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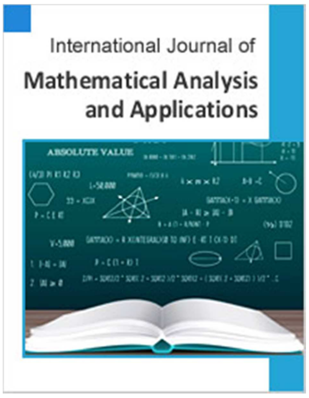

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Fixed Point Theorems in Left and Right Dislocated Quasi-Multiplicative Metric Spaces

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Abstract
In this work, we introduce generalized types of multiplicative metric spaces so-called left and right dislocated quasi multiplicative metric spaces (abbrev. l d q-multiplicative metric space and r d q-multiplicative metric space respectively). We state and prove two fixed point theorems in these spaces.

1. Introduction

In 1906, Frechet introduced the notion of metric space. We can see recent generalizations of metric space and concern topological structures in literature. (See, e.g., [1-5]).

In 2001, the concept of dislocated metric spaces was introduced by Hitzler [5]. He generalized metric spaces in the sense that self-distance of points need not to be zero necessarily. Also, he established Banach's contraction principle in these spaces.

In 2005, Zeyada et al. [8] introduced the notion of dislocated quasi metric spaces and generalized the result of Hitzler's result to dislocated quasi metric spaces. On the other hand, in 2013, Ahmed et al. [1] introduced two generalized types of dislocated metric spaces and proved some fixed point theorems in these spaces. Recently Ahmed et al. [2] introduce the concept of dislocated quasi multiplicative metric metric spaces. Also, Ahmed et al. [3] established the fixed point results for two rational contraction self-mappings on dislocated quasi multiplicative metric spaces.

Definition 1.1. [7] Let X be a nonempty set. A distance on X is a function $d: X \times X \to [0, \infty)$. A pair (X, d) is called a distance space. If d satisfies the following conditions for all $x, y, z \in X$:

(1) If $d(x, y) = 0$, then $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, y) \le d(x, z) + d(z, y)$,

for all $x, y, z \in X$, then it is called a dislocated metric on X (or simply d-metric). It is obvious that if d satisfies (1) − (3) and $d(x, x) = 0$ for all $x \in X$, then d is a metric on X. It's clear that a metric is a d-metric and the converse is not true (see, e.g., [5]).

Definition 1.2. [1] Let X be a nonempty set and a $d: X \times X \to [0, \infty)$ be a function satisfying the conditions for all $x, y, z \in X$:

(1) If $d(x, y) = d(y, x) = 0$, then $x = y$;
(2) $d(x, y) \le d(x, z) + d(z, y)$,

then it is called a dislocated quasi metric on X (d q-metric).

On the third hand, Bashirov et al. [4] defined the concept of multiplicative metric spaces, in the following.

Definition 1.3. Let X be a nonempty set. A function $d: X \times X \to R$, R:= the set of all real numbers, is called multiplicative metric iff d satisfies the following conditions for all x, y, z ∈ X:

(1) $d(x, y) \geq 1$;

(2) $d(x, y) = 1$ if and only if $x = y$;

(3) $d(x, y) = d(y, x)$;

(4) $d(x, y) \leq d(x, z) \cdot d(z, y)$.

In this case, (X, d) is called a multiplicative metric space.

Example 1.1. [7] Let R^n+, $R^n+=(0, \infty)$ be the collection of all n-tubles of positive real numbers.

Let d: Rn × Rn → R be defined as follows

$$d(x,y) = \left|\frac{x_1}{y_1}\right| \cdot \left|\frac{x_2}{y_2}\right| \cdot \ldots \cdot \left|\frac{x_n}{y_n}\right|,$$

where $x = (x_1, x_2,..., x_n) \in R$, y = (y1, y2,..., yn) ∈ R and $|\cdot|: R+ \to R+$ is defined as follows

$$|a| = \begin{cases} a: if\ a \geq 1 \\ a: if\ a < 1 \end{cases}$$

Then it is obvious that the all conditions of multiplicative metric are satisfied.

After that Özavsar et al. [6] were the first researchers who discussed multiplicative metric mapping by giving some topological properties of the relevant multiplicative metric space. Furthermore, Özavsar [6] et al. mentioned the concept of multiplicative contraction.

Definition 1.4. [6] Let (X, d) be a multiplicative metric space. A mapping $f: X \to X$ is called multiplicative contraction iff there exists a real constant $\lambda \in [0, 1)$ such that

$$d(f(x_1), f(x_2)) \leq d(x_1, x_2)^{\lambda}\ \forall x, y \in X.$$

Özavsar et al. [7] state and proved Banach contraction principle in multiplicative metric spaces as follows.

Theorem 1.1. Let (X, d) be a multiplicative metric space and let $F: X \to X$ be a multiplicative contraction. If (X, d) is complete, then f has a unique fixed point.

The present paper is arranged as follows. In Section (2), we introduce the concept of left dislocated quasi-multiplicative metric spaces. Also, we present a fixed point theorem in these spaces. After that, we give the concept of right dislocated quasi-multiplicative metric spaces in Section 3. Finally a fixed point theorem in these spaces is established.

2. Left Dislocated Quasi-Multiplicative Metric Space

First we introduce the following concept.

Definition 2.1. Let X be a nonempty set. A function $d: X \times X \to R$ is called left dislocated quasi- multiplicative metric (abbrev l d q-multiplicative metric) iff it satisfies the following conditions for all x, y, z ∈ X:

(1) $d(x, y) \geq 1$;

(2) If $d(x, y) = 1$, then $x = y$;

(3) $d(x, y) \leq d(z, x) \cdot d(z, y)$.

Then the pair (X, d) is called left dislocated quasi-multiplicative metric space.

Example 2.1. Let $X = [0, \infty)$. Define a mapping $d: X \times X \to R$, by

$$d(x,y) = a^{|x-\frac{2(x-y)}{3}|}, \text{ and } d(y,x) = a^{|y-\frac{2(y-x)}{3}|}.$$

where a > 1 is any fixed number. Then for each a > 1, d is satisfy (X, d) is called left dislocated quasi-multiplicative metric space.

Now, we state the following Definition.

Definition 2.2. A sequence $\{x_n\}$ in l d q-converges to x iff

$$\lim_{n\to\infty} d(xn, x) = \lim_{n\to\infty} d(x, xn) = 1$$

In this case x is called a l d q-limit of $\{x_n\}$.

Definition 2.3. A sequence $\{x_n\}$ in l d q-multiplicative metric space (X, d) is called Cauchy if for all $\epsilon > 1$, there exist $n_0 \in N$ such that for all $n, m \geq n_0$, we have

$$d(x_n, x_m) < \epsilon \text{ and } d(x_m, x_n) < \epsilon$$

Definition 2.4. A l d q-multiplicative metric space (X, d) is called complete if every Cauchy sequence in it is l d q-convergent.

Lemma 2.1. Let (X, d) be a l d q-multiplicative metric space, every subsequence of any convergent sequence is convergent.

Proof. Let $\{x_n\}$ be a l d q-convergent sequence in l d q-multiplicative metric space. Thus there exist $x \in X$ such for any ϵ there exists n0 ∈ N such that $d(x_n, x) < \epsilon$ and $d(x, x_n) < \epsilon$ for all n > n0. Let $\{x_{nk}\}$ be a subsequence of $\{x_n\}$. Then for every $\epsilon > 0$, there exist n0 ∈ N such that $d(x_{nk}, x) < \sqrt{\epsilon}$ and $d(x, x_{nk}) < \sqrt{\epsilon}$ for all $n > n_0$. Then $\{x_{nk}\}$ is convergent.

Lemma 2.2. Let (X, d) be a l d q-multiplicative metric space. If a sequence $\{x_n\}$ is a l d q- convergent, then the l d q-limit point is unique.

Proof. Suppose that $x_n \to x$ and $x_n \to y$, then for each >1, we have $n_0 \in N$ such that $d(x_n, x) < \epsilon$ and $d(x, x_n) < \epsilon$.

Also $d(x_n, y) < \sqrt{\epsilon}$ and $d(y, x_n) < \sqrt{\epsilon}$. By using the multiplicative triangular inequality, we have

$$d(x,y) \leq d(x,x_n).d(x_n,y) < \sqrt{\epsilon}.\sqrt{\epsilon} = \epsilon$$

Since $\epsilon > 1$ is arbitrary. Thus, we have $d(x, y) = 1$, therefor $x = y$.

Theorem 2.1. Let (X, d) be complete l d q-multiplicative metric space and $F: X \to X$ be a map such that

$$d(F x, F y) \leq \max\{d(x, y), d(x, F x), d(y, F y), d(x, F y), d(y, F x)\}^{\lambda}. \quad (1)$$

for all x, y ∈ X, where $\lambda \in [0, \frac{1}{2})$, and F is continuous. Then F has unique fixed point

Proof. Let $x_0 \in X$, an arbitrary point. Define a sequence $\{x_n\}$ in X by $x_n = F x_{n-1}$, for n = 1, 2,.... Firstly if $x_n0 = x_n0 +1$ for some n0 = 1, 2,..., then it's clear that $\{x_n\}$ is fixed point Secondly, assume x_n0 6= x_n0 +1, ∀n. Then we have from (3.1)

$$d(x_n, x_{n+1}) = d(F x_{n-1}, F x_n)$$
$$\leq \max \{d(x_{n-1}, x_n), d(x_{n-1}, F x_{n-1}), d(x_n, F x_n), d(x_{n-1}, F x_n), d(x_n, F x_{n-1})\}^{\lambda}$$
$$\leq \max \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}^{\lambda}$$
$$\leq \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1}), [d(x_{n-1}, x_n) \cdot d(x_n, x_{n+1})], d(x_n, x_n)\}^{\lambda}$$
$$= d(x_n, x_{n-1})^{\lambda} \cdot d(x_n, x_{n+1})^{\lambda}$$
$$\leq d(x_n, x_{n-1}{}^{\lambda} \cdot d(x_{n-1}, x_n)^{\lambda 2} \quad (2)$$

Thus

$$d(x_n, x_{n+1}) \leq d(x_n, x_n\text{-}1)^{\lambda} \cdot d(x_{n-1}, x_n)^{\lambda 2}, \forall n \in N. \quad (3)$$

Similarly

$$d(x_{n+1}, x_n+2) \leq d(x_{n+1}, x_n)^{\lambda} \cdot d(x_n, x_{n+1})^{\lambda 2} \quad (4)$$

Therefor

$$d(x_n, x_{n+1}) \leq d(x_n, x_n\text{-}1)^{\lambda} \cdot d(x_{n-1}, x_n)^{\lambda 2}$$
$$\leq d(x_{n-1}, x_n-2)^{\lambda 2} \cdot d(x_n-2, x_{n-1})^{\lambda 3}$$
$$\leq d(x_n-2, x_n-3)^{\lambda 3} \cdot d(x_n-3, x_n-2)^{\lambda 4}$$
$$\leq d(x_1, x_0)^{\lambda n} \cdot d(x_0, x_1)^{\lambda n-1} \; \forall n \in N. \quad (5)$$

Now we claim that the sequence $\{x_n\}$ satisfies the multiplicative Cauchy convergence, then let n, m ∈ N be such that m > n, then by multiplicative triangular inequality, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m-1}, x_{n+m}) \cdot d(x_{n+m-2}, x_{n+m-1}) \cdot \ldots \cdot d(x_n, x_{n+1})$$
$$\leq [d(x_1, x_0)^{\lambda n+m-1} \cdot d(x_0, x_1)^{\lambda n+m}] \cdot [d(x_1, x_0)^{\lambda n+m-2} \cdot d(x_0, x_1)^{\lambda n+m-1}[d(x_1, x_0)^{\lambda} \cdot d(x_0, x_1)^{\lambda}]$$

Letting n → ∞, in (2.6), we have $d(x_n, x_{n+m}) \to 1$. Hence the sequence $\{x_n\}$ is satisfies multi-

plicative Cauchy sequence.

By the completeness of X, there exist x ∈ X such that

$x_n \to x$. From lemma (2.1), we obtain $x_{n+1} = F x_n$. Since F is continuous, then $x_{n+1} = F x_n \to F x$. Since the multiplicative limit is unique, thus x = F x.

3. Right Dislocated Quasi Multiplicative Metric Spaces

Definition 3.1. Let X, be a nonempty set. A function d: X × X → R is called right dislocated quasi-multiplicative metric (abbrev r d q-multiplicative metric) iff it satisfies the following

Conditions for all x, y, z ∈ X

(1) d(x, y) ≥ 1;

(2) If d(x, y) = 1, then x = y;

(3) d(x, y) ≤ d(x, z) · d(y, z).

Here R:= [0, ∞) denotes the set Then the (X, d) is called left dislocated quasi multiplicative-metric space.

Example 3.1. Let X = [0, ∞). Define a mapping d: X × X → R, by

$$d(x,y) = a^{|x-\frac{2(x-y)}{3}|}, \text{ and } d(y,x) = a^{|y-\frac{2(y-x)}{3}|}.$$

where a > 1 is any fixed number. Then for each a > 1, d is satisfy (X, d) is called right dislocated quasi-multiplicative metric space.

Now, we state the following Definition.

Definition 3.2. A sequence $\{x_n\}$ in l d q-converges to x iff

$$\lim_{n\to\infty} d(xn, x) = \lim_{n\to\infty} d(x, xn) = 1$$

In this case x is called a r d q-limit of $\{x_n\}$

Definition 3.3. A sequence $\{x_n\}$ in r d q-multiplicative metric space (X, d) is called Cauchy if for all ϵ> 1, there exist n_0 ∈ N such that for all n, m ≥ n0, we have

$$d(x_n, x_m) < \epsilon, \text{ and } d(x_m, x_n) < \epsilon.$$

Definition 3.4. A r d q-multiplicative metric space (X, d) is called complete if every Cauchy sequence in it is d q-convergent.

Now, we state the following lemma without proof.

Lemma 3.1. Let (X, d) be a l d q-multiplicative metric space, every subsequence of any conver- gent sequence is convergent.

Lemma 3.2. Let (X, d) be a r d q-multiplicative metric space. If a sequence $\{x_n\}$ is a r d q- convergent, then the r d q-limit point is unique.

Theorem 3.1. Let (X, d) be complete r d q-multiplicative metric space and F: X → X be a map such that

$$d(F x, F y) \leq \max \{d(x, y), d(x, F x), d(y, F y), d(x, F y), d(y, F x)\}\lambda. \quad (6)$$

for all x, y ∈ X, where $\lambda \in [0, \frac{1}{2})$, and F is continuous. Then F has unique fixed point.

Corollary 3.1. Let (X, d) be a r d q-multiplicative metric space and let F: X → X be a multi- plicative contraction. If (X, d) is complete, then f has a unique fixed point.

Remark 3.1. Corollary (3.1) generalizes Theorem (1.1).

References

[1] M. A. Ahmed, F. M. Zeyada and G. F. Hassan, Fixed Point Theorems in Generalized Types of Dislocated Metric Space and Its Applications. Thai j. math. ISSIN 168-0209, 11 (2013), 67-73.

[2] M. A. Ahmed and A. M. Zidan, Fixed point theorems in a generalized type of multiplicative metric spaces. International Journal of Advances in Mathematics. Volume 2016, Number 1, Pages 49-54, 2016.

[3] M. A. Ahmed, A. H. Soliman, and A. M. Zidan, On The Existence of Fixed Points of Two Rational Type Contractions in A Dislocated Quasi Multiplicative Metric Spaces and It's Application. Journal of Advanced Studies in Topology, [S. l.], v. 8, n. 1, p. 59-66, (2017). ISSN 2090-388X.

[4] A. E. Bashirov, E. M. Kurpinar and A. Özyapici, Multiplicative calculus and its applica- tions, J. Math. Anal. Appl., 337 (2008) 36-48.

[5] P. Hitzler, Generalized Metrics and Topology in Logic Programming Semantics, Ph. D The- sis, National Univeristy of Ireland, University College Cork, (2001).

[6] M. Özavsar and A. C. Cevikel, Fixed point of multiplicative contraction mapping on mul- tiplicative metric spaces. http://arxiv.org/abs/1205.5131v1.

[7] P. Waszkiewicz, The local triangle axioms in topology and domain theory, Applied Gen- eral Topology 4 (1) (2003) 47-70.

[8] Zeyada F M, Hassan G H, Ahmed M A. A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces[J]. Arabian journal for Science and Engineering, 2006, 31(1A): 111.