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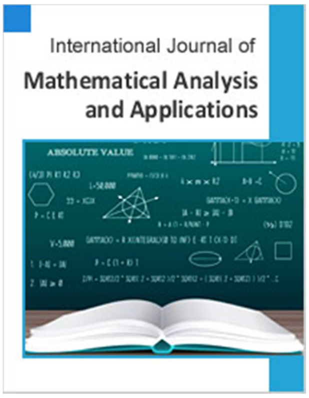

An Alternative Method for Deriving Infinite Series Using Long Division for School-Aged Students

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Abstract
The equivalence of an infinite series, such as (0.999… = 1) to a finite number is often not intuitively obvious to understand for most students. Yet, it is helpful to use an intuitive method to derive the equivalence that is familiar to most elementary school students. In this paper, a simple proof of the equality of 0.999… and 1 is provided by a non-traditional method using long division with an alternate quotient. This alternate method that uses long division to prove an infinite series of numbers can stimulate the imagination of students to use abstraction to understand the mathematical concept of infinity. This can be done without the use of geometric series.

1. Introduction

Infinity is a concept that stimulates the imagination of students to explore the usefulness of mathematical abstraction. Infinity is often taught as being a number that extends forever without any endpoint. Yet, in mathematics, most infinite numbers often asymptote to a finite number at infinity. The equivalence of an infinite series to a finite number at infinity is one of the most difficult concepts to reconcile in the mind of the students.

Thus, it is parliament to provide a pedagogical methodology that is easily understandable to students in order to stimulate their imagination in mathematical abstraction. The ability to abstract complex concepts (especially mathematical concepts) is unique to intelligent species, such as humans and primates. Infants as young as 6 to 8 months old [1] have been shown to acquire the ability to abstract numerical quantities. More specifically, the strategy used to solve mathematical problems also varies with developmental age [2, 3]. Thus, it is important to provide the most intuitive mathematical strategies to teach how to solve complex problems that are appropriate for their developmental age as the brain matures. [4-6].

The ability to understand the number system requires abstraction and mentalization of the mathematical concepts that may not have equivalence in the physical world, such as negative numbers and imaginary numbers. Furthermore, the concept of infinity requires imagination that expands the physical world into the imaginary world as a continuum. The asymptotic approach of an infinite series to a concrete finite number at infinity provides the link for students to bridge between a physical entity and an abstract entity.

Using an age-appropriate, easily understandable strategy to solve mathematical problems can facilitate the mental development of children in mental imagination and abstraction. In other words, the methodologies for teaching mathematical and scientific concepts are often based on the principle of reduction and simplification. By reducing complex phenomena into simplistic entities, the inter-relationships between complex concepts and principles can be established intuitively.

2. Intuitive Understanding of Concepts and Mental Abstraction

The ability to understand the concept of numbers is an innate ability in infants [7]. If school-aged students are able to understand the relational properties of numbers to solve arithmetical problems, they can be taught to generalize the same concept to solve other problems using relational properties [8]. They can use the same strategies to solve other problems, such as order-indifferent, decomposition, and retrieval problems. Thus, the specific strategies for solving problems in the number system can provide the foundation for solving other similar problems in school-aged students that are intuitive to them.

The term "intuitive" is often as ill-defined as the term "common sense" because there does not seem to have any explicit definable rules to quantify such mental processes. Yet, if such intuitive concepts and common sense can be quantified by explicit definitions, then educators can apply a consistent and an effective pedagogy to demonstrate the problem-solving strategies to students.

Intuitive learning in neuroscience is the mental process that is compatible with the neurobiological mechanisms in learning and knowledge acquisition. One of the basic biological mechanisms in learning is associative learning in which new knowledge is gained by associating it with previous knowledge [9, 10]. Furthermore, another basic biological mechanism in learning is imitation learning in which new knowledge is gained by mimicking an example demonstrated to the learner [11, 12]. These mechanisms essentially generalize the knowledge by forming a conceptual framework by association and by mimicking similar processes for solving problems. For example, the intuitive way to learn how to cook is to observe the demonstration performed by the chef. The strategy to solve the complex problems in producing a product from scratch (from the raw ingredients to a cooked meal) can be taught without reading any instructions or following any recipes. By the same token, the strategy for solving mathematical problems can also be learned intuitively by demonstrating the procedures for using familiar methodologies and applying them to new situations. When a student is able to apply the learned concepts and principles to a new situation, he/she has achieved a higher competence level of learning in Bloom's taxonomy [13-17] that is above and beyond simple comprehension/understanding.

3. An Intuitive Approach to Demonstrate the Concept of Infinity, Infinite Series, and Finite Numbers

The proof of equality of infinite series of numbers was well described by Euler [18] in 1840. One of the most common proofs for the equality of repeating decimal, such as 0.999… is equal to 1 (i.e., 0.999… = 1), is the use of infinite series analysis [19]. However, a more intuitive approach to demonstrate the equality of this infinite series is the use of long division. This provides a useful tool for elementary students to understand the equality of an infinite series of numbers and a finite number.

One of the simple proofs of the equality of 0.999… and 1 (0.999… = 1) can be provided by expanding repeated decimals as an infinite series in a Taylor series as shown below:

$$\begin{aligned}
0.999... &= 0.9 + 0.09 + 0.009 + ... \\
&= 9*10^{-1} + 9*10^{-2} + 9*10^{-3} + ... \\
&= 9*\left(10^{-1} + 10^{-2} + 10^{-3} + ...\right) \\
&= \left(10-1\right)*\left(10^{-1} + 10^{-2} + 10^{-3} + ...\right) \\
&= \left(10^{0} + 10^{-1} + 10^{-2} + ...\right) \\
&\quad -\left(10^{-1} + 10^{-2} + 10^{-3} + ...\right) \\
&= 10^{0} \\
&= 1
\end{aligned} \tag{1}$$

This strategy for proving the equality (0.999… = 1) is rather intuitive and straightforward for most elementary school students.

For a more advanced high-school student, the same infinite series can be represented in terms of the limits of an infinite series (which requires more complex mathematical notation than the previous formulation):

$$\begin{aligned}
0.999... &= \lim_{n\to\infty}\left[9*\sum_{k=-1}^{n}\left(10^{-k}\right)\right] \\
&= \lim_{n\to\infty}\left[(10-1)*\sum_{k=-1}^{n}\left(10^{-k}\right)\right] \\
&= \lim_{n\to\infty}\left[\sum_{k=0}^{n}\left(10^{-k}\right) - \sum_{k=-1}^{n}\left(10^{-k}\right)\right] \\
&= 10^{0} \\
&= 1
\end{aligned} \tag{2}$$

However, there is a much more intuitive representation of the proof using an alternate, non-traditional/novel strategy in a long division for dividing 1 by 1.

4. Methods

The traditional method of long division dividing 1 by 1 ($\frac{1}{1} = 1$) is given by:

$$\begin{array}{r} 1 \\ 1\overline{)1} \\ \underline{1} \\ 0 \end{array} \tag{3}$$

Most elementary school students are familiar with this long division method.

However, the novel approach to do long division is not to use the quotient of 1, even though it may seem intuitively obvious to do so. The alternative strategy is to divide using the quotient of 0, which carries to the next decimal to use the quotient of 0.9 instead. Thus, rather than dividing the fraction using the quotient of 1, the quotient of 0.9 can be used instead, which gives a remainder of 0.1:

$$\begin{array}{r} 0.9 \\ 1\overline{)1.0} \\ \underline{9} \\ 1 \end{array} \tag{4}$$

Continue the long division for the next decimal digit gives:

$$\begin{array}{r} 0.99 \\ 1\overline{)1.00} \\ \underline{90} \\ 10 \\ \underline{9} \\ 1 \end{array} \tag{5}$$

Repeat the long division for the next decimal digit gives:

$$\begin{array}{r} 0.999 \\ 1\overline{)1.000} \\ \underline{90} \\ 10 \\ \underline{90} \\ 10 \\ \underline{9} \\ 1 \end{array} \tag{6}$$

The above long division can continue to infinity, which is also equal to the traditional long division of $\frac{1}{1}=1$:

$$\begin{array}{r} 0.999 \\ 1\overline{)1.000} \\ \underline{90} \\ 10 \\ \underline{90} \\ 10 \\ \underline{9} \\ 1 \end{array} = \begin{array}{r} 1 \\ 1\overline{)1} \\ \underline{1} \\ 0 \end{array} \tag{7}$$

Thus, it essentially proves that 0.999… is indeed equal to 1 at an infinite place of decimal, i.e., 0.999… = 1.

5. Results

The above non-traditional method of long division can be used to prove that 0.999… is equal to 1 (0.999… = 1) at an infinite place of decimal, even for elementary school students to understand an infinite series intuitively. The pedagogical strategy is to use an existing methodology that is familiar to an average student and apply it to a new situation in this mathematical proof.

The novelty of this alternate long division method is unfamiliar to most students because they were taught to divide a number based on an existing method of 1/1 = 1. Yet, there is no logical reason why the method in a long division has to use a quotient that matches the expected value (or expected result). This requires an alternate approach to solve a mathematical problem in a different way, which requires "thinking outside of the box." By breaking the traditional mathematical rule, applying a familiar method unconventionally, the complex concept of infinity can be demonstrated to exist with the asymptotic value of an infinite series that approaches to 1 at infinity, which an elementary school student can understand intuitively.

6. Discussions

The use of a non-traditional method of long division to prove the equality (0.999… = 1) can be understood by most elementary school students intuitively by using an alternate quotient other than 1. It is easy to understand without the use of infinite series of numbers or the use of the convergence theorem for geometric series. It is also important to demonstrate to students that the quotient of a long division does not always need to be exact. It is the novel use of a different quotient in a long division to divide a number without violating any mathematical principle that provides the proof of equality between an infinite series (0.999… = 0.9 + 0.09 + 0.009 + …, in this example) and a finite number (1, in this example). The concept of the asymptotic value merging to an exact finite number at infinity can be easily understood intuitively even for elementary school students. This example of an application of familiar problem-solving strategy to solve a new problem using an unconventional approach can be taught to students to generalize creatively to solve other novel problems.

7. Conclusion

The proof of the equivalence of an infinite series to a finite number can be established and derived intuitively using an alternate non-traditional method for school-aged students to comprehend the simplicity of mathematical proofs without relying on other advanced mathematical concepts of infinite series for abstraction of infinity in a concrete way using a familiar long-division method.

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