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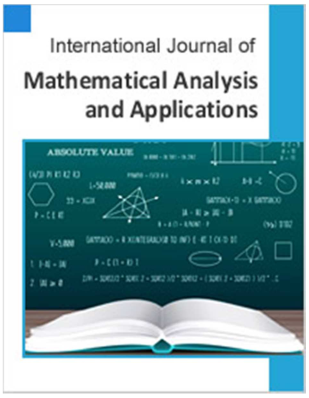

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Common Fixed Point Theorems in Metric Spaces

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Abstract
In this paper, we rewrite the notion of CLR-property and occasionally coincidentally idempotent for hybrid pairs of L-fuzzy and crisp mappings in metric spaces, we also establish common fixed point theorems for L-fuzzy and crisp mappings in metric spaces. Further, we give an example to support our results. Finally, we prove our results for sequence of L-fuzzy mappings in metric spaces. Both of integral type and usual contractive condition may be used. These theorems extend, generalize and improve corresponding previous results.

1. Introduction

In 1965, Zadeh [11] introduced the notion of fuzzy set which generalizes classical set. As a generalization of Nadler's contraction principle for fuzzy mappings, in 1981, Heilpern [5] defined the concept of fuzzy mapping, he also proved fixed point theorem for fuzzy contractive mappings in metric linear spaces. Further, in 1967, Goguen [6] introduced the notion of ℓ-Fuzzy sets as a generalization of fuzzy sets [11].

Definition 1.1 *[6]* An ℓ-fuzzy set A on a nonempty set X is a function $A: X \to L$, where ℓ is complete distributive lattice with 1_ℓ and 0_ℓ. In ℓ-fuzzy sets if $L=[0,1]$, then we obtained fuzzy sets.

In 2014, Rashid et al. [10] defined the concept of ℓ-*fuzzy* mapping. Further, he established a common ℓ-fuzzy fixed point theorem in complete metric spaces.

Definition 1.2 [10] Let X and Y be two arbitrary nonempty sets, $\Im_\ell(Y)$ denote the collection of all ℓ-fuzzy sets in Y. A mapping F is called ℓ-fuzzy mapping if F is a mapping from X into $\Im_\ell(Y)$. An ℓ-fuzzy mapping F is an ℓ-fuzzy subset on $X \times Y$ with membership function $F(x)(y)$. The function $F(x)(y)$ is the grade of membership of y in $F(x)$.

Definition 1.3 [10] Let F, G are ℓ-fuzzy mappings from an arbitrary nonempty set X into $\Im_\ell(X)$. A point $z \in X$ is called an ℓ-fuzzy fixed point of F if $z \in \{Fz\}_{\alpha_\ell}$, where $\alpha_\ell \in L \setminus \{0_\ell\}$. The point $z \in X$ is called a common ℓ-fuzzy fixed point of F and G if $z \in \{Fz\}_\alpha \cap \{Gz\}_{\alpha_\ell}$.

On the other hand, in 2014, Ahmed and Nafadi [2] introduced the notion of CLR property for single-valued and multi-valued mappings in metric spaces.

Definition 1.4[2] Let (X,d) be a metric space, $f: X \to X$ and $F: X \to CL(X)$.

Then the hybrid pair (f,F) is said to have (CLR) property if there exist a sequence $\{x_n\}$ in X and $A\in CL(X)$ such that $\lim_{n\to\infty} fx_n = u\in A = \lim_{n\to\infty} Fx_n$, with $u = fv$, for some $u,v\in X$.

Some common fixed point results with CLR property have emerged. One may rewrite this notion for a pair of ℓ-Fuzzy and crisp mappings in metric spaces. In this case, we may prove our results without completeness of the involved space.

In this paper, we prove common fixed point theorems using implicit relation with integral type and usual contractive condition. We defined an integral type contraction and obtained a generalization of Banach's contraction principle. So, integral contractive type mappings is a generalization of contractive mappings. Further, we rewrite the notion of CLR property, weakly commuting and occasionally coincidentally idempotent mappings. Our results extend, generalize and improve corresponding previous results.

Let (X,d) be a metric space, $CL(X)$ denote the collection of all closed subsets of X and $\Re^+$ the set of non negative real numbers, define the metrics $d: X\times CL(X)\to\Re^+$, $H: CL(X)\times CL(X)\to\Re^+$ as $d(x,A)=\inf_{y\in A} d(x,y)$ and $H(A,B)=\max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$.

2. Preliminaries

Definition 2.1 [6] A partially ordered set $\ell=(L,\le_L)$ is called:

i. a lattice, if $a\vee b\in L$ and $a\wedge b\in L$ for any $a,b\in L$,
ii. complete lattice, if $\vee A\in L$ and $\wedge A\in L$ for any $A\subseteq L$,
iii. distributive if $a\vee(b\wedge c)=(a\vee b)\wedge(a\vee c)$, $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$ for any $a,b,c\in L$.

Definition 2.2 [10] The α_ℓ-level set of ℓ-fuzzy set A is denoted by A_{α_ℓ} and is defined as follows: $A_{\alpha_\ell}=\{x:\alpha_\ell \prec_\ell A\}$ if $\alpha_\ell\in L\setminus\{0_\ell\}$, $A_{0_\ell}=\overline{\{x:0_\ell\prec_\ell A\}}$. Here $\overline{B}$ denotes the closure of the set B.

Definition 2.3 [7] A map $f: Y\subseteq X\to X$ is said to be coincidentally idempotent w.r.t. a mapping $F: Y\to CL(X)$ if $ffx = fx$ for all $x\in C(f,F)$, where $C(f,F)$ denote the set of coincidence points of f and F.

Definition 2.4[9] A map $f: Y\subseteq X\to X$ is said to be occasionally coincidentally idempotent w.r.t. a mapping $F: Y\to CL(X)$ if $ffx = fx$ for some $x\in C(f,F)$.

Remark 2.1 [9] Coincidentally idempotent pairs of mappings are occasionally coincidentally idempotent, but the converse is not necessarily true.

Definition 2.5 [1] Let (X,d) be a metric space, $Y\subseteq X$ and $F: Y\to CL(X)$. A map $f: Y\to X$ is said to be F-weakly commuting at $x\in Y$ if $ffx\in Ffx$ provided that $fx\in Y$ for all $x\in Y$.

Let Φ be the family of all continuous mappings $\varphi:[0,\infty)^6\to[0,\infty)$, which are non-increasing in the 3^{rd}, 4^{th}, 5^{th}, 6^{th} and non decreasing in the 1^{st} coordinate variable and satisfying the following properties:

$$(\varphi_1)\ \varphi(u,0,0,u,u,0)\le 0 \text{ or } \varphi(u,0,u,0,0,u)\le 0,$$

$$(\varphi_2)\ \int_0^{\varphi(u,0,0,u,u,0)}\phi(s)ds\le 0 \text{ or } \int_0^{\varphi(u,0,u,0,0,u)}\phi(s)ds\le 0,$$

$$(\varphi_3)\ \int_0^{\varphi(\int_0^u\psi(p)dp,0,\int_0^u\psi(p)dp,0,0,\int_0^u\psi(p)dp)}\phi(s)ds\le 0 \text{ or}$$

$$\int_0^{\varphi(\int_0^u\psi(p)dp,0,0,\int_0^u\psi(p)dp,\int_0^u\psi(p)dp,0)}\phi(s)ds\le 0.$$

$\forall u\ge 0$ implies $u=0$, where $\phi,\psi:[0,\infty)\to[0,\infty)$ is a summable non negative lebesgue integrable functions such that for each $\varepsilon>0$, $\int_0^\varepsilon\phi(s)ds>0$ and $\int_0^\varepsilon\psi(p)dp>0$. Note that if $\psi(p)=1$, then $(\varphi_3)\Rightarrow(\varphi_2)$, if $\phi(s)=1$, then $(\varphi_2)\Rightarrow(\varphi_1)$ and if $\phi(s)=\psi(p)=1$, then $(\varphi_3)\Rightarrow(\varphi_1)$.

3. Main Results

Let (X,d) be a metric space, $Y\subseteq X$, $f,g: Y\to X$ and F,G are two ℓ*-fuzzy* mappings from Y into $\Im_\ell(X)$ such that for each $x\in Y$, $\alpha_\ell\in L\setminus\{0_\ell\}$, $\{Fx\}_{\alpha_\ell}$ and $\{Gx\}_{\alpha_\ell}$ are nonempty closed subsets of X. Suppose that $\zeta:[0,\infty)\to[0,\infty)$ is a summable non negative lebesgue integrable functions such that for each $\varepsilon\ge 0$, $\int_0^\varepsilon\zeta(e)de\ge 0$.

First, we rewrite some definitions for hybrid pairs of ℓ*-fuzzy* and crisp mappings in metric spaces.

Definition 3.1 *The hybrid pairs* (f,F) *and* (g,G) *are said to have* (CLR) *property if there exist two sequences* $\{x_n\},\{y_n\}$ *in* Y *and* $A,B\in CL(X)$ *such that* $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gy_n = u, \lim_{n\to\infty}\{Fx_n\}_{\alpha_\ell}=A, \lim_{n\to\infty}\{Gy_n\}_{\alpha_\ell}=B$, with $u=fv=gw, u\in A\cap B$, for som $u,v,w\in Y$.

Definition 3.2 A map f is said to be occasionally coincidentally idempotent w.r.t. F if f is idempotent at some coincidence points of (f,F), i.e., $ffx = fx$ for some $x\in C(f,F)$.

Definition 3.3 A map $f: Y\to X$ is said to be F-weakly commuting at $x\in Y$ if $ffx\in\{Ffx\}_{\alpha_\ell}$.

Now, we state and prove the following result.

Theorem 3.1 Let (X,d) be a metric space, $Y\subseteq X$, $f,g: Y\to X$ and F,G are two ℓ-fuzzy mappings from

Y into $\Im_\ell(X)$ such that for each $x \in Y$, $\alpha_\ell \in L \setminus \{0_\ell\}$, $\{Fx\}_{\alpha_\ell}$ and $\{Gx\}_{\alpha_\ell}$ are nonempty closed subsets of X which satisfy the following condition, for all $x, y \in Y$ and $E \geq 0$ there exist $\varphi \in \Phi$ such that

$$\int_0^{\varphi(Q)} \phi(s)ds + E\int_0^{\min(Q)} \zeta(e)de \leq 0. \quad (1)$$

where

$$Q = \left(H(\{Fx\}_{\alpha_\ell}, \{Gy\}_{\alpha_\ell}), d(fx, gy), d(fx, \{Fx\}_{\alpha_\ell}), d(gy, \{Gy\}_{\alpha_\ell}), d(fx, \{Gy\}_{\alpha_\ell}), d(gy, \{Fx\}_{\alpha_\ell}) \right)$$

If (f, F) and (g, G) are satisfy (CLR) property, weakly commuting and occasionallycoincidentally idempotent, then f, g, F, G have a common fixed point.

Proof. Since (f, F) and (g, G) are satisfy (CLR) property, there exist two sequences $\{x_n\}, \{y_n\}$ in Y and $A, B \in CL(X)$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u, \lim_{n\to\infty} \{Fx_n\}_{\alpha_\ell} = A, \lim_{n\to\infty} \{Gy_n\}_{\alpha_\ell} = B$, with $u = fv = gw$ and $u \in A \cap B$, for some $u, v, w \in Y$. We show that $A = B$. As

$$\int_0^{\varphi(Q)} \phi(s)ds + E\int_0^{\min(Q)} \zeta(e)de \leq 0.$$

where

$$Q = \left(H(\{Fx_n\}_{\alpha_\ell}, \{Gy_n\}_{\alpha_\ell}), d(fx_n, gy_n), d(fx_n, \{Fx_n\}_{\alpha_\ell}), d(gy_n, \{Gy_n\}_{\alpha_\ell}), d(fx_n, \{Gy_n\}_{\alpha_\ell}), d(gy_n, \{Fx_n\}_{\alpha_\ell}) \right)$$

As $n \to \infty$, we have $\int_0^{\varphi(H(A,B),0,d(fv,A),d(fv,B),d(fv,B),d(fv,A))} \phi(s)ds \leq 0.$ Which give

$$\int_0^{\varphi(H(A,B),0,0,H(A,B),H(A,B),0)} \phi(s)ds \leq \int_0^{\varphi(H(A,B),0,d(fv,A),d(fv,B),d(fv,B),d(fv,A))} \phi(s)ds \leq 0.$$

From (φ_2), we have $H(A, B) = 0$, so that $A = B$. Now, $fv \in \{Fv\}_{\alpha_\ell}$, to prove this, since $fv \in A$, we show that $A = \{Fv\}_{\alpha_\mathcal{L}}$. As $\int_0^{\varphi(Q)} \phi(s)ds + E\int_0^{\min(Q)} \zeta(e)de \leq 0.$ where

$$Q = \left(H(\{Fv\}_{\alpha_\ell}, \{Gy_n\}_{\alpha_\ell}), d(fv, gy_n), d(fv, \{Fv\}_{\alpha\ell}), d(gy_n, \{Gy_n\}_{\alpha_\ell}), d(fv, \{Gy_n\}_{\alpha_\ell}), d(gy_n, \{Fv\}_{\alpha_\ell}) \right).$$

When $n \to \infty$, we have

$$\int_0^{\varphi(H(\{Fv\}_{\alpha_\ell},A),0,d(fv,\{Fv\}_{\alpha_\ell}),d(fv,A),d(fv,A),d(fv,\{Fv\}_{\alpha_\ell}))} \phi(s)ds \leq 0.$$

So that

$$\int_0^{\varphi(H(\{Fv\}_{\alpha_\ell},A),0,d(A,\{Fv\}_{\alpha_\ell}),0,0,d(A,\{Fv\}_{\alpha_\ell}))} \phi(s)ds$$

$$\leq \int_0^{\varphi(H(\{Fv\}_{\alpha_\ell},A),0,d(fv,\{Fv\}_{\alpha_\ell}),d(fv,A),d(fv,A),d(fv,\{Fv\}_{\alpha_\ell}))} \phi(s)ds$$

$$\leq 0.$$

By (φ_2), then $H(A, \{Fv\}_{\alpha_\ell}) = 0$, then we have $A = \{Fv\}_{\alpha_\ell}$, and $fv \in \{Fv\}_{\alpha_\ell}$. By a similar way one canfind $gw \in \{Gw\}_{\alpha_\ell}$ for $w \in Y$. Further, since $ffv = fv$ and $ffv \in \{Ffv\}_{\alpha_\ell}$ so that we have $u = fu \in \{Fu\}_{\alpha_\ell}$.

Also, since $ggw = gw$ and $ggw \in \{Ggw\}_{\alpha_\ell}$ implies $u = gu \in \{Gu\}_{\alpha_\ell}$. Then f, g, F, G have a commonfixed point.

Example 3.1 *Let* (X, d) *be a metric space,* $Y \subseteq X = [0,1]$. *Define the maps* f, g, F, G *on* Y *as*

$$fx = \frac{2x}{3}, \quad gx = \frac{x}{4},$$

$$(Fx)(y) = \begin{cases} 0, & \text{if } 0 \leq y \leq \frac{1}{5}, \\ \frac{1}{3}, & \text{if } \frac{1}{5} < y < \frac{2x}{3}, \\ \frac{2}{3}, & \text{if } \frac{2x}{3} \leq y \leq 1. \end{cases}$$

and

$$(Gx)(y) = \begin{cases} 0, & \text{if } 0 \leq y \leq \frac{1}{4}, \\ \frac{1}{6}, & \text{if } \frac{1}{4} < y < \frac{x}{4}, \\ \frac{1}{4}, & \text{if } \frac{x}{4} \leq y \leq 1. \end{cases}$$

for all $x, y \in Y$. Define two sequences $\{x_n\}$ and $\{y_n\}$ in Y such that $\{x_n\} = \{\frac{1}{n}\}, \{y_n\} = \{\frac{1}{2n}\}, n \in \mathbb{N}$. Since $\{Fx\}_{\frac{2}{3}} = [\frac{2x}{3}, 1]$, then $\lim_{n\to\infty} \{Fx_n\}_{\frac{2}{3}} = \lim_{n\to\infty} [\frac{2}{3n}, 1] = [0,1]$. Similarly, $\{Gx\}_{\frac{1}{4}} = [\frac{x}{4}, 1]$, then $\lim_{n\to\infty} \{Gy_n\}_{\frac{1}{4}} = \lim_{n\to\infty} [\frac{1}{8n}, 1] = [0,1]$.

Now, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = 0 \in [0,1] = \lim_{n\to\infty} \{Fx_n\}_{\frac{2}{3}} = \lim_{n\to\infty} \{Gy_n\}_{\frac{1}{4}}$, that is, the hybrid pairs (f, F) and (g, G) are satisfy the property (CLR). Further, let $\varphi(t_1, \ldots, t_6) = t_1 - \min\{t_2, t_3, t_4, t_5\}$ and $\int_0^{\varphi(Q)} \phi(s)ds + E\int_0^{\min(Q)} \zeta(e)de \le 0$, where

$$Q = \left(H(\{Fx_n\}_{\alpha_\ell}, \{Gy_n\}_{\alpha_\ell}), d(fx_n, gy_n), d(fx_n, \{Fx_n\}_{\alpha_\ell}), d(gy_n, \{Gy_n\}_{\alpha_\ell}), d(fx_n, \{Gy_n\}_{\alpha_\ell}), d(gy_n, \{Fx_n\}_{\alpha_\ell}) \right)$$

As $n \to \infty$

$$\int_0^{\varphi\begin{pmatrix} H(\{Fx_n\}_{\alpha_\ell}, \{Gy_n\}_{\alpha_\ell}), d(fx_n, gy_n), d(fx_n, \{Fx_n\}_{\alpha_\ell}), \\ d(gy_n, \{Gy_n\}_{\alpha_\ell}), d(fx_n, \{Gy_n\}_{\alpha_\ell}), d(gy_n, \{Fx_n\}_{\alpha_\ell}) \end{pmatrix}} \phi(s)ds \le 0.$$

Now, f, g, F, G satisfy all conditions of theorem 3.1 and $f0 = g0 = 0 \in [0,1] = \{F0\}_{\frac{2}{3}} = \{G0\}_{\frac{1}{4}}$ is a common fixed point.

If $f = g$ and $F = G$ in theorem 3.1 we have the following result.

Corollary 3.1 Let (X, d) be a metric space, $Y \subseteq X$, $f : Y \to X$ and F be an ℓ-fuzzy mapping from Y into $\Im_\ell(X)$ such that for each $x \in Y$, $\alpha_\ell \in L \setminus \{0_\ell\}$, $\{Fx\}_{\alpha_\ell}$ is nonempty closed subset of X and satisfies condition (1) where

$$Q = \left(H(\{Fx\}_{\alpha_\ell}, \{Fy\}_{\alpha_\ell}), d(fx, fy), d(fx, \{Fx\}_{\alpha_\ell}), d(fy, \{Fy\}_{\alpha_\ell}), d(fx, \{Fy\}_{\alpha_\ell}), d(fy, \{Fx\}_{\alpha_\ell}) \right).$$

If (f, F) satisfies (CLR) property, weakly commuting and occasionally coincidentallyidempotent, then f and F have a common fixed point.

If $L = I = [0,1]$ in theorem 3.1 we have the following Corollary.

Corollary 3.2 Let (X, d) be a metric space, $Y \subseteq X$, $f, g : Y \to X$ and F, G are two fuzzy mappings from Y into $\Im(X)$ (the collection of all fuzzy sets of X) such that for each $x \in Y$, $\alpha \in (0,1]$, $\{Fx\}_\alpha$ and $\{Gx\}_\alpha$ are nonempty closed subsets of X and satisfies condition (1) where

$$Q = \left(H(\{Fx\}_\alpha, \{Gy\}_\alpha), d(fx, gy), d(fx, \{Fx\}_\alpha), d(gy, \{Gy\}_\alpha), d(fx, \{Gy\}_\alpha), d(gy, \{Fx\}_\alpha) \right).$$

If (f, F) and (g, G) are satisfy (CLR) property, weakly commuting and occasionally coincidentally idempotent, then f, g, F, G have a common fixed point.

If $f = g$ and $F = G$ in corollary 3.2 we have the following result.

Corollary 3.3 Let (X, d) be a metric space, $Y \subseteq X$, $f : Y \to X$ *and* F *is a fuzzy mappings from* Y *into* $\Im(X)$ *such that for each* $x \in Y$, $\alpha \in (0,1]$, $\{Fx\}_\alpha$ *is a nonempty closed subset of* X *and satisfies condition* (3.1) *where*

$$Q = \left(H(\{Fx\}_\alpha, \{Fy\}_\alpha), d(fx, fy), d(fx, \{Fx\}_\alpha), d(fy, \{Fy\}_\alpha), d(fx, \{Fy\}_\alpha), d(fy, \{Fx\}_\alpha) \right).$$

If (f, F) satisfies (CLR) property, weakly commuting and occasionally coincidentally idempotent, then f and F have a common fixed point.

Theorem 3.2 *Let* (X, d) *be a metric space,* $Y \subseteq X$, $f, g : Y \to X$, $\{F_n\}$ *be a sequence of* ℓ*-fuzzy mapping from* Y *into* $\Im_\ell(X)$ *such that for each* $x \in Y$, $\alpha_\ell \in L \setminus \{0_\ell\}$, $\{Fx\}_{\alpha_\ell}$ *is nonempty closed subset of* X *and satisfies condition* (1) *where* $k = 2n - 1$, $l = 2n$, $n \in \mathbb{N}$ *and*

$$Q = \left(H(\{F_k x\}_{\alpha_\ell}, \{F_l y\}_{\alpha_\ell}), d(fx, gy), d(fx, \{F_k x\}_{\alpha_\ell}), d(gy, \{F_l y\}_{\alpha_\ell}), d(fx, \{F_l y\}_{\alpha_\ell}), d(gy, \{F_k x\}_{\alpha_\ell}) \right).$$

If the pairs (f, F_k) and (g, F_l) are satisfy the property (CLR), weakly commuting and occasionally coincidentally idempotent, then (f,F_k) and (g,F_l) have a common fixed point.

Proof. Since (f,F_k) and (g,F_l) are satisfy the property (CLR), there exist two sequences $\{x_{kn}\},\{y_{kn}\}$ in Y and $A_k,B_k \in CL(X)$ such that

$$\lim_{n\to\infty} fx_{kn} = \lim_{n\to\infty} gy_{kn} = u_k, \lim_{n\to\infty}\{F_k x_{kn}\}_{\alpha_\ell} = A_k, \lim_{n\to\infty}\{F_l y_{kn}\}_{\alpha_\ell} = B_k.$$

with $u_k = fv_k = gw_l$, $u_k \in A_k \cap B_k$, for some $u_k, v_k, w_k \in Y$. Now, we show that $A_k = B_k$. As $\int_0^{\varphi(Q)}\phi(s)ds + E\int_0^{\min(Q)}\zeta(e)de \le 0.$ where

$$Q = \begin{pmatrix} H(\{F_k x_{kn}\}_{\alpha_\ell},\{F_l y_{kn}\}_{\alpha_\ell}), d(fx_{kn}, gy_{kn}), d(fx_{kn},\{F_k x_{kn}\}_{\alpha_\ell}), \\ d(gy_{kn},\{F_l y_{kn}\}_{\alpha_\ell}), d(fx_{kn},\{F_l y_{kn}\}_{\alpha_\ell}), d(gy_{kn},\{F_k x_{kn}\}_{\alpha_\ell}) \end{pmatrix}.$$

As $n\to\infty$, we find that $\int_0^{\varphi(H(A_k,B_k,t),0,0,d(fv_k,B_k),d(fv_k,B_k),0)}\phi(s)ds \le 0$

So that

$$\int_0^{\varphi(H(A_k,B_k),0,0,d(A_k,B_k),d(A_k,B_k),0)}\phi(s)ds \le \int_0^{\varphi(H(A_k,B_k),0,d(fv_k,A_k),d(fv_k,B_k),d(fv_k,B_k),d(fv_k,A_k))}\phi(s)ds \le 0.$$

By (φ_2), we have $H(A_k,B_k)=0$, i.e., $A_k = B_k$. As $u_k \in A_k$, we show that $\{F_k v_k\}_{\alpha_{\mathcal{L}}} = A_k$. As $\int_0^{\varphi(Q)}\phi(s)ds + E\int_0^{\min(Q)}\zeta(e)de \le 0.$ where

$$Q = \begin{pmatrix} H(\{F_k v_k\}_{\alpha_\ell},\{F_l y_{kn}\}_{\alpha_\ell}), d(fv_k, gy_{kn}), d(fv_k,\{F_k v_k\}_{\alpha_\ell}), \\ d(gy_{kn},\{F_l y_{kn}\}_{\alpha_\ell}), d(fv_k,\{F_l y_{kn}\}_{\alpha_\ell}), d(gy_{kn},\{F_k v_k\}_{\alpha_\ell}) \end{pmatrix}.$$

which on making $n\to\infty$ reduces to

$$\int_0^{\varphi(H(\{F_k v_k\}_{\alpha_\ell},A_k),0,0,d(\{F_k v_k\}_{\alpha_\ell},A_k),d(\{F_k v_k\}_{\alpha_\ell},A_k),0)}\phi(s)ds \le 0,$$

so that $\{F_k v_k\}_{\alpha_\ell} = A_k$. The remaining parts are easy to prove. This concludes the proof.

If $f=g$ in theorem 3.2 we have

Corollary 3.4 *Let (X,d) be a metric space, $Y\subseteq X$, $f:Y\to X$, $\{F_n\}$ be a sequence of ℓ-fuzzy mappings from Y into $\Im_\ell(X)$ such that for each $x\in Y$ $\alpha_\ell \in L\setminus\{0_\ell\}$, $\{Fx\}_{\alpha_\ell}$ isnonempty closed subset of X and satisfies condition* (1) *where $k=2n-1$, $l=2n$, $n\in\mathbb{N}$ and*

$$Q = \left(H(\{F_k x\}_{\alpha_\ell},\{F_l y\}_{\alpha_\ell}), d(fx,fy), d(fx,\{F_k x\}_{\alpha_\ell}), d(fy,\{F_l y\}_{\alpha_\ell}), d(fx,\{F_l y\}_{\alpha_\ell}), d(fy,\{F_k x\}_{\alpha_\ell})\right).$$

If the pairs (f,F_n) satisfies the property (CLR), weakly commuting and occasionally coincidentally idempotent, then (f,F_n) have a common fixed point.

If $L=I=[0,1]$ in theorem 3.2 we have the following Corollary.

Corollary 3.5 *Let (X,d) be a metric space, $Y\subseteq X$, $f,g:Y\to X$, $\{F_n\}$ be a sequence of fuzzy mappings from Y into $\Im(X)$ such that for each $x\in Y$, $\alpha\in(0,1]$ and $\{F_n x\}_\alpha$ are nonempty closed subsets of X and satisfies condition* (1) *where $k=2n-1$, $l=2n$, $n\in\mathbb{N}$ and*

$$Q = \left(H(\{F_k x\}_\alpha,\{F_l y\}_\alpha), d(fx,gy), d(fx,\{F_k x\}_\alpha), d(gy,\{F_l y\}_\alpha), d(fx,\{F_l y\}_\alpha), d(gy,\{F_k x\}_\alpha)\right).$$

If the pairs (f,F_k) and (g,F_l) are satisfy the property (CLR), weakly commuting and occasionally coincidentally idempotent, then (f,F_k) and (g,F_l) have a common fixed point.

If $f = g$ in corollary 3.5 we have

Corollary 3.6 *Let* (X,d) *be a metric space,* $Y \subseteq X$ *,* $f : Y \to X$ *,* $\{F_n\}$ *be a sequence of fuzzy mappings from* Y *into* $\Im(X)$ *such that for each* $x \in Y$ *,* $\alpha \in (0,1]$ *and* $\{F_n x\}_\alpha$ *are nonempty closed subsets of* X *and satisfies condition* (1) *where* $k = 2n-1$ *,* $l = 2n$ *,* $n \in \mathbb{N}$ *and*

$$Q = \big(H(\{F_k x\}_\alpha, \{F_l y\}_\alpha), d(fx, fy), d(fx, \{F_k x\}_\alpha), d(fy, \{F_l y\}_\alpha), d(fx, \{F_l y\}_\alpha), d(fy, \{F_k x\}_\alpha)\big).$$

If the pairs (f, F_n) are satisfy the property (CLR), weakly commuting and occasionally coincidentally idempotent, then (f, F_n) have a common fixed point.

4. FromIntegral Type to Usual Contractive Condition

In theorem 3.1, condition (1), one may put $\zeta(e) = 1$ or $\phi(s) = 1$ or both, which give the following respectively:

$$\int_0^{\varphi(Q)} \phi(s)ds + E\min(Q) \le 0. \tag{2}$$

$$\varphi(Q) + E\int_0^{\min(Q)} \zeta(e)de \le 0. \tag{3}$$

$$\varphi(Q) + E\min(Q) \le 0. \tag{4}$$

Now, we will replace the integral contractive condition by another, then we can obtained the following:

Corollary 4.1 *Let* (X,d) *be a metric space,* $Y \subseteq X$ *,* $f, g : Y \to X$ *and* F, G *are two* ℓ*-fuzzy mappings from* Y *into* $\Im_\ell(X)$ *such that for each* $x \in Y$ *,* $\alpha_\ell \in L \setminus \{0_\ell\}$ *,* $\{Fx\}_{\alpha_\ell}$ and $\{Gx\}_{\alpha_\ell}$ *are nonempty closed subsets of* X *, suppose that for all* $x, y \in Y$ *and* $E \ge 0$ *there exist* $\varphi \in \Phi$ *such that one of (*(2), (3), (4)*) is satisfied, where*

$$Q = \Big(H(\{Fx\}_{\alpha_\ell}, \{Gy\}_{\alpha_\ell}), d(fx, gy), d(fx, \{Fx\}_{\alpha_\ell}), d(gy, \{Gy\}_{\alpha_\ell}), d(fx, \{Gy\}_{\alpha_\ell}), d(gy, \{Fx\}_{\alpha_\ell})\Big).$$

If (f, F) and (g, G) are satisfy (CLR) property, weakly commuting and occasionally coincidentally idempotent, then f, g, F, G have a common fixed point.

Corollary 4.2 *Let* (X,d) *be a metric space,* $Y \subseteq X$ *,* $f : Y \to X$ *and* F *be an* ℓ*-fuzzy mappings from* Y *into* $\Im_\ell(X)$ *such that for each* $x \in Y$ *,* $\alpha_\ell \in L \setminus \{0_\ell\}$ *,* $\{Fx\}_{\alpha_\ell}$ is *nonempty closed subsets of* X *, suppose that for all* $x, y \in Y$ *and* $E \ge 0$ *there exist* $\varphi \in \Phi$ *such that one of (*(2), (3), (4)*)is satisfied, where*

$$Q = \Big(H(\{Fx\}_{\alpha_\ell}, \{Fy\}_{\alpha_\ell}), d(fx, fy), d(fx, \{Fx\}_{\alpha_\ell}), d(fy, \{Fy\}_{\alpha_\ell}), d(fx, \{Fy\}_{\alpha_\ell}), d(fy, \{Fx\}_{\alpha_\ell})\Big).$$

If (f, F) are satisfy (CLR) property, weakly commuting and occasionally coincidentally idempotent, then f and F have a common fixed point.

Corollary 4.3 *Let* (X,d) *be a metric space,* $Y \subseteq X$ *,* $f, g : Y \to X$ *and* F, G *are two fuzzy mappings from* Y *into* $\Im(X)$ *such that for each* $x \in Y$ *,* $\alpha \in (0,1]$*,* $\{Fx\}_\alpha$ *and* $\{Gx\}_\alpha$ *are nonempty closed subsets of* X *, suppose that for all* $x, y \in Y$ *and* $E \ge 0$ *there exist* $\varphi \in \Phi$ *such that one of (*(2), (3), (4)*) is satisfied, where*

$$Q = \big(H(\{Fx\}_\alpha, \{Gy\}_\alpha), d(fx, gy), d(fx, \{Fx\}_\alpha), d(gy, \{Gy\}_\alpha), d(fx, \{Gy\}_\alpha), d(gy, \{Fx\}_\alpha)\big).$$

If (f, F) and (g, G) are satisfy (CLR) property, weakly commuting and occasionally coincidentally idempotent, then f, g, F, G have a common fixed point.

Corollary 4.4 *Let* (X,d) *be a metric space,* $Y \subseteq X$ *,* $f : Y \to X$ *and* F *is a fuzzy mapping from* Y *into* $\Im(X)$ *such that for each* $x \in Y$ *,* $\alpha \in (0,1]$*,* $\{Fx\}_\alpha$ *is a nonempty closed subset of* X *, suppose that for all* $x, y \in Y$ *and* $E \ge 0$ *there exist* $\varphi \in \Phi$ *such that one of (*(2), (3), (4)*) is satisfied, where*

$$Q = \left(H(\{Fx\}_\alpha, \{Fy\}_\alpha), d(fx, fy, t), d(fx, \{Fx\}_\alpha), d(fy, \{Fy\}_\alpha), d(fx, \{Fy\}_\alpha), d(fy, \{Fx\}_\alpha)\right).$$

If (f, F) satisfies (CLR) property, weakly commuting and occasionally coincidentally idempotent, then f and F have a common fixed point.

Theorem 4.1 *Let* (X,d) *be a metric space,* $Y \subseteq X$ *,* $f, g: Y \to X$ *,* $\{F_n\}$ *be a sequence of* ℓ *-fuzzy mappings from* Y *into* $\Im_\ell(X)$ *such that for each* $x \in Y$ *,* $\alpha_\ell \in L\setminus\{0_\ell\}$ *,* $\{Fx\}_{\alpha_\ell}$ is *nonempty closed subset of* X *,* $n \in \mathbb{N}$ *, suppose that for all* $x, y \in Y$ *and* $E \ge 0$ *there exist* $\varphi \in \Phi$ *such that one of (*(2), (3), (4)*) is satisfied, where*

$$Q = \left(H(\{F_k x\}_{\alpha_\ell}, \{F_l y\}_{\alpha_\ell}), d(fx, gy), d(fx, \{F_k x\}_{\alpha_\ell}), d(gy, \{F_l y\}_{\alpha_\ell}), d(fx, \{F_l y\}_{\alpha_\ell}), d(gy, \{F_k x\}_{\alpha_\ell})\right).$$

If the pairs (f, F_k) and (g, F_l) satisfy the property (CLR), weakly commuting and occasionally coincidentally idempotent, then (f, F_k) and (g, F_l) have a fixed point.

Theorem 4.2 *Let* $f, g: Y \subseteq X \to X$ *be two mappings from a subset* Y *of a metric space* (X,d) *into* X *and* F, G *are two* ℓ *-fuzzy mappings from* Y *into* $\Im_\ell(X)$ *such that for each* $x \in Y$ *,* $\alpha_\ell \in L\setminus\{0_\ell\}$ *,* $\{Fx\}_{\alpha_\ell}$ and $\{Gx\}_{\alpha_\ell}$ *are nonempty closed subsets of X , suppose that for all* $x, y \in Y$ *and* $E \ge 0$ *there exist* $\varphi \in \Phi$ *such that one of (*(2), (3), (4)*) is satisfied, where*

$$Q = \varphi\begin{pmatrix} \int_0^{H(\{Fx\}_{\alpha_\ell}, \{Gy\}_{\alpha_\ell})} \psi(p)dp, \int_0^{d(fx,gy)} \psi(p)dp, \int_0^{d(fx,\{Fx\}_{\alpha_\ell})} \psi(p)dp, \\ \int_0^{d(gy,\{Gy\}_{\alpha_\ell})} \psi(p)dp, \int_0^{d(fx,\{Gy\}_{\alpha_\ell})} \psi(p)dp, \int_0^{d(gy,\{Fx\}_{\alpha_\ell})} \psi(p)dp \end{pmatrix}.$$

If (f, F) and (g, G) are satisfy the (CLR) property,weakly commuting and occasionally coincidentally idempotent, then (f, F) and (g, F) have a common fixed point.

Theorem 4.3 *Let* (X,d) *be a metric space,* $Y \subseteq X$ *,* $f, g: Y \to X$ *,* $\{F_n\}$ *be a sequence of* ℓ *-fuzzy mappings from* Y *into* $\Im_\ell(X)$ *such that for each* $x \in Y$ *,* $\alpha_\ell \in L\setminus\{0_\ell\}$ *,* $\{Fx\}_{\alpha_\ell}$ is *nonempty closed subset of* X *, suppose that for all* $x, y \in Y$ *and* $E \ge 0$ *there exist* $\varphi \in \Phi$ *such that one of (*(2), (3), (4)*) is satisfied, where* $k = 2n-1$*,* $l = 2n$*,* $n \in \mathbb{N}$ *and*

$$Q = \varphi\begin{pmatrix} \int_0^{H(\{F_k x\}_{\alpha_\ell}, \{F_l y\}_{\alpha_\ell})} \psi(p)dp, \int_0^{d(fx,gy)} \psi(p)dp, \int_0^{d(fx,\{F_k x\}_{\alpha_\ell})} \psi(p)dp, \\ \int_0^{d(gy,\{F_l y\}_{\alpha_\ell})} \psi(p)dp, \int_0^{d(fx,\{F_l y\}_{\alpha_\ell})} \psi(p)dp, \int_0^{d(gy,\{F_k x\}_{\alpha_\ell})} \psi(p)dp \end{pmatrix}.$$

If the pairs (f, F_k) and (g, F_l) are satisfy the property (CLR), weakly commuting and occasionally coincidentally idempotent, then (f, F_k) and (g, F_l) have a common fixed point.

Remark 4.1 Put $\psi(p) = 1$ in theorem 4.3 we have theorem 4.1, Put $\psi(p) = 1$ and $F_k = F, F_l = G$ in theorem 4.3 we have corollary 4.1, Put $F_k = F, F_l = G$ in theorem 4.3 we have theorem 4.2, Put $\psi(p) = \phi(s) = \zeta(e) = 1$ and $F_k = F, F_l = G$ in theorem 4.2 andtheorem 3.1, then the two theorems are the same.

5. Conclusions

In this paper, we study the existence of common fixed point theorems for ℓ -fuzzy and crisp mappings in metric spaces, we use an integral type of contractive condition with implicit relation of real valued values. In our results, we introduce the notion of CLR property, weakly commuting and occasionally coincidentally idempotent mappings for two hybrid pairs of ℓ -fuzzy and crisp mappings in metric spaces. Finally, we prove these results in usual contractive condition as a special case of our results.

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