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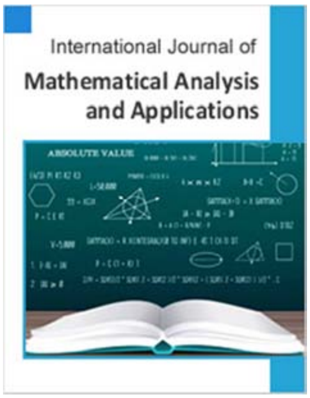

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Differential Transform Method (DTM) for Nonlinear Initial Value Problems

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Abstract
The differential transform method (DTM) is based on the Tylor series for all variable. In this paper, we implemented the differential transform method (DTM) for solving various forms of nonlinear Klein-Gordon type equations. The suggested algorithm is helpful for finding the approximate analytical solution of nonlinear Klein-Gordon type equations. Several illustrative examples are given and the solutions of our model equations are calculated in the form of convergent series with easily computable components. The results reveals that the approach is quite easy to implement and proves to be very effective for such type of equations. The method introduces a promising tool for solving many linear and nonlinear differential equations.

1. Introduction

Non-linear partial differential equations are of great importance in various scientific fields. The nonlinear models of real-life problems are still difficult to solve either numerically or analytically. A broad class of expository analytical solution techniques and numerical strategies were utilized to deal with these issues. For some nonlinear problems, although exact solution can be obtained but often appears in complex terms and not useful for applications.

For example, the nonlinear Klein-Gordon equations arise in a various physical circumstances and the initial-value problem of the one-dimensional non-linear Klein-Gordon equation is given by the following

$$f_{tt} + af_{xx} + \psi(f) = \phi(x,t) \tag{1}$$

Where $f = f(x,t)$ represented wave displacement at position x and time t, a is a known constant and $\psi(u)$ is a nonlinear force. In different physical applications, the nonlinear force $\psi(f)$ has also other forms, such as Klein–Gordon Equation with a Power Law Nonlinearity [2]. Some numerical methods for solving equation (1) are given in [2-6] and the references therein.

Several techniques based on series expansion, variational iteration method [10-12] Adomian decomposition method [7-9], Homotopy perturbation method [13], Homotopy analysis method [14] and auxiliary equation method [15] has been applied for the solution of these.

In this paper, differential transform Method (DTM) based on Taylor's series is applied to solve various forms of nonlinear Klein-Gordon type equations. This method does not require any symbolic computation like traditional high order Taylor's series. Traditional Taylor series method is computationally taken long time for large orders and obtaining highly order accuracy is much difficult. By the use of suggested method it is possible to obtain highly accurate results or exact solutions for differential equations. The solutions of equations under discussion are calculated in the form of convergent series with easily computable components by using symbolic computation which is in suitable form for application.

2. Differential Transform Method

Differential transform method (DTM) first introduced by Zhou [1] and its main purpose therein was to compute both linear and nonlinear initial value problems arising in electrical circuit theory. Differential transform method (DTM) is a semi-analytical numerical technique that basically uses Taylor series for the solution of differential equations in the form of a polynomial with a distinct algorithm. Consequently, the DTM is an alternative choice for getting Taylor series solution of the given differential equations. There are also other methods in the literature based on Taylor series expansion such as Restricted Taylor series method [4] and Adomian decomposition method [7-9]. But differential transform method (DTM) formulizes the Taylor series in a totally different way. With this method, the given differential equation and related boundary conditions are transformed into a recurrence relations which leads to the solution of a system of algebraic equations as coefficients of a power series solution and is easily carried out in computer. Because of this property, the method is no need of linearization of the nonlinear problems and as a result avoids the large computational works and the round-off errors. Differential Transform method has been successfully applied to various problems [21-29] recently. In this section we briefly describe differential transform method as follow

2.1. One-Dimensional Differential Transform

Definition 2.1. If $f(t)$ is analytic in the domain T, then it will be differentiated continuously with respect to time t,

$$\frac{\partial^k f(t)}{\partial t^k} = \eta(t,k) \text{, for all } t \in T \tag{2}$$

For $t = t_i$, then $\eta(t,k) = \mu(t_i,k)$, where k belongs to the set of nonnegative integers, denoted as the $K-$ domain. Therefore, equation (2) can be rewritten as

$$F(k) = \eta(t_i,k) = \left[\frac{\partial^k f(t)}{\partial t^k}\right]\bigg|_{t=t_i} \tag{3}$$

Where $F(k)$ is called the spectrum of $f(t)$ at $t = t_i$.

Definition 2.2. If $f(t)$ can be expressed by Taylor's series, then $f(t)$ can be represented as

$$f(t) = \sum_{k=0}^{\infty}\left[\frac{(t-t_i)^k}{k!}\right]\mathrm{F}(k) . \tag{4}$$

Equation (4) is called the inverse of $f(t)$, with the symbol D denoting the differential transformation process. Combining (3) and (4), we obtain

$$f(t) = \sum_{k=0}^{\infty}\left[\frac{(t-t_i)^k}{k!}\right]\mathrm{F}(k) \equiv D^{-1}F(k) \tag{5}$$

Applying the differential transformation, a differential equation in the domain of interest can be transformed to algebraic equation in the $K-$domain and the $u(t)$ can be obtained by finite-term Taylor's series plus the remainder, as

$$f(t) = \sum_{k=0}^{\infty}\left[\frac{(t-t_i)^k}{k!}\right]\mathrm{F}(k) + R_{n+1}(t) \tag{6}$$

In order to speed up the convergent rate and the accuracy of calculation, the entire domain of t needs to be split into sub-domains [3].

2.2. Two-Dimensional Differential Transform

Let $u(x,y)$ a function of two variables such that $u(x,y)$ is an analytic function in the domain K and let $(x,y) = (x_0,y_0)$ in this domain. The function $w(x,y)$ is then represented by a power series whose center located at (x_0,y_0) . The differential transform of function $w(x,y)$ is

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,y_0)} \tag{7}$$

Where $u(x,y)$ is the original function and $U(k,h)$ is the transformed function. The differential inverse transform of $U(k,h)$ is defined as follows

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)(x-x_0)^k (y-y_0)^h \tag{8}$$

Combining equations (7) and (8), it can be obtained that

$$u(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,y)}{\partial x^k \partial y^h}\right]_{(x_0,y_0)} (x-x_0)^k (y-y_0)^h . \tag{9}$$

From the above definitions, it can be found that the concept of the two-dimensional differential transform is derived from the two-dimensional Taylor series expansion with equations (7) and (8), the fundamental mathematical operations performed by two-dimensional differential transform can readily be obtained and are listed in Table 1.

Table 1. *Operations for the two-dimensional differential transformation.*

Original function	Transformed function
$u(x,y)=f(x,y)\pm g(x,y)$	$U(k,h)=F(k,h)\pm G(k,h)$
$w(x,y)=af(x,y)$	$U(k,h)=aF(k,h)$
$u(x,y)=\dfrac{\partial f(x,y)}{\partial x}$	$U(k,h)=(k+1)f(k+1,h)$
$u(x,y)=\dfrac{\partial f(x,y)}{\partial y}$	$U(k,h)=(h+1)f(k,h+1)$
$u(x,y)=\dfrac{\partial f^{r+s}(x,y)}{\partial x^r \partial y^s}$	$U(k,h)=(k+1)(k+2)...(k+r)(h+1)(h+2)...(h+s)F(k+r,h+s)$
$u(x,y)=f(x,y)g(x,y)$	$U(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}F(r,h-s)G(k-r,s)$
$u(x,y)=x^m y^n$	$U(k,h)=\delta(k-m,h-n)=\delta(k-m)\delta(h-n)=\begin{cases}1, & for\ k=m \quad and\ h=n\\ 0, & otherwise\end{cases}$
$u(x,y)=f(x,y)g(x,y)\dfrac{\partial c^2(x,y)}{\partial x^2}$	$U(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}(k-r-t+2)(k-r-t+1)F(r,h-s-p)G(t,s)C(k-r-t+2,p)$
$u(x,y)=\dfrac{\partial f(x,y)}{\partial x}\dfrac{\partial g(x,y)}{\partial x}$	
$u(x,y)=f(x,y)\dfrac{\partial g^2(x,y)}{\partial x^2}$	$U(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+2)(k-r+1)F(r,h-s)G(k-r+2,s)$
$u(x,y)=f(x,y)g(x,y)q(x,y)$	$U(k,h)=\sum_{r=0}^{k}\sum_{t=0}^{k-r}\sum_{s=0}^{h}\sum_{p=0}^{h-s}F(r,h-s-p)G(t,s)Q(k-r-t,p)$

3. Applications

To illustrate the effectiveness of the present method, several test examples are considered in this section.

Example 1. We first consider the Sine-Gordon equation

$$f_{tt}-f_{xx}=\sin f \tag{10}$$

subject to initial conditions

$$f(x,0)=\frac{\pi}{2},\ f_t(x,0)=0 \tag{11}$$

Taking in to consideration $\sin f=f-\frac{f^3}{3!}+\frac{f^5}{5!}-...$, then the transformed version of equation (10) is

$$(h+1)(h+2)F(k,h+2)-(k+1)(k+2)F(k+2,h)=$$
$$F(k,h)-\frac{1}{6}\left(\sum_{r=0}^{k}\left(\sum_{t=0}^{k-r}\left(\sum_{s=0}^{h}\left(\sum_{p=0}^{h-s}F(r,h-s-p)F(t,s)F(k-r-t,p)\right)\right)\right)\right) \tag{12}$$

The transformed initial conditions are

$$F(k,0)=\frac{\pi}{2}\quad k=0,1,2,... \tag{13}$$

and

$$F(k,1)=0 \tag{14}$$

respectively. Substituting (13) and (14) into (12), we obtained the closed form solution as

$$f(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U(k,h)x^k t^h$$

$$=0.5000000000\pi+0.4624161147t^2-0.009005575050t^4$$
$$-0.005527867713t^6-0.0001544012065t^8+0.00005500492296t^{10}$$

or

$$=\frac{\pi}{2}+\frac{1}{2}t^2-\frac{1}{240}t^6+\frac{1}{172800}t^{10}+... \tag{15}$$

which is the solution of the problem.

Table 2. *Comparison of the present method with ADM [40] for various t values.*

t	DTM	ADM
0.01	1.570842569	1.570846327
0.02	1.570981292	1.570996327
0.03	1.571212494	1.571246327
0.04	1.571536170	1.571596327
0.05	1.571952311	1.572046327
0.06	1.572460908	1.572596327
0.07	1.573061949	1.573246327
0.08	1.573755420	1.573996326
0.09	1.574541304	1.574846325
0.1	1.575419582	1.575796323

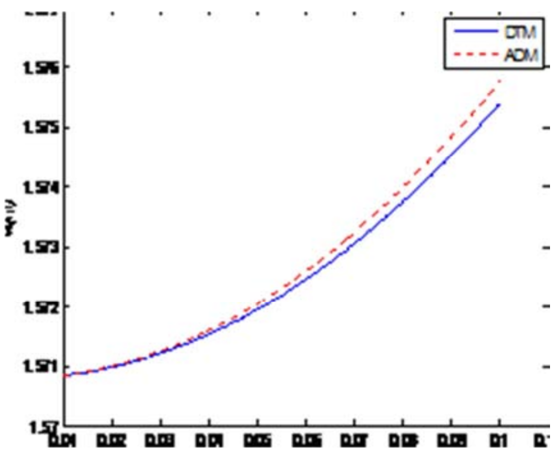

Figure 1. *Comparison between ADM and DTM.*

Example 2: Consider the sine-Gordon equation

$$f_{tt} - f_{xx} + \sin f = 0 \qquad (16)$$

subject to initial conditions

$$f(x,0) = 0,\ f_t(x,0) = 4\sec h(x) \qquad (17)$$

Taking into consideration $\sin f = f - \frac{f^3}{3!} + \frac{f^5}{5!} - \dots$, then the transformed version of equation (16) is

$$(h+1)(h+2)F(k,h+2) - (k+1)(k+2)F(k+2,h) = F(k,h) - \frac{1}{6}\left(\sum_{r=0}^{k}\left(\sum_{t=0}^{k-r}\left(\sum_{s=0}^{h}\left(\sum_{p=0}^{h-s} F(r,h-s-p)F(t,s)F(k-r-t,p)\right)\right)\right)\right) \qquad (18)$$

The transformed initial conditions are

$$F(k,0) = 0 \quad k = 0,1,2,\dots \qquad (19)$$

$$F(k,1) = 4\sec h(x) \quad k = 0,1,2,\dots \qquad (20)$$

respectively. Substituting (19) and (20) into (18), we obtain the closed form solution as

$$f(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h$$

$$= 4t - \frac{4}{3}t^3 + \frac{4}{5}t^5 - \frac{116}{315}t^7 - 2x^2t + 2x^2t^3 - 2x^2t^5 + \frac{97}{63}x^2t^7 + \frac{5}{6}x^4t - \frac{11}{6}x^4t^3 + \frac{17}{6}x^4t^5 - \frac{1177}{378}x^4t^7$$
$$- \frac{61}{180}x^6t + \frac{241}{180}x^6t^3 - \frac{541}{180}x^6t^5 + \frac{51887}{11340}x^6t^7 + \frac{277}{2016}x^8t - \frac{8651}{10080}x^8t^3 + \frac{26837}{10080}x^8t^5 - \frac{3412537}{635040}x^8t^7 \dots$$

or

$$= \left(4 - 2x^2 + \frac{5}{6}x^4 - \frac{61}{180}x^6 + \frac{277}{2016}x^8\right)t$$
$$+ \left(-\frac{4}{3} + 2x^2 - \frac{11}{6}x^4 + \frac{241}{180}x^6 - \frac{8651}{10080}x^8\right)t^3$$
$$+ \left(\frac{4}{5} - 2x^2 + \frac{17}{6}x^4 - \frac{541}{180}x^6 + \frac{26837}{10080}x^8\right)t^5$$
$$+ \left(-\frac{116}{315} + \frac{94}{63}x^2 - \frac{1177}{378}x^4 + \frac{51887}{11340}x^6 - \frac{3412537}{635040}x^8\right)t^7 + \dots \qquad (21)$$

which gives the closed form solution by [10]

$$g(x,t) = 4\arctan(t\sec h(x)) \qquad (22)$$

Table 3. Comparison of the differential transform method and exact solution for values of (x,t).

(x, t)	DTM	Exact solution	Absolute Error
(0,0)	0	0	0
(0.1,0,1)	0.3967025516	0.3967025318	$0.198.10^{-7}$
(0.2,0.2)	0.7744407152	0.7744383605	$0.23547.10^{-5}$
(0.3,0.3)	1.117938909	1.117903310	0.000035599
(0.4,0.4)	1.417703903	1.417478390	0.000225513
(0.5,0.5)	1.670046492	1.669172441	0.000874051
(0.6,0.6)	1.875503673	1.873003332	0.002500341
(0.7,0.7)	2.034159224	2.027881163	0.006278061
(0.8,0.8)	2.119859965	2.101787350	0.018072615
(0.9,0.9)	1.935206961	1.863811562	0.071395399
(1,1)	0.4479024943	0.1202380952	0.3276643991

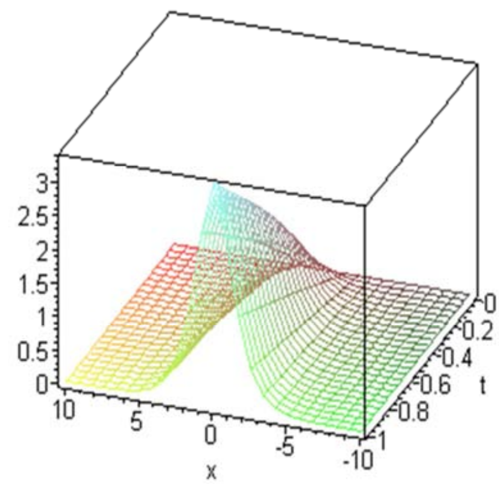

Figure 2. *Comparison between exact solution and differential transform method for $f(x,t)$.*

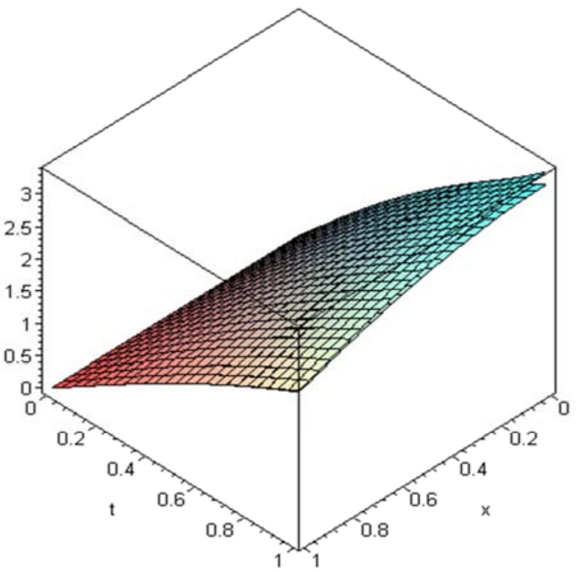

Figure 3. *Comparison between exact solution and differential transform method for $x = 0,..1$ and $t = 0,..,1$*

Table 4. Comparison between exact solution for x=0.01 and 5-iterative MADM, 2-iterative VIM, 3-iterative HPM and 4-iterative DTM.

t	$\lvert Exact - MADM\rvert$	$\lvert Exact - VIM\rvert$	$\lvert Exact - HPM\rvert$	$\lvert Exact - DTM\rvert$
0.01	1.320E-6	4.999E-7	6.000E-16	6.564E-25
0.02	1.045E-5	3.997E-6	8.110E-14	1.344E-21
0.03	3.491E-5	1.348E-5	1.384E-12	1.162E-19
0.04	8.191E-5	3.192E-5	1.035E-12	2.753E-18
0.05	1.583E-4	6.226E-5	4.922E-11	3.205E-17
0.06	2.707E-4	1.074E-4	1.759E-10	2.382E-16
0.07	4.253E-4	1.702E-4	5.155E-10	1.297E-15
0.08	6.280E-4	2.535E-4	1.307E-9	5.638E-15
0.09	8.844E-4	3.600E-4	2.969E-9	2.059E-14
0.1	1.200E-3	4.924E-4	6.175E-9	6.564E-14

Table 5. Comparison between exact solution for x=0.01 and 5-iterative MADM, 2-iterative VIM, 3-iterative HPM and 4-iterative DTM.

t	$\lvert Exact - MADM\rvert$	$\lvert Exact - VIM\rvert$	$\lvert Exact - HPM\rvert$	$\lvert Exact - DTM\rvert$
0.01	1.925E-4	4.974E-7	5.000E-16	6.278E-25
0.02	3.926E-4	3.998E-6	7.330E-14	1.285E-21
0.03	6.079E-4	1.341E-5	1.252E-12	1.112E-19
0.04	8.453E-4	3.176E-5	9.360E-12	2.633E-18
0.05	1.112E-3	6.195E-5	4.452E-11	3.065E-17
0.06	1.413E-3	1.069E-4	1.590E-10	2.277E-16
0.07	1.757E-3	1.694E-4	4.661E-10	1.241E-15
0.08	2.147E-3	2.523E-4	1.182E-9	5.393E-15
0.09	2.591E-3	3.583E-4	2.683E-9	1.970E-14
0.1	3.092E-3	4.901E-4	5.581E-9	6.278E-14

Example 3. Consider the nonlinear non-homogeneous Klein-Gordon equation

$$f_{tt} - f_{xx} + f^2 = x^2 t^2 \tag{23}$$

subject to initial conditions

$$f(x,0) = 0 \quad , \quad f_t(x,0) = x \tag{24}$$

The transformed version of Eq.(23) is

$$(h+1)(h+2)F(k,h+2) - (k+1)(k+2)F(k+2,h) + \sum_{r=0}^{k}\sum_{s=0}^{h} F(r,h-s)F(k-r,s) = \delta(k-2)\delta(h-2) \tag{25}$$

The transformed initial conditions are

$$F(k,0) = 0 \tag{26}$$

and

$$F(k,1) = \begin{cases} 1, & k=1 \\ 0, & otherwise \end{cases} \tag{27}$$

respectively.

Substituting (26) and (27) in (25), we obtained the solution as

$$f(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} F(k,h) x^k t^h$$

$$u(x,t) = xt \tag{28}$$

which is the exact solution of (23).

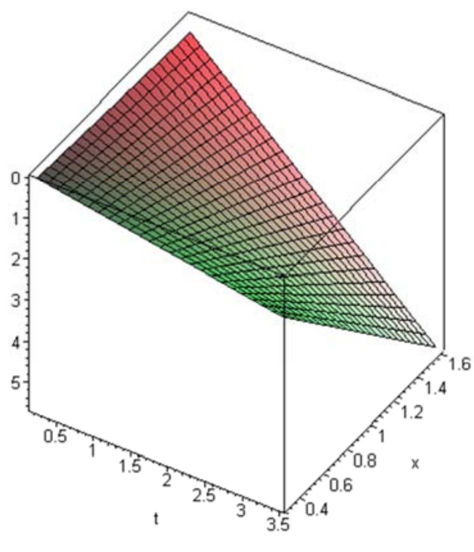

Figure 4. Solution of Klein-Gordon equation for $x = 0.3,\ldots,1.6$ and $t = 0.2,..,3.5$.

4. Conclusions

In this paper, differential transform method (DTM) is extended to solve the nonlinear Klein–Gordon equations. The present study reveals that the suggested algorithm is quite easy to impliment and gives highly accuracy or exact solution to nonlinear Klein–Gordon equations.

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References

[1] J. K. Zhou, Differential transformation and its application for electrical circuits. Wuuhahn, China: Huarzung University Press (1986).

[2] A. D. Polyanin, V. F. Zaitsev, Handbook of Nonlinear partial Differential Equations, Chapman & Hall/CRC, Boca Raton, 2004.

[3] M. J. Jang, C. L. Chen, Y. C. Liu, Two-dimensional differential transform for partial differential equations, Applied Mathematics and Computation, 121 (2001) 261-270.

[4] M. Gülsu, T. Öziş, Numerical solution of Burgers'equation with restrictive Taylor approximation, Applied Mathematics and Computation, 171 (2005), 1192-1200.

[5] Lu X, Schmid R. Symplectic integration of sine-Gordon type systems, Mathematics and Computers in Simulation, 50 (1999), 255–63.

[6] A. Dehghan, A. Shokri, Numerical solution of the Klein-Gordon equation using radial basis functions, Journal of Computational and Applied Mathematics, 230 (2009), 400-410.

[7] E. Y. Deeba, S. A. Khuri, A decomposition method for solving the nonlinear Klein-Gordon equation, Journal of Computational Physics, 124 (1996), 442-448.

[8] S. M. El-Sayed, The decomposition method for studying the Klein-Gordon equation, Chaos, Solitons& Fractals 18 (2003), 1025-1030.

[9] D. Kaya, S. M. El-Sayed, A numerical solution of the Klein-Gordon equation and convergence of the decomposition method, Applied mathematics and computation, 156 (2004), 341-353.

[10] B. Batiha, M. S. M. Noorani and I. Hashim, Numerical solution of sine-Gordon equation by variational iteration method, Physics Letters A, 370 (2007), 437-440.

[11] S. Abbasbandy, Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method, International Journal for Numerical Methods Engineering, 70 (2007), 876-881.

[12] S. T. Mohyud-Din, M. A. Noor and K. I. Noor, Modified variational iteration method for solving sine-Gordon equations, World Applied Sciences Journal 6 (2009), 999-1004.

[13] L. Jin, Analytical Approach to the sine-Gordon equation using homotopy perturbation method, Int. J. Contemp. Math. Sciences, 4(5) (2009), 225-231.

[14] U. Yücel, Homotopy analysis method for the sine-Gordon equation with initial conditions, Applied Mathematics and Computation, 203 (2008), 387-395.

[15] Lv, X, Lai, S., Wub, Y. H., An auxiliary equation technique and exact solutions for a nonlinear Klein–Gordon equation, Chaos, Solitons and Fractals 41 (2009) 82–90.

[16] M. S. H. Chowdhury, I. Hashim, Application of homotopy-perturbation method to Klein-Gordon and sine-Gordon equations, Chaos, Solitons and Fractals, 39 (2007), 1928-1935.

[17] A. S. V. Ravi Kanth, K. Aruna, Differential transform method for solving the linear and nonlinear Klein-Gordon equation, Computer Physics Communications, 180 (2009), 708-711.

[18] Lu X, Schmid R. Symplectic integration of sine-Gordon type systems, Mathematics and Computers in Simulation, 50 (1999), 255–63.

[19] E. Yusufoğlu, The variational iteration method for studying the Klein-Gordon equation, Applied Mathematics Letters, 21 (2008), 669-674.

[20] A. A. Hemeda, Variational iteration method for solving wave equation, Computers and Mathematics with Applications, 56 (2008), 1948-1953.

[21] C. K. Chen, S. H. Ho, Solving partial differential equations by two-dimensional differential transform method, Applied Mathematics and Computation, 106 (1999), 171-179.

[22] M. J. Jang, C. L. Chen and Y. C. Liu, On solving the initial-value problems using the differential transformation method, 115 (2000), 145-160.

[23] I. H. Abdel-Halim Hassan, Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems, Chaos, Solitons and Fractals 36 (2008) 53-65.

[24] I. H. Abdel-Halim Hassan, Different applications for the differential transformation in the differential equations, Applied Mathematics and Computation, 129 (2002), 183-201.

[25] F. Ayaz, On two-dimensional differential transform method, Applied Mathematics and Computation, 143 (2003) 361-374.

[26] S. H. Chang, I. L. Chang, A new algorithm for calculating one-dimensional differential transform of nonlinear functions, Applied Mathematics and Computation, 195(2008) 799-808.

[27] A. Arikoglu, I. Ozkol, Solution of differential-difference equations by using differential transform method, Applied Mathematics and Computation, 181(2006) 153-162.

[28] C. K. Chen, S. H. Ho, Application of differential transformation to eigenvalue problems, Applied Mathematics and Computation 147 (1996) 173-188.

[29] C. K. Chen, S. H. Ho, Solving partial differential equations by two dimensional differential transform, Appl. Math. Comput. 106 (1999) 171-179.

[30] S. Liao, Notes on the homotopy analysis method: Some definitions and theorems, Communications in Nonlinear Science and Numerical Simulation, 14 (2009), 983-997.

[31] A. M. Wazwaz, A study on linear and nonlinear Schrodinger equations by the variational iteration method, Chaos, Solitons and Fractals, 37 (2008), 1136-1142.

[32] S. U. Islam, S. Haq and J. Ali, Numerical solition of special 12^{th}-order boundary value problems using differential transform method, Communications in Nonlinear Science and Numerical Simulation, 14 (2009), 1132-1138.

[33] S. Liu, Z. Fu and S. Liu, Exact solutions to sine-Gordon-type equations, Physics Letters A, 351 (2006), 59-63.

[34] A. G. Bratsos, A modified explicit numerical scheme for the two-dimensional sine-Gordon equation, International Journal of Computer Mathematics, 85 (2008), 241-252.

[35] S. Liao, Homotopy analysis method: A new analytical technique for nonlinear problems, Communications in Nonlinear Science and Numerical Simulation, 2 (1997), 95-100.

[36] S. Liao, On the homotopy analysis method for nonlinear problems, Applied Mathematics and Computation, 147 (2004), 499-513.

[37] A. D. Polyanin, V. F. Zaitsev, Handbook of Nonlinear partial Differential Equations, Chapman & Hall/CRC, Boca Raton, 2004.

[38] M. A. M. Lynch, Large amplitude instability in finite difference approximations to the Klein–Gordon equation, Applied Numerical Mathematics, 31 (1999), 173–82.

[39] C. L. Chen, Y. C. Liu, Solution of two-point boundary-value problems using the differential transformation method, Journal of Optimization Theory and Applications, 99 (1998), 23-35.

[40] A. M. Wazwaz, Partial differential equations methods and applications, Saint Xavier University, USA, 459p, 2002.