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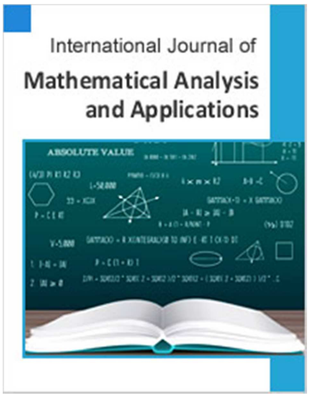

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Investigations of the Green Function and Discreteness of Spectrum of Higher Order Operator-Differential Equations on Semi-Axis

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Abstract
In the present paper the Green's function and the spectrum of even higher order operator-differential equations are studied. As first the Green's function of the principal part of the equation with frozen coefficients is constructed. By using the Levy method, the integral equation is obtained for the Green's function coefficients. In the Banach spaces of operator valued functions the solution of the obtained integral equations is studied. The uniform estimation of the Green function from which in particular the discreteness of the spectrum is derived.

1. Introduction

Let H be a separable Hilbert space. Suppose that $L_2\left[H,[0,\infty)\right]$ the spaces of strongly measurable Hilbert space valued functions on $[0,\infty)$ such that

$$\|f\|_{L_2\left[H,[0,\infty)\right]}=\left(\int_0^\infty \|f(x)\|_H^2\,dx\right)^{\frac{1}{2}}<\infty,$$

where $\|f(x)\|_H=(f,f)$ is an inner product on H. Let us consider in the space $H_1=L\left[H,[0,\infty)\right]$ a differential operator L, generated by the expression

$$l(y)=(-1)^n\left(P(x)y^{(n)}\right)^{(n)}+\sum_{j=2}^{2n}Q_j(x)y^{(2n-j)} \tag{1}$$

with the boundary conditions

$$y^{(l_1)}(0) = y^{(l_2)}(0) = ... = y^{(l_n)}(0) = 0, \qquad (2)$$

here $0 \le l_1 < l_2 < ... < l_n \le 2n-1$, $y \in H_1$ and derivatives are understood in the strong sense.

Let $D(\Omega)$ the space of test functions. Let D' be a totality of all the functions of the form $\sum_{k=1}^{l_j} \phi_k(x) f_k$, where $\phi_k(x)$ are finite, $2n$ -times continuously-differentiable scalar functions and $f_k \in D(Q)$.

Now define the operator L' generated by the expression (1) and boundary conditions (2) with domain of definition D'. Under some conditions on the coefficients of expressions (1) with boundary conditions (2) the operator L' is a positive and symmetric operator in the space H_1.

Assume that the closure L of the operator L' is a self-adjoint and lower semi-bounded operator in H_1.

In this paper authors study the Green's functions and discreteness of the spectrum of the operator L. Notice that the Green function of the Sturm-Liouville equation with a self-adjoint operator-coefficient was first studied by B. M. Levitan [1], while the asimptotic distribution of the eigenvalues of the operator L was studied in the paper of B. M. Levitan and A. G. Kostyuchenko [2]. The Green's function and asimptotic behavior of the eigenvalues of the operator L generated by the expression $l(y) = -(P(x)y')' + Q(x)y$ in the self-adjoint case was studied by E. Abdukadyrov [3]. The Green's function and asimptotic behavior of eigenvalues of higher even order operator given on the whole axis is studied by M. Bayramoglu [4]. The case of a semi-axis was considered in [5]-[8] and others recently published papers [9]-[18]. Note that many problems of mechanics, mathematical physics, theory of partial differential equations, etc are reduced to the study of boundary value problems for operator-differential equations in different spaces. The asymptotic distribution of eigenvalues for boundary-value problems with operator coefficients was first considered by A. G. Kostyuchenko and B. M. Levitan [1] and [2]. There followed a lot of papers dedicated to the investigation of spectrum of differential operators with operator coefficients. The asymptotic distribution of the eigenvalues of operators defined on the whole space and having a discrete spectrum may be interesting for those who specialize in quantum mechanics. The theory of operator-differential equations with unbounded operator-coefficient is a common tool for studying infinite systems of ordinary differential-operators, partial differential equations and integrodifferential equations. The main task in this theory is to determine the behavior of the eigenvalues and eigenfunctions of the associated differential operators.

Well assume that coefficients of the operator L satisfy the following conditions:

1. For all $x \in [0, +\infty)$ and for all $h \in H$

$$m(h,h)L_H \le (P(x)h,h)_H \le M(h,h)_H, \quad m, M > 0;$$

2. The operator-function $P(x)$ is $n-$ time differentiable for all $x \in [0, +\infty)$;

3. The operators $Q(x)$ are self-adjoint in H almost for all x, moreover, in H there exists common for all x and dense everywhere in H the set $D\{Q(x)\} = D(Q)$, on which $Q(x)$ are defined and symmetric (This, we assume, that operators $Q(x)$ can be unbounded in H);

4. Operators $Q(x)$ are uniformly bounded below, i.e. there exists such a number $d > 0$ that for all x and $f \in D(Q)$, $(Q(x)f, f)_H > d(f, f)_H$;

5. The exists a constant number $c > 0$, $0 < a < \frac{2n+1}{2n}$ such that for all x and $|x-\xi| \le 1$ the following inequality is true; $\left\| [Q(\xi) - Q(x)] \cdot Q^{-a}(a) \right\| \le c|x-\xi|$;

6. For $|x-\xi| > 1$

$$\left\| K(\xi) \exp\left\{ -\frac{J_m \varepsilon_1}{2} |x-\xi| \omega \right\} \right\| < C,$$

where $K(x) = P^{-\frac{1}{2}}(x) Q(x) P^{-\frac{1}{2}}(x)$, $\omega = \{K(x) + \mu P^{-1}(x)\}^{\frac{1}{2n}}$, $\mu > 0$;

7. For all $x, \xi \in [0, +\infty)$

$$\left\| Q(x) P^{\pm\frac{1}{2}}(x) Q^{-1}(x) \right\|_H < C, \quad \left\| Q(\xi) P^{-\frac{1}{2}}(x) Q^{-1}(\xi) \right\|_H < C;$$

8. Operator-functions $Q_j(x), j = 3, 4, ..., 2n-1$ are self-adjoint in H and for all $x \in [0, +\infty)$

$$\left\| Q_j(x) Q^{\frac{1-j}{2n}}(x) \right\|_H < C, \ (j = 3, 4, ..., 2n-1), \ \varepsilon > 0.$$

The main results of the paper is that the operator $R_\mu = (L + \mu E)^{-1}$, $\mu > 0$ is an integral operator with an operator kernel $G(x, \eta; \mu)$, which is called the Green operator-function of the operator L. By definition of the Green function $G(x, \eta; \mu)$ is an operator function in H dependent on two variables x and $\eta (0 \le x, \eta < \infty)$ the parameter μ and satisfy the conditions:

a) $\frac{\partial^k G(x, \eta; \mu)}{\partial \eta^k} (k = 0, 1, 2, ..., 2n-2)$ is a strongly continuous operator valued function with respect to variables (x, η);

b) $\frac{\partial^{2n-1} G(x, x+0; \mu)}{\partial \eta^{2n-1}} - \frac{\partial^{2n-1} G(x, x-0; \mu)}{\partial \eta^{2n-1}} = (-1)^n E$;

v) $(-1)^n G_\eta^{(2n)} + \sum_{j=2}^{2n-1} G_\eta^{(2n-j)} Q_j(\eta) + \mu G = 0$;

q) $G^{(l_1)}(x,0;\mu) = G^{(l_2)}(x,0;\mu) = ... = G^{(l_n)}(x,0;\mu) = 0$;

d) $G^*(x,\eta;\mu) = G(\eta;x;\mu)$;

e) $\int_0^\infty \|G(\eta;x;\mu)\|_H^2 d\eta < \infty$.

The Green function of the operator L is studied in three stages.

In the first stage the Green function of the operator L_1 generated by the differential expression

$$l_1(y) = (-1)^n \left(P(\xi) y^{(n)}\right)^{(n)} + Q(\xi) y + \mu y \tag{3}$$

and boundary conditions (2) is constructed. Here ξ is a fixed point.

In the second stage we construct and study some properties of the Green function of the operator L_0 generated by the differential expression

$$l_0(y) = (-1)^n \left(P(x) y^{(n)}\right)^{(n)} + Q(x) y + \mu y \tag{4}$$

and boundary conditions (2).

In the third stage the Green function of the operator L, generated by the differential expression (1) and boundary conditions (2) is studied.

The present paper is organized as follows. In Section 2, a Green function of the operator L_1 is constructed. After that, in Section 3 a Green function $G_0(x,\eta;\mu)$ of the operator a L_0 and solution of the obtained integral equations is proved.

2. Construction of the Green Function of the Operator L_1

The Green function $G_1(x,\eta,\xi,\mu)$ of the operator L_1 we'll seek in the form

$$G_1(x,\eta,\xi,\mu) = g(x,\eta,\xi,\mu) + V(x,\eta,\xi,\mu), \tag{5}$$

where $g(x,\eta,\xi,\mu)$ is the Green function of the equation $l_1(y) = 0$ on the whole *axis*. As is known [4], it is of the form:

$$g(x,\eta,\xi,\mu) = \frac{1}{2\pi} P^{-\frac{1}{2}}(\xi) \omega_\xi^{1-2n} \sum_{k=1}^n \varepsilon_k \exp\left(i\varepsilon_k |x-\eta| \omega_\xi\right) P^{-\frac{1}{2}}(\xi). \tag{6}$$

Here by ε_k we determine roots from $\sqrt[2n]{-1}$ lying on the upper half plane.

The function $V(x,\eta,\xi,\mu)$ is bounded as $x \to +\infty$ by solution of the following problem:

$$l_1(V) = 0 \tag{7}$$

$$V^{(l_j)}(x,\eta,\xi,\mu)\Big|_{x=0} = -g^{(l_j)}(x,\eta,\xi,\mu)\Big|_{x=0}, \quad j = 1,2,...,n. \tag{8}$$

Solution of problem (7), (8) is represented in the form:

$$V(x,\eta,\xi,\mu) = \frac{1}{2ni} P^{-\frac{1}{2}}(\xi) \omega_\xi^{1-2n} \sum_{k=1}^n \varepsilon_k e^{i\varepsilon_k \omega_\xi (x+\eta)} P^{-\frac{1}{2}}(\xi). \tag{9}$$

Then, Greens function on problem (3), (2) will take the form:

$$G_1(x,\eta,\xi,\mu) = \frac{1}{2ni} P^{-\frac{1}{2}}(\xi) \omega_\xi^{1-2n} \sum_{k=1}^n \varepsilon_k e^{i\varepsilon_k \omega_\xi (x+\eta)} P^{-\frac{1}{2}}(\xi) - -\frac{1}{2ni} P^{-\frac{1}{2}}(\xi) \omega_\xi^{1-2n} \sum_{k=1}^n \varepsilon_k e^{i\varepsilon_k \omega_\xi (x+\eta)} P^{-\frac{1}{2}}(\xi). \tag{10}$$

The function $G_1(x,\eta,\xi,\mu)$ can be transformed in the following form:

$$G_1(x,\eta,\xi,\mu) = \begin{cases} \frac{1}{2ni} P^{-\frac{1}{2}}(\xi) \omega_\xi^{1-2n} \sum_{k=1}^n \varepsilon_k e^{i\varepsilon_k \omega_\xi (\eta-x)} \left\{E - e^{2i\varepsilon_k \omega_\xi \eta}\right\} P^{-\frac{1}{2}}(\xi), \; x > \eta, \\ \frac{1}{2ni} P^{-\frac{1}{2}}(\xi) \omega_\xi^{1-2n} \sum_{k=1}^n \varepsilon_k e^{i\varepsilon_k \omega_\xi (x-\eta)} \left\{E - e^{2i\varepsilon_k \omega_\xi \eta}\right\} P^{-\frac{1}{2}}(\xi), \; x > \eta. \end{cases}$$

Since $\left\|e^{i\varepsilon_k\omega_\xi\eta}\right\|_H \to 0$, as $\mu\to\infty$, we have:

$$G_1(x,\eta,\xi,\mu)=\frac{1}{2ni}P^{-\frac{1}{2}}(\xi)\omega_\xi^{1-2n}\sum_{k=1}^{n}\varepsilon_k e^{i\varepsilon_k\omega_\xi|x-\eta|}\left\{E-r(x,\eta,\xi,\mu)\right\}P^{-\frac{1}{2}}(\xi) \tag{11}$$

moreover, as $\mu\to\infty$ we have $\left\|r(x,\eta,\xi,\mu)\right\|=0(1)$ uniformly by (x,η).

3. Construction of the Green Function $G_0(x,\eta;\mu)$ of the Operator a L_0 and Solution of the Obtained Integral Equations

Now, let's investigate Green's function of equation

$$(-1)^n\left(P(x)y^{(n)}\right)^{(n)}+Q(x)y+\mu y=0 \tag{12}$$

with boundary conditions (2).

Therefore rewrite equation (12) in the following form:

$$(-1)^n\left(P(x)y^{(n)}\right)^{(n)}+Q(x)y+\mu y=(-1)^n\left(P(\xi)y^{(n)}\right)^{(n)}+Q(\xi)y+\mu y$$

$$+(-1)^n\left\{\left(P(x)y^{(n)}\right)^{\omega_n}-\left(P(\xi)y^{(n)}\right)\right\}^{(n)}+\left\{Q(x)-Q(\xi)\right\}y=0. \tag{13}$$

Formally search Green function of operator $L_0, G_0(x,\eta;\xi,\mu)$ in the form:

$$G_0(x,\eta;\xi,\mu)=G_1(x,\eta;\xi,\mu)+G_2(x,\eta;\xi,\mu).$$

In the last equation, putting $G_1(x,\eta;\xi,\mu)+G_2(x,\eta;\xi,\mu)$ instead of y and applying to the both hand sides the operator, generated by kernel $G_1(x,\eta;\xi,\mu)$ (supposing $x=\xi$), we get

$$G_0(x,\eta;\xi,\mu)=G_1(x,\eta;\xi,\mu)-\int_0^\infty G_2(x,\xi;\mu)\left[Q(\xi)-Q(x)\right]G_0(\xi,\eta;\mu)\,d\xi$$

$$+\frac{1}{2ni}\int_0^\infty P_{(x)}^{-1\frac{1}{2}}\omega\sum_{k=1}\varepsilon_k\exp\left(i\varepsilon_k|x-\xi|\omega\right)^n\left(E-r(x,\xi;\mu)\right)P_{(x)}^{-\frac{1}{2}}\times\left[P(\xi)-P(x)\right]G_0(\eta,\xi;\mu)\,d\xi$$

$$+(-1)^n\sum_{m=1}^{n}C_n^m\int_0^\infty G_1^{(2n-m)}(x,\xi;\mu)P_\xi^{(m)}(\xi)G_0(\eta,\xi;\mu)\,d\xi.$$

(Here $G_0(x,\eta;\mu)\equiv G_0(x,\eta,x,\mu), G_1(x,\eta;\mu)=G_1(x,\eta,x,\mu)$).

For studying the solution of integral equation (13), according to the paper

[1], we introduce the Banach spaces $X_1, X_2, X_3^{(p)}, X_2^{(s)}, X_4^{(s)}$ and $X_5\ (p\ge 1, s\ge 0)$, whose elements are operator functions $A(x,\eta)$ in H for $\eta\in(0,\infty)$ and the norms are determined in the following form:

$$\left\|A(x,\eta)\right\|_{X_1}^2=\int_0^\infty\left\{\int_0^\infty\left\|A(x,\eta)\right\|_H^2\,d\eta\right\}dx,$$

$$\left\|A(x,\eta)\right\|_{x_2}^2 = \int_0^\infty \left\{\int_0^\infty \left\|A(x,\eta)\right\|_2^2 d\eta\right\} dx .$$

Here by $\left\|A(x,\eta)\right\|_2^2$ we denote the Hilbert-Schmidt norm (absolute-norm) of the operator-function $A(x,\eta)$ in H .

$$\left\|A(x,\eta)\right\|_{x_3^{(p)}} = \left[\sup_{0\le x<\infty}\int_0^\infty \left\|A(x,\eta)\right\|_H^p d\eta\right]^{1/p},$$

$$\left\|A(x,\eta)\right\|_{x_2^{(p)}} = \int_0^\infty dx\left\{\int_0^\infty \left\|A(x,\eta)Q^s(\eta)\right\|_2^2 d\eta\right\},$$

$$\left\|A(x,\eta)\right\|_{x_4^{(s)}} = \sup_{0\le x<\infty}\int_0^\infty \left\|A(x,\eta)Q^s(\eta)\right\|_H d\eta ,$$

$$\left\|A(x,\eta)\right\|_{x_5} = \sup_{0\le x<\infty}\sup_{0\le \eta<\infty}\left\|A(x,\eta)\right\|_H .$$

Definition and proof of their completeness in the case $-\infty < x,\ \eta < \infty$ are given by B. M. Levitan [1].

Determine the following integral operator:

$$NA(x,\eta) = \int_0^\infty G_1(x,\xi;\mu)\left[Q(\xi)-Q(x)\right]A(x,\eta)\,d\xi + \frac{1}{2\pi i}\int P^{-\frac{1}{2}}(x)\,\omega\sum_{m=1}^{n}\varepsilon_k \exp\left(i\varepsilon_k\left|x-\xi\right|\omega\right)\left(E-r(x,\eta;\mu)\right)P_{(x)}^{-\frac{1}{2}}$$

$$\times\left[P(\xi)-P(x)\right]A(\xi,\eta)\,d\xi + (-1)^n\int_0^\infty\sum_{m=1}^{n} C_n^m C_1^{(2n-m)}(x,\xi;\mu)P_{(\xi)}^{(m)}A(\xi,\eta)\,d\xi . \tag{14}$$

Theorem 3.1. If operator functions $P(x)$ and $Q(x)$ satisfy conditions 1-7, then for sufficiently large $\mu>0$ the operator N is contracting operator in the spaces $X_1, X_2, X_3^{(p)}, X_2^{(s)}, X_4^{(s)}, X_5$, therefore equation (12) can be solved by iteration method and the following estimation is valid

$$\left\|NA(x,\eta)\right\|_X = \frac{1}{\mu^\beta}\left\|A(x,\eta)\right\|_X ,\ \ \beta>0,\ \mu\to\infty . \tag{15}$$

Proof. In all above considered Banach spaces, equation (14) has a unique solution that may be obtained with the help of iterations method if only the operator-function $G_1(x,\eta;\mu)$ belongs to the appropriate space. Form estimation (15) it follows that the norm of operator N as $\mu\to\infty$ tends to zero. Therefore, as $\mu\to\infty$ we get following asymptotic equality

$$G_0(x,\eta;\mu) = G_1(x,\eta;\mu)\left[1+\theta(x,\eta;\mu)\right], \tag{16}$$

where $\left\|\theta(x,\eta;\mu)\right\| = 0(1)$ as $\mu\to\infty$, uniformly by $(x,\eta)\in(0,\infty)$.

We'll search Green's function of problem (1) and (2) in the form of:

$$G(x,\eta;\mu) = G_0(x,\eta;\mu) + \int_0^\infty G_0(x,\xi;\mu)\rho(\xi,\eta)\,d\xi . \tag{17}$$

Using the basic properties of Green function $G_0(x,\eta;\mu)$ for defining $\rho(x,\eta)$ we get the following integral equation

$$\rho(x,\eta) + \sum_{j=2}^{2n} Q_j(x)\frac{\partial^{2n-j}G_0(x,\eta,\mu)}{\partial\eta^{2n-j}}$$

$$-\sum_{j=2}^{2n}\int_0^\infty Q_j(x)\frac{\partial^{2n-j}G_0(x,\eta,\mu)}{\partial\eta^{2n-j}}\rho(\xi,\eta)\,d\xi . \tag{18}$$

Let

$$F(x,\eta,\mu) = -\sum_{j=1}^{2n}\int_0^\infty Q_j(x)\frac{\partial^{2n-j}G_0(x,\eta,\mu)}{\partial\eta^{2n-j}} .$$

Then the following equation

$$\rho(\xi,\eta) = F(x,\eta,\mu) - \int_0^\infty F(x,\xi,\mu)\rho(\xi,\eta)\,d\xi \tag{19}$$

holds.

Using the explicit form (10) of the Green function $G_1(x,\eta,\mu)$ it is easy to get the following estimation for the norm of the operator function $F(x,\eta,\mu)$

$$\|F(x,\eta,\mu)\| \le C\mu^{-\gamma} e^{-J_m \omega_1 \sqrt{\mu}|x-\eta|}.$$

Hence

$$\sup_{0\le\eta<\infty} \int_0^\infty \|F(x,\eta,\mu)\|_H^2 \, d\eta \le C \cdot \mu^{-2\gamma},$$

i.e. the function $F(x,\eta,\mu)$ is an element of the space $X_3^{(2)}$ as $\mu \to \infty$ tends (with respect to the norm of the space $X_3^{(2)}$) to zero. Therefore equation (19) in the space $X_3^{(2)}$ has a solution and this solution is unique. Hence in particular it follows the fact that for sufficiently large $\mu > 0$ the solution $\rho(\xi,\eta)$ of equation (19) behaves in the same way as $F(x,\eta,\mu)$.

For sufficiently large $\mu > 0$ the integral operator, contained in equation (17) is contractive (and as $\mu \to \infty$ convergent to zero) and therefore as $\mu \to \infty$ we have:

$$G(x,\eta,\mu) = G_0(x,\eta,\mu)\left[E + \alpha(x,\eta,\mu)\right],$$

where

$$\|\alpha(x,\eta,\mu)\| = 0(1). \tag{20}$$

Using asimptotical equality (10), (16), (20) we get the following important equality

$$G(x,\eta,\mu) = g(x,\eta,\mu)\left[E + \beta(x,\eta,\mu)\right],$$

where

$$\|\beta(x,\eta,\mu)\| = 0(1). \tag{21}$$

Using the expression for the function $g(x,\eta,\mu)$ for the function $G(x,\eta,\mu)$ we get

$$G(x,\eta,\mu) = \frac{1}{2ni} P^{-\frac{1}{2}}(x)\, \omega_\xi^{1-2n} \sum_{k=1}^{n} \varepsilon_k e^{i\varepsilon_k \omega_\xi |x-\eta|} P^{-\frac{1}{2}}(x)\left[E + \Omega(x,\eta,\mu)\right], \tag{22}$$

where

$$\|\Omega(x,\eta,\mu)\|_H = 0(1)$$

as $\mu \to \infty$.

From (21) and belonging of $g(x,\eta,\mu)$ to the space X_2 it follows that the integral operator generated by the kernel $G(x,\eta,\mu)$ is a Hilbert-Schmidt type operator, i.e

$$\int_0^\infty \int_0^\infty \|G(x,\eta,\mu)\|_2^2 \, dx d\eta < \infty$$

Since the function $G(x,\eta,\mu)$ is a kernel of the operator $R_\mu = (L + \mu E)^{-1}$ we get, that operator L has a discrete spectrum $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n, ...$ with a unique limit point in infinity.

This completes the proof of Theorem 3.1.

4. Conclusion

Thus, the Green's function and the spectrum of even higher order operator-differential equations is constructed and solvability of the obtained integral equations is proved.

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