$$

Consider the integral

$$\iint_{I2} F(x,y)\,dx\,dy \le \left| \iint_{I2} F(x,y)\,dx\,dy \right.$$

$$= \left| \int_0^p \left(\int_p^\infty x^{-a} \left\{ \left(\left(\frac{1}{x}\right)\right) - ((x)) \, x^{2a-2} \right\} \cos(b\log xy)\,dx \right) \frac{1}{y^a}\,dy \right|$$

$$\le \int_0^p \left| \int_p^\infty \left(x^{-a} \left\{ \left(\left(\frac{1}{x}\right)\right) - ((x)) \, x^{2a-2} \right\} \cos(b\log xy)\,dx \right) \right| \frac{1}{y^a}\,dy$$

$$\le \int_0^p \left(\int_p^\infty \left| x^{-a} \left\{ \left(\left(\frac{1}{x}\right)\right) - ((x)) \, x^{2a-2} \right\} \right| \left| \cos(b\log xy) \right| dx \right) \frac{1}{y^a}\,dy$$

$$\leq \int_P^\infty \left| x^{-a}\{((\frac{1}{x}))-((x))x^{2a-2}\}\right| dx \times \int_0^P \frac{1}{y^a} dy$$

(This is because in this region ((y)) = y). It is evident that the integral $\int_P^\infty \left| x^{-a}\{((\frac{1}{x}))-((x))x^{2a-2}\}\right| dx$ is bounded,this was proved in the course of proving that the integral $\iint_{I1} F(x,y)\,dx\,dy$ is bounded .Also it is evident that the integral $\int_0^P \frac{1}{y^a}dy$ is bounded. Thus we deduce that the integral

(40) $\iint_{I2} F(x,y)\,dx\,dy$ is bounded

Hence, according to equation(39), the integral

$$\iint_{I3} F(x,y)\,dx\,dy \text{ is bounded} \tag{40}$$

Now consider the integral

$$\iint_{I3} F(x,y)\,dx\,dy \tag{41}$$

We write it in the form

$$\iint_{I3} F(x,y)\,dx\,dy = \int_0^p \left(\int_0^\infty y^{-1-a}((y))\cos(b\log xy)\,dy\right) \frac{\{((\frac{1}{x}))-x^{2a-1}\}}{x^a}dx$$

(This is because in this region ((x)) = x)

$$\leq \left| \int_0^p \left(\int_0^\infty y^{-1-a}((y))\cos(b\log xy)\,dy\right) \frac{\{((\frac{1}{x}))-x^{2a-1}\}}{x^a}dx \right|$$

$$\leq \int_0^p \left| \left(\int_0^\infty y^{-1-a}((y))\cos(b\log xy)\,dy\right) \right| \left| \frac{\{((\frac{1}{x}))-x^{2a-1}\}}{x^a} \right| dx$$

$$\leq \int_0^p \left(\int_0^\infty y^{-1-a}((y))\,dy\right) \left| \frac{\{((\frac{1}{x}))-x^{2a-1}\}}{x^a} \right| dx \tag{42}$$

Now we consider the integral with respect to y

$$\int_0^\infty y^{-1-a}((y))\,dy$$

$$= (\lim t\to 0)\int_0^{1-t} y^{-1-a}\times y\,dy + (\lim t\to 0)\int_{1+t}^\infty y^{-1-a}((y))\,dy$$

(where t is a very small arbitrary positive number). (Note that ((y))=y whenever $0 \leq y<1$).

Thus we have $(\lim t\to 0)\int_{1+t}^\infty y^{-1-a}((y))\,dy < (\lim t\to 0)\int_{1+t}^\infty y^{-1-a}dy = \frac{1}{a}$

and $(\lim t\to 0)\int_0^{1-t} y^{-1-a}\times y\,dy = \frac{1}{1-a}$

Hence the integral (43) $\int_0^\infty y^{-1-a}((y))\,dy$ is bounded. (43)

Since $\left|\int_0^\infty y^{-1-a}((y))\cos(b\log xy)dy\right| \leq \int_0^\infty y^{-1-a}((y))\,dy$, we conclude that the integral $\left|\int_0^\infty y^{-1-a}((y))\cos(b\log xy)dy\right|$ is a bounded function of x . Let this function be H(x). Thus we have

$$\left|\int_0^\infty y^{-1-a}((y))\cos(b\log xy)dy\right| = H(x) \leq K$$

(K is a positive number) (44)

Now equation (44) gives us

$$-K \leq \int_0^\infty y^{-1-a}((y))\cos(b\log xy)dy \leq K \tag{45}$$

According to equation (42) we have

$$\iint_{I3} F(x,y)\,dx\,dy = \int_0^p \left(\int_0^\infty y^{-1-a}((y))\cos(b\log xy)\,dy\right) \frac{\{((\frac{1}{x}))-x^{2a-1}\}}{x^a}dx$$

$$\geq \int_0^p (-K)\frac{\{((\frac{1}{x}))-x^{2a-1}\}}{x^a}dx$$

$$= K \int_p^0 \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a} dx \quad (46)$$

Since $\iint_{I3} F(x,y)\,dx\,dy$ is bounded, then $\int_p^0 \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a}$ dx is also bounded. Therefore the integral

$$G = \int_0^p \frac{\{((\frac{1}{x})) - x^{2a-1}\}}{x^a} dx \text{ is bounded} \quad (47)$$

We denote the integrand of (47) by

$$F = \frac{1}{x^a} \{((\frac{1}{x}))- x^{2a-1}\} \quad (48)$$

Let δ G [F] be the variation of the integral G due to the variation of the integrand δ F.

Since

$$G[F] = \int F\,dx \text{ (the integral (49) is indefinite)} \quad (49)$$

(here we do not consider a as a parameter, rather we consider it as a given exponent)

We deduce that $\frac{\delta G[F]}{\delta F(x)} = 1$

that is

$$\delta G[F] = \delta F(x) \quad (50)$$

But we have

$$\delta G[F] = \int dx \frac{\delta G[F]}{\delta F(x)} \delta F(x) \text{ (the integral (51) is indefinite)} \quad (51)$$

Using equation (50) we deduce that

$$\delta G[F] = \int dx\, \delta F(x) \text{ (the integral (52) is indefinite)} \quad (52)$$

Since G[F] is bounded across the elementary interval [0,p], we must have that

$$\delta G[F] \text{ is bounded across this interval} \quad (53)$$

From (52) we conclude that

$$\delta G = \int_0^P dx\, \delta F(x) = \int_0^P dx \frac{dF}{dx} \delta x = [F\, \delta x] \text{ (at x = p)-} [F\, \delta x] \text{ (at x = 0)} \quad (54)$$

Since the value of [F δx] (at x = p)is bounded,we deduce from equation (54) that

$$\lim (x \to 0)\, F\, \delta x \text{ must remain bounded.} \quad (55)$$

Thus we must have that

$$(\lim x \to 0)\, [\delta x \frac{1}{x^a} \{((\frac{1}{x}))- x^{2a-1}\}] \text{ is bounded.} \quad (56)$$

First we compute

$$(\lim x \to 0)\, \frac{\delta x}{x^a} \quad (57)$$

Applying L 'Hospital ' rule we get

$$(\lim x \to 0)\, \frac{\delta x}{x^a} = (\lim x \to 0) \frac{1}{a} \times x^{1-a} \times \frac{d(\delta x)}{dx} = 0 \quad (58)$$

We conclude from (56) that the product

$$0 \times (\lim x \to 0)\, \{((\frac{1}{x}))- x^{2a-1}\} \text{must remain bounded.} \quad (59)$$

Assume that a =0.5. (remember that we considered a as a given exponent)This value a =0.5 will guarantee that the quantity $\{((\frac{1}{x}))- x^{2a-1}\}$ will remain bounded in the limit as (x $\to 0$).Therefore, in this case (a=0.5) (56) will approach zero as (x $\to 0$) and hence remain bounded.

3. Second Part of the Proof of the Hypothesis

Now suppose that a< 0.5. In this case we consider a as a parameter.Hence we have

$$G_a[x] = \int dx \frac{F(x,a)}{x} x \text{ (the integral (60) is indefinite)} \quad (60)$$

Thus

$$\frac{\delta G_a[x]}{\delta x} = \frac{F(x,a)}{x} \quad (61)$$

But we have that

$$\delta G_a[x] = \int dx \frac{\delta G_a[x]}{\delta x} \delta x \text{ (the integral (62) is indefinite)} \quad (62)$$

Substituting from (61) we get

$$\delta G_a[x] = \int dx \frac{F(x,a)}{x} \delta x \text{ (the integral (63) is indefinite)} \quad (63)$$

We return to equation (49) and write

G = lim (t $\to 0$) $\int_t^p Fdx$ (t is a very small positive number 0<t<p)

$$= \{F\,x(\text{at } p) - \lim (t \to 0)\, Fx\,(\text{at } t)\} - \lim (t \to 0) \int_t^p x\,dF \tag{64}$$

Let us compute

$$\lim (t \to 0)\, Fx\,(\text{at } t) = \lim (t \to 0)\, t^{1-a} ((\frac{1}{t})) - t^{a} = 0 \tag{65}$$

Thus equation (64) reduces to

$$G - Fx\,(\text{at } p) = -\lim (t \to 0) \int_t^p x\,dF \tag{66}$$

Note that the left – hand side of equation (66) is bounded. Equation (63) gives us

$$\delta G_a = \lim (t \to 0) \int_t^p dx \frac{F}{x} \delta x \tag{67}$$

(t is the same small positive number 0<t<p)

We can easily prove that the two integrals $\int_t^p x\,dF$ and $\int_t^p dx \frac{F}{x}\delta x$ are absolutely convergent .Since the limits of integration do not involve any variable , we form the product of (66) and (67)

$$K = \lim(t \to 0) \int_t^p \int_t^p x dF \times dx \frac{F}{x} \delta x = \lim(t \to 0) \int_t^p F dF \times \int_t^p \delta x dx \tag{68}$$

(K is a bounded quantity)

That is

$$K = \lim(t \to 0) [\frac{F^2}{2} (\text{at } p) - \frac{F^2}{2} (\text{at } t)] \times [\delta x\,(\text{at } p) - \delta x\,(\text{at } t)] \tag{69}$$

We conclude from this equation that

$$\{[\frac{F^2}{2} (\text{at } p) - \lim(t \to 0) \frac{F^2}{2} (\text{at } t)] \times [\delta x\,(\text{at } p)]\} \text{ is bounded.} \tag{70}$$

(sincelim (x $\to 0$) δ x = 0, which is the same thing as lim(t $\to 0$) δ x = 0)

Since $\frac{F^2}{2}$ (at p) is bounded, we deduce at once that $\frac{F^2}{2}$ must remain bounded in the limit as (t $\to 0$), which is the same thing as saying that F must remain bounded in the limit as (x $\to 0$) . Therefore.

$$\lim (x \to 0) \frac{((\frac{1}{x})) - x^{2a-1}}{x^a} \text{ must remain bounded} \tag{71}$$

But

$$\lim (x \to 0) \frac{((\frac{1}{x})) - x^{2a-1}}{x^a} = \lim(x \to 0) \frac{x^{1-2a}}{x^{1-2a}} \times \frac{((\frac{1}{x})) - x^{2a-1}}{x^a}$$

$$= \lim(x \to 0) \frac{x^{1-2a}((\frac{1}{x})) - 1}{x^{1-a}} = \lim(x \to 0) \frac{-1}{x^{1-a}} \tag{72}$$

It is evident that this last limit is unbounded. This contradicts our conclusion (71) that

$$\lim (x \to 0) \frac{((\frac{1}{x})) - x^{2a-1}}{x^a} \text{ must remain bounded (for a< 0.5)}$$

Therefore the case a<0.5 is rejected. We verify here that,for a = 0.5 (71)remains bounded as (x $\to 0$).

We have that

$$((\frac{1}{x})) - x^{2a-1} < 1 - x^{2a-1} \tag{73}$$

Therefore

$$\lim(a \to 0.5)\,(x \to 0) \frac{((\frac{1}{x})) - x^{2a-1}}{x^a} < \lim(a \to 0.5)\,(x \to 0)$$

$$\frac{1-x^{2a-1}}{x^a} \quad (74)$$

We consider the limit

$$\lim(a \to 0.5)\ (x \to 0)\ \frac{1-x^{2a-1}}{x^a} \quad (75)$$

We write

$$a = (\lim x \to 0)\ (0.5 + x) \quad (76)$$

Hence we get

$$\lim(a \to 0.5)\ (x \to 0)\ x^{2a-1} = \lim\ (x \to 0)\ x^{2(0.5+x)-1} = \lim\ (x \to 0)\ x^{2x} = 1 \quad (77)$$

(Since $\lim(x \to 0)\ x^{x} = 1$)

Therefore we must apply L 'Hospital ' rule with respect to x in the limiting process (75)

$$\lim(a \to 0.5)\ (x \to 0)\ \frac{1-x^{2a-1}}{x^a} = \lim(a \to 0.5)\ (x \to 0)\ \frac{-(2a-1)x^{2a-2}}{ax^{a-1}}$$

$$= \lim(a \to 0.5)\ (x \to 0)\ \frac{(\frac{1}{a}-2)}{x^{1-a}} \quad (78)$$

Now we write again

$$a = (\lim x \to 0)\ (0.5 + x) \quad (79)$$

Thus the limit (78) becomes

$$\lim(a \to 0.5)\ (x \to 0)\ \frac{(\frac{1}{a}-2)}{x^{1-a}} = \lim(x \to 0)\ \frac{(0.5+x)^{-1}-2}{x^{0.5-x}} = \lim(x \to 0)\ \frac{(0.5+x)^{-1}-2}{x^{0.5}\times x^{-x}}$$

$$= \lim(x \to 0)\ \frac{(0.5+x)^{-1}-2}{x^{0.5}} \text{ (Since } \lim\ (x \to 0)\ x^{-x} = 1) \quad (80)$$

We must apply L 'Hospital ' rule

$$\lim(x \to 0)\ \frac{(0.5+x)^{-1}-2}{x^{0.5}} = \lim(x \to 0)\ \frac{-(0.5+x)^{-2}}{0.5x^{-0.5}} =$$

$$\lim(x \to 0)\ \frac{-2\times x^{0.5}}{(0.5+x\ )^2} = 0 \quad (81)$$

Thus we have verified here that, for a = 0.5 (71) approaches zero as ($x \to 0$) and hence remains bounded.

We consider the case a >0.5. This case is also rejected, since according to the functional equation, if ($\zeta(s)$=0) (s = a+ bi) has a root with a>0.5, then it must have another root with another value of a < 0.5. But we have already rejected this last case with a<0.5

Thus we are left with the only possible value of a which is a = 0.5

Therefore a = 0.5

This proves the Riemann Hypothesis.

References

[1] Titch marsh, E. C. (1999) “The Theory of the Riemann zeta – function”, London: Oxford University Press.

[2] Apostol, Tom M. (1974) “Mathematical Analysis”, Reading, Massachusetts; Addison –wesley Publishing Company.

[3] Edwards, H. M. (1974) “Riemann 's zeta function”, New York: Academic Press, Inc.

[4] Apostol, Tom M. (1976) “Introduction to Analytic Number Theory”, New York: Springer – Verlag.

[5] Koblits, Neal (1984) “P- adic Numbers, P – adic Analysis, and Zeta – Functions”, New York: Springer – Verlag.

[6] Geiner, Walter. Reinhardt, Joachim (1996)” Field Quantization”, Berlin: Springer – Verlag.

[7] Ribenboim, Paulo (1988) "The Book of Prime Number Records", Berlin: Springer – Verlag.

[8] Larson, Loren C. (1983) "Problem-Solving Through Problems", Berlin: Springer – Verlag.

[9] Wilf, Herbert S. (1994) "Generatingfunctionology", New York: Academic Press.

[10] Guy, Richard K. (1994) "Unsolved Problems in Number Theory", Berlin: Springer – Verlag.