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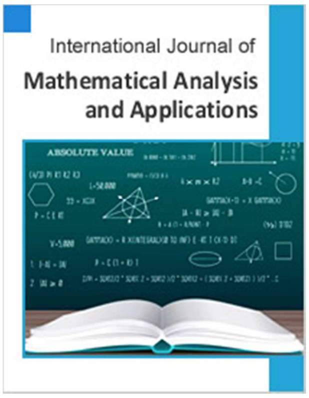

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Disjoint Variation, (s)-Boundedness and Brooks-Jewett Theorems for Lattice Group-Valued k-Triangular Set Functions

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Abstract
We consider some basic properties of the disjoint variation of lattice group-valued set functions and (s)-boundedness for k-triangular set functions, not necessarily finitely additive or monotone. Using the Maeda-Ogasawara-Vulikh representation theorem of lattice groups as subgroups of continuous functions, we prove a Brooks-Jewett-type theorem for k-triangular lattice group-valued set functions, in which (s)-boundedness is intended in the classical like sense, and not necessarily with respect to a single order sequence. To this aim, we deal with the disjoint variation of a lattice group-valued set function and study the basic properties of the set functions of bounded disjoint variation. Furthermore we show that our setting includes the finitely additive case.

1. Introduction

In the literature there have been several recent researches, about limit theorems for lattice-group or vector lattice-valued set functions. For a historical survey and related results see also [1-3] and their bibliographies. In this paper we deal with k-triangular lattice group-valued set functions. Some examples of such set functions are the M-*measures*, that is monotone set functions $m$with $m(\emptyset) = 0$, continuous from above and from below and compatible with respect to finite suprema and infima, which are 1-triangular set functions. The *measuroids* are examples of 1-triangular set functions, not necessarily monotone (see also [4]).

In this paper, using the Maeda-Ogasawara-Vulikh representation theorem for lattice groups as subgroups of some suitable spaces of continuous functions, we extend to k-triangular set functions some Brooks-Jewett-type theorems, proved in [5] in the finitely additive setting. Note that, in our context, (s)-boundedness is intended in the classical like sense, and not necessarily with respect to a single order sequence. Observe that, differently than in the finitely additive setting, boundedness of k-triangular set functions, in general, does not imply (s)-boundedness. Thus, we consider the *disjoint variation* of a lattice group-valued set function m and prove that boundedness of the disjoint variation of m is a sufficient condition (in general, not necessary) for (s)-boundedness of m.

2. Preliminaries

Let R be a Dedekind complete lattice group, G be an infinite set, Σ be a σ-algebra of subsets of G, $m: \Sigma \to R$ be a bounded set function, and k be a fixed positive integer.

A sequence $(\sigma_p)_p$ in R is called (O)-*sequence* iff it is decreasing and $\bigwedge_{p=1}^{\infty} \sigma_p = 0$. A sequence $(x_n)_n$ in R is *order convergent* (or (O)-*convergent*) to x iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x| \leq \sigma_p$ for each $n \geq n_0$, and in this case we write $(O)\lim_n x_n = x$.

The *positive* and *negative part* of m are defined by $m^+(A) := \bigvee\{m(B): B \in \Sigma,\ B \subset A\}$,
$m^-(A) := \bigvee\{-m(B): B \in \Sigma,\ B \subset A\}$, $A \in \Sigma$, respectively.

The semivariation of m is defined by

$$v(m)(A) := \bigvee\{|m(B)|: B \in \Sigma,\ B \subset A\},\ A \in \Sigma.$$

A set function $m: \Sigma \to R$ is (s)-*bounded* iff $(O)\lim_h v(m)(C_h) = 0$ for every disjoint sequence $(C_h)_h$ in Σ. The set functions $m_j: \Sigma \to R$, $j \in \mathbb{N}$, are *uniformly* (s)-*bounded* iff $(O)\lim_h(\bigvee_{j=1}^{\infty} v(m_j)(C_h)) = 0$ for any disjoint sequence $(C_h)_h$ in Σ.

The set functions $m_j: \Sigma \to R$, $j \in \mathbb{N}$, are *equibounded* iff there is $u \in R$ with $|m_j(A)| \leq u$ whenever $j \in \mathbb{N}$ and $A \in \Sigma$.

We say that $m: \Sigma \to R$ is k-*triangular* iff $0 = m(\emptyset) \leq m(A)$ for any $A \in \Sigma$ and $m(A) - k\, m(B) \leq m(A \cup B) \leq m(A) + k\, m(B)$ for all $A, B \in \Sigma$, $A \cap B = \emptyset$.

It is easy to prove the following

Proposition 2.1 If $m: \Sigma \to R$ is k-triangular, then also $v(m)$ is k-triangular.

3. The Main Results

We begin with observing that it is well-known that, if $m_j: \Sigma \to R, j \in \mathbb{N}$, are equibounded set functions, then the union of the ranges of the m_j's can be embedded in the space

$$\mathcal{C}(\Omega) := \{f: \Omega \to \mathbb{R},\ f \text{ is continuous}\}, \quad (1)$$

where Ω is a suitable compact extremely disconnected Hausdorff topological space, existing thanks to the Maeda-Ogasawara-Vulikh representation theorem. Every lattice supremum and infimum in $\mathcal{C}(\Omega)$ coincides with the respective pointwise supremum and infimum in the complement of a meager subset of Ω (see also [6] and [7, p. 69]).

We will prove a Brooks-Jewett-type theorem for a sequence $(m_j)_j$ of lattice group-valued set functions. The technique we will use is to find a meager set $N \subset \Omega$ such that the real-valued "components" $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$, are (s)-bounded and pointwise convergent for any $\omega \in \Omega \setminus N$, and then to apply the corresponding classical results existing for real-valued k-triangular set functions (see also [2]). We require pointwise convergence of the m_j's with respect to a single (O)-sequence, in order to find a single corresponding meager set N, to obtain pointwise convergence of the "components" in $\Omega \setminus N$. Concerning (s)-boundedness of the "components", observe that, differently from the finitely additive case, a bounded k-triangular set function, even monotone, in general is not (s)-bounded, as we will see in (2). So, in our setting, we will give a condition which implies (s)-boundedness of the "components". To this aim, we deal with the *disjoint variation* of a lattice group-valued set function (see also [2, 8-9]) and prove that boundedness of the disjoint variation implies (s)-boundedness of the "components". Furthermore, we will show that our context includes the finitely additive case.

Now we give the following technical proposition.

Proposition 3.1. Let $m_j: \Sigma \to R, j \in \mathbb{N}$, be a sequence of equibounded set functions. If there is a meager set $N_* \subset \Omega$ such that the set functions $m_j(\cdot)(\omega)$ are real-valued and k-triangular for every $\omega \in \Omega \setminus N_*$ and $j \in \mathbb{N}$, then the m_j's are k-triangular. Moreover, if the m_j's are k-triangular, then the set functions $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$, are real-valued and k-triangular for every $\omega \in \Omega$.

Proof: Thanks to (1), for every $\omega \in \Omega$ and $j \in \mathbb{N}$ the set function $m_{j,\omega}$ defined by $m_{j,\omega}(A) := m_j(A)(\omega)$, $A \in \Sigma$, is real-valued. Now we prove the first part. Let N_* be as in the hypothesis, then

$$m_j(A)(\omega) - k\, m_j(B)(\omega) \leq m_j(A \cup B)(\omega) \leq m_j(A)(\omega) + k\, m_j(B)(\omega)$$

for every $j \in \mathbb{N}$, $A, B \in \Sigma$ with $A \cap B = \emptyset$ and $\omega \in \Omega \setminus N_*$, and

$$0 = m_j(\emptyset)(\omega) \leq m_j(A)(\omega)$$

for all $j \in \mathbb{N}$, $A \in \Sigma$ and $\omega \in \Omega \setminus N_*$. Since N_* is meager, by a density argument it follows that

$$m_j(A) - k\, m_j(B) \leq m_j(A \cup B) \leq m_j(A) + k\, m_j(B)$$

for every $j \in \mathbb{N}$, $A, B \in \Sigma$ with $A \cap B = \emptyset$, and $0 = m_j(\emptyset) \leq m_j(A)$ for all $j \in \mathbb{N}$ and $A \in \Sigma$, that is m_j is k-triangular for every $j \in \mathbb{N}$. The proof of the last part is straightforward.

Now we deal with (s)-boundedness of k-triangular set functions. In general, differently from the finitely additive setting, it is not true that every bounded k-triangular capacity is (s)-bounded. Indeed, let $G = [1, 2]$, set

$$m(\emptyset) = 0, \text{ and } m(A) = \sup A \quad (2)$$

if $A \subset G$, $A \neq \emptyset$. It is not difficult to see that m is bounded, positive, monotone and 1-triangular. For each disjoint sequence $(A_n)_n$ of nonempty subsets of G it is $m(A_n) \geq 1$ for every $n \in \mathbb{N}$, and so it is not true that $\lim_n m(A_n) = 0$. So, m is not (s)-bounded. So, we consider the disjoint variation of a lattice group-valued set function.

Definitions 3.2. Let us add to R an extra element $+\infty$, obeying to the usual rules, and for any set function $m: \Sigma \to R$ let us define the *disjoint variation* $\overline{m}: \Sigma \to R \cup \{+\infty\}$ of m by

$$\overline{m}(A) := \bigvee_I \Big(\sum_{i \in I} |m(D_i)|\Big), \quad A \in \Sigma,$$

where the involved supremum is taken with respect to all finite disjoint families $\{D_i: i \in I\}$ such that $D_i \in \Sigma$ and $D_i \subset A$ for each $i \in I$.

A set function m is *of bounded disjoint variation* (or BDV) iff $\overline{m}(G) \in R$.

Examples 3.3. We give an example of a 1-triangular monotone set function, which is not BDV. Let m be as in (2). It is easy to check that $v(m)(G) = 2$. Pick arbitrarily $n \in \mathbb{N}$ and put $D_i = [1 + \frac{i-1}{n}, 1 + \frac{i}{n}[$, $i = 1, \dots, n$. It is $m(D_i) = \sup D_i \geq 1$, and so $\sum_{i=1}^{n} m(D_i) \geq n$. From this and arbitrariness of n we get $\overline{m}(G) = +\infty$, and hence m is not BDV. Thus, boundedness does not imply BDV, though it is easy to see that the converse implication holds.

We give an example of a 1-triangular monotone set function, which is BDV but not finitely additive. Let $m_0(A) := \sum_{n \in A} \frac{(-1)^n}{n^2}$, $A \subset \mathbb{N}$, $m^*(A) := |m_0(A)|$, $m(A) := v(m^*)(A) = \sup\{|m_0(B)|: B \subset A\} = \sup\{|\sum_{n \in B} \frac{(-1)^n}{n^2}|: B \subset A\}$, $A \subset \mathbb{N}$.

Note that m^* is not increasing, since $m^*(\{1,3\}) = \frac{10}{9} > \frac{31}{36} = m^*(\{1,2,3\})$. It is easy to see that m^* is 1-triangular. Hence, by Proposition 2.1, m is 1-triangular.

Note that m is positive and monotone, $m(\emptyset) = 0$ and

$$0 \leq \overline{m}(\mathbb{N}) = \sup_I (\sum_{i \in I} m(D_i)) = \sup_I (\sum_{i \in I} (\max_{B \subset D_i} |\sum_{n \in B} \frac{(-1)^n}{n^2}|)) \leq$$

$$\leq \sup_I (\sum_{i \in I} (\sum_{n \in D_i} \frac{1}{n^2})) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad (3)$$

where the involved supremum is taken with respect to all finite disjoint families $\{D_i: i \in I\}$ such that $D_i \subset \mathbb{N}$ for every $i \in I$, and hence m is BDV. Note that the supremum in (3) is exactly equal to $\frac{\pi^2}{6}$: indeed, it is enough to consider, for each $n \in \mathbb{N}$, the family $\{D_j := \{j\}: j = 1, \dots, n\}$, and to take into account that $m(\{j\}) = \frac{1}{j^2}$ for any $j \in \mathbb{N}$. Finally, it is $m(\{1,2\}) = \max\{1, \frac{1}{4}, \frac{3}{4}\} = 1 < \frac{5}{4} = 1 + \frac{1}{4} = m(\{1\}) + m(\{2\})$. Thus, m is not finitely additive.

We now show that, in general, (s)-boundedness does not imply BDV. Let $G = [1,1]$, Σ be the σ-algebra of all Borel subsets of G, $m_0(A) = \int_A \operatorname{sgn} x\, dx$, $A \in \Sigma$, where $\operatorname{sgn}(x) = 1$ if $x \in]0,1]$, $\operatorname{sgn}(x) = -1$ if $x \in [-1,0]$ and $\operatorname{sgn}(0) = 0$, and set $m^*(A) = \sqrt{|m_0(A)|}$, $A \in \Sigma$. Note that m^* is not monotone: indeed,

$$m^*(G) = \sqrt{|m_0(G)|} = \sqrt{|\int_{-1}^{1} \operatorname{sgn} x\, dx|} = 0$$
$$= m^*(\emptyset), m^*([0,1]) = \sqrt{|m_0([0,1])|}$$
$$= \sqrt{|\int_{0}^{1} \operatorname{sgn} x\, dx|} = 1.$$

Now, fix arbitrarily $n \in \mathbb{N}$ and pick $D_i = [\frac{i-1}{n}, \frac{i}{n}[$, $i = -n+1, -n+2, \dots, -1, 0, 1, \dots, n$. It is

$\overline{m^*}(G) \geq \sum_{i=-n+1}^{n} \sqrt{\frac{1}{n}} = \frac{2n}{\sqrt{n}} = 2\sqrt{n}$, and hence, by arbitrariness of n, it follows that m^* is not BDV.

We now prove that m^* is 1-triangular. Pick any two disjoint sets $A, B \in \Sigma$. Then, it is

$$m^*(A \cup B) = \sqrt{|m_0(A \cup B)|} = \sqrt{\left|\int_{A \cup B} \operatorname{sgn} x\, dx\right|} = \sqrt{\left|\int_A \operatorname{sgn} x\, dx + \int_B \operatorname{sgn} x\, dx\right|} \leq \sqrt{\left|\int_A \operatorname{sgn} x\, dx\right| + \left|\int_B \operatorname{sgn} x\, dx\right|}$$
$$= \sqrt{|m_0(A)| + |m_0(B)|} \leq \sqrt{|m_0(A)|} + \sqrt{|m_0(B)|} = m^*(A) + m^*(B);$$

$$m^*(A) = \sqrt{|m_0(A)|} = \sqrt{|\int_A \operatorname{sgn} x\, dx|} = \sqrt{|\int_{A \cup B} \operatorname{sgn} x\, dx - \int_B \operatorname{sgn} x\, dx|} \leq \sqrt{|\int_{A \cup B} \operatorname{sgn} x\, dx| + |\int_B \operatorname{sgn} x\, dx|}$$
$$= \sqrt{|m_0(A \cup B)| + |m_0(B)|} \leq \sqrt{|m_0(A \cup B)|} + \sqrt{|m_0(B)|} = m^*(A \cup B) + m^*(B),$$

getting 1-triangularity of m^*.

Set $m(A) := v(m^*)(A) = \sup\{m^*(B): B \in \Sigma, B \subset A\}$, $A \in \Sigma$. Note that m is positive and increasing. Since m^* is not BDV, then a fortiori m is not. By Proposition 2.1, m is 1-triangular, since m^* is. Moreover, it is not difficult to see that m^* is (s)-bounded. Hence, m is (s)-bounded (see also [9, Theorem 2.2]). Thus, property BDV is not a necessary condition for (s)-boundedness of k-triangular set functions.

Now we prove that BDV is a sufficient condition for (s)-boundedness of a set function m with values in a lattice group R and of its real-valued "components".

Proposition 3.4. *Let $m: \Sigma \to R$ be a BDV set function, and Ω be as in* (1). *Then the set function $m_\omega := m(\cdot)(\omega)$ is real-valued, BDV and (s)-bounded for every $\omega \in \Omega$. Moreover m is (s)-bounded.*

Proof. Since m is bounded, arguing analogously as at the beginning of the proof of Proposition 3.1, for any $\omega \in \Omega$ the set function m_ω defined by $m_\omega(A) := m(A)(\omega)$, $A \in \Sigma$, is real-valued. For each $\omega \in \Omega$ it is

$$\overline{m_\omega}(G) = \sup_I (\sum_{i \in I} |m(D_i)(\omega)|) = \sup_I ((\sum_{i \in I} |m(D_i)|)(\omega)) \leq$$
$$\leq (\bigvee_I (\sum_{i \in I} |m(D_i)|))(\omega) = (\overline{m}(G))(\omega) \in \mathbb{R},$$

since the pointwise supremum is less or equal than the corresponding lattice supremum in $\mathcal{C}(\Omega)$. So, m_ω is BDV for each $\omega \in \Omega$. By [8, Theorem 3.2], for each disjoint sequence $(H_n)_n$ in Σ and $\omega \in \Omega$ it is $\lim_n \overline{m_\omega}(H_n) = 0$, and a fortiori $\lim_n v(m_\omega)(H_n) = 0$. This proves the first part.

Now, choose any disjoint sequence $(H_n)_n$ in Σ. By the Maeda-Ogasawara-Vulikh representation theorem (see also [6]) there is a meager set N_* with

$$[\bigwedge_{n=1}^{\infty}(\bigvee_{s=n}^{\infty} v\,(m)(H_s))](\omega) = [\bigwedge_{n=1}^{\infty}(\bigvee_{s=n}^{\infty}(\bigvee_{A\in\Sigma, A\subset H_s} |m(A)|))](\omega) = \inf_n(\sup_{s\ge n}(\sup_{A\in\Sigma, A\subset H_s} |m(A)(\omega)|))$$
$$= \inf_n(\sup_{s\ge n} v\,(m(\cdot)(\omega))(H_s)) = 0 = \sup_n(\inf_{s\ge n} v\,(m(\cdot)(\omega))(H_s) = \sup_n(\inf_{s\ge n}(\sup_{A\in\Sigma, A\subset H_s} |m(A)(\omega)|)$$
$$= [\bigvee_{n=1}^{\infty}(\bigwedge_{s=n}^{\infty}(\bigvee_{A\in\Sigma, A\subset H_s} |m(A)|))](\omega) = [\bigvee_{n=1}^{\infty}(\bigwedge_{s=n}^{\infty} v\,(m)(H_s))](\omega)$$

for every $\omega \in \Omega \setminus N_*$. From this we obtain $[(O)\lim_n v\,(m)(H_n)](\omega) = 0$ for each $\omega \in \Omega \setminus N_*$. By a density argument, we get $[(O)\lim_n v\,(m)(H_n)](\omega) = 0$ for every $\omega \in \Omega$, namely $(O)\lim_n v\,(m)(H_n) = 0$. By arbitrariness of the chosen sequence $(H_n)_n$, we have (s)-boundedness of m.

Now we show that our setting includes the finitely additive case. Indeed we have the following

Proposition 3.5. *Every bounded finitely additive measure $m: \Sigma \to R$ is BDV.*

Proof: First of all consider the case in which m is positive. Then, thanks to finite additivity, m is also increasing. If $\{D_i: i \in I\}$ is any disjoint finite family of subsets of G, whose union we denote by B, then we get

$$\sum_{i\in I} m\,(D_i) = m(\bigcup_{i\in I} D_i) = m(B) \le m(G) \qquad (4)$$

(see also [2, Proposition 3.4]). From (4) and boundedness of m we deduce that m is BDV, at least when m is positive. In the general case, $m = m^+ - m^-$, where m^+ and m^- are the positive and the negative part of m, respectively. Proceeding analogously as in [10, Theorem 2.2.1], it is possible to check that m^+ and m^- are finitely additive. Then, by the previous case, m^+ and $m^-$are BDV, and

$$\sum_{i\in I} |m(D_i)| = \sum_{i\in I}(m(D_i))^+ + \sum_{i\in I}(m(D_i))^- \le \qquad (5)$$
$$\le \sum_{i\in I} m^+\,(D_i) + \sum_{i\in I} m^-\,(D_i) \le \overline{m^+}(G) + \overline{m^-}(G).$$

Taking in (5) the supremum with respect to I, we get the assertion.

Now we are in position to prove the following Brooks-Jewett-type theorem, which extends [5, Theorem 3.1] to the context of k-triangular set functions.

Theorem 3.6. Let Ω be as in (1), $m_j: \Sigma \to R$, $j \in \mathbb{N}$, be a sequence of BDV k-triangular equibounded set functions. Suppose that there is a set function $m_0: \Sigma \to R$ such that the sequence $(m_j)_j$ (O)-converges to m_0 with respect to a single (O)-sequence. Then there is a meager subset $N \subset \Omega$ such that for each $\omega \in \Omega \setminus N$ the real-valued set functions $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$, are uniformly (s)-bounded (with respect to j). Moreover the m_j's are uniformly (s)-bounded.

Proof: Observe that, since the m_j's are equibounded and k-triangular, for every $\omega \in \Omega$ the functions $m_j(\cdot)(\omega)$, $j \in \mathbb{N}$, are real-valued, k-triangular and BDV, and hence (s)-bounded on Σ, thanks to [8, Theorem 3.2]. Moreover there is an (O)-sequence $(\sigma_p)_p$ such that for every $p \in \mathbb{N}$ and $A \in \Sigma$ there is $j_0 \in \mathbb{N}$ with $|m_j(A) - m_0(A)| \le \sigma_p$ for all $j \ge j_0$. By the Maeda-Ogasawara-Vulikh representation theorem (see also [6]) there is a meager set $N \subset \Omega$, such that the sequence $(\sigma_p(\omega))_p$ is an (O)-sequence in $\mathbb{R}$ for each $\omega \in \Omega \setminus N$. Thus for every $p \in \mathbb{N}$ and $A \in \Sigma$ there is $j_0 \in \mathbb{N}$ with

$$|m_j(A)(\omega) - m_0(A)(\omega)| \le \sigma_p(\omega) \qquad (6)$$

for each $\omega \in \Omega \setminus N$ and $j \ge j_0$. This implies that $\lim_j\ m_j(A)(\omega) = m_0(A)(\omega)$ for any $A \in \Sigma$ and $\omega \in \Omega \setminus N$. Thus for such ω's the real-valued set functions $m_j(\cdot)(\omega)$ satisfy the hypotheses of the Brooks-Jewett-type theorem (see also [2]), and so they are uniformly (s)-bounded. This concludes the first part of the assertion.

Now we prove that the set functions m_j, $j \in \mathbb{N}$, are uniformly (s)-bounded. Pick arbitrarily any disjoint sequence $(C_h)_h$ in Σ and let us show that

$$\bigwedge_{s=1}^{\infty}[\bigvee_{h=s}^{\infty}(\bigvee_{j=1}^{\infty}[\bigvee_{B\in\Sigma, B\subset C_h}\ m_j(B)])] = 0. \qquad (7)$$

As the set functions $m_j(\cdot)(\omega)$ are uniformly (s)-bounded for any $\omega \in \Omega \setminus N$, where N is as in (6), it is

$$\inf_s[\sup_{h\ge s}\{\sup_j[v(m_j(\cdot)(\omega))(C_h)]\}] = \lim_h\{\sup_j[v(m_j(\cdot)(\omega))(C_h)]\} = 0 \qquad (8)$$

for all $\omega \in \Omega \setminus N$. As any countable union of meager subsets of Ω is still meager, then there is a meager subset Q of Ω, without loss of generality containing N, such that for any $h \in \mathbb{N}$ and $\omega \in \Omega \setminus Q$ it is

$$\sup_j[\sup_{B\in\Sigma, B\subset C_h}\ m_j(B)(\omega)] = (\bigvee_j[\bigvee_{B\in\Sigma, B\subset C_h}\ m_j(B)])(\omega). \qquad (9)$$

From (8) and (9) it follows that

$$\bigwedge_{s=1}^{\infty}[\bigvee_{h=s}^{\infty}(\bigvee_{j=1}^{\infty}[\bigvee_{B\in\Sigma, B\subset C_h}\ m_j(B)])](\omega) = 0 \qquad (10)$$

for every $\omega \in \Omega \setminus Q$. Thus, (7) follows from (10) and a density argument. From (7) we deduce that $(O)\lim_h(\bigvee_{j=1}^{\infty}[\bigvee_{B\in\Sigma, B\subset C_h}\ m_j(B)]) = 0$, that is $(O)\lim_h(\bigvee_j v\,(m_j)(C_h)) = 0$. Hence, by arbitrariness of the chosen sequence $(C_h)_h$, the m_j's are uniformly (s)-bounded.

4. Conclusions

We proved a Brooks-Jewett-type theoremfor Dedekind complete lattice group-valued *k*-triangular set functions, not necessarily finitely additive, extending [5, Theorem 3.1]. We used the corresponding classical results for real-valued set functions. Note that, in the non-additive setting, boundedness of a set function is not sufficient to have (s)-boundedness or (s)-boundedness of its real-valued "components". So, we dealt with the disjoint variation of a lattice group-valued set function and we studied the property BDV (bounded disjoint variation). We showed that there exist bounded monotone k-triangular set functions not BDV and not finitely additive, that there are bounded monotone k-triangular set functions satisfying BDV but not finitely additive, that property BDV is a sufficient but not necessary condition for (s)-boundedness and allows to prove our Brooks-Jewett-type theorem without assuming finite additivity. Furthermore, we proved that our setting includes the finitely additive case, since every bounded finitely additive lattice group-valued set function satisfies property BDV.

Prove similar results with respect to other kinds of convergence.

Prove other types of limit theorems in different abstract contexts.

Prove some kinds of limit theorems without assuming condition BDV.

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