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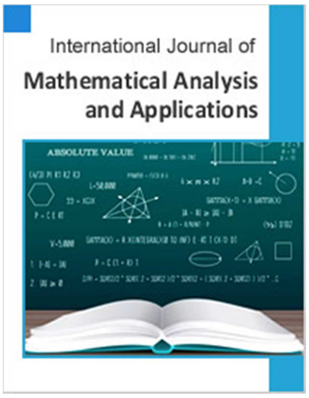

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Vorticity Transport in Magnetic Maxwellian Viscoelastic Fluid

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Abstract
Transport of vorticity in a magnetic Maxwellian viscoelastic fluid in the presence of suspended magnetic particles is considered here. Equations governing the transport of vorticity are obtained from the equations of magnetic fluid flow proposed by Wagh and Jawandhia in their 1996 study on the transport of vorticity in magnetic fluid. It follows from the analysis of these equations that the transport of solid vorticity is coupled with the transport of fluid vorticity. Further, we find that because of a thermo-kinetic process, fluid vorticity can exist in the absence of solid vorticity, but when fluid vorticity is zero then solid vorticity is necessarily zero. We also study a two-dimensional case.

1. Introduction

Magnetic fluids are suspensions of small magnetic particles in liquid carrier. Thus it is a two-phase system, consisting of solid and liquid phases. The net effect of the particles suspended in the fluid is extra dragging force acting on the system. This is due to relative velocity between the solid and fluid particles. Saffman (1962) proposed the equations of the flow of suspension of non-magnetic particles. These equations were modified by Wagh (1991) to describe the flow of a magnetic fluid, by including the magnetic body force $\mu_0 M \nabla H$. The transport of vorticity in a magnetic fluid has been studied by Wagh and Jawandhia (1996). Yan and Koplik (2009) have studied the transport and sedimentation of suspended particles in inertial pressure-driven flow.

With the growing importance of non-Newtonian fluids in modern technology and in various manufacturing and processing industries, considerable effort has been directed towards understanding their flow. Widely used theoretical models (models A and B, respectively) for certain classes of viscoelastic fluids have been proposed by Oldroyd (1958). The stability of a horizontal layer of Maxwell's viscoelastic fluid heated from below was investigated by Vest and Arpaci (1969). The thermal instability of Maxwellian viscoelastic fluid in the presence of a uniform rotation has been considered by Bhatia and Steiner (1972), where rotation is found to have a destabilizing effect. This is in contrast to the thermal instability of a Newtonian fluid where rotation has a stabilizing effect. In another study, Bhatia and Steiner (1973) have studied the problem of thermal instability of a viscoelastic fluid in hydromagnetics and have found that the magnetic field has the stabilizing influence on Maxwell fluid just as in the case of Newtonian fluid. Sharma and Sharma (1977) have considered the thermal instability of a rotating Maxwell fluid through porous medium and found that, for stationary convection, the rotation has stabilizing effect where as the permeability of the medium has both stabilizing as well as destabilizing effect, depending on the magnitude of rotation. Sharma and Kumar (1996) have considered the Hall effect on thermosolutal instability in a Maxwellian viscoelastic fluid in porous medium. The thermal instability of a rotating Maxwellian viscoelastic fluid permeated with suspended particles in porous medium has been studied by Kumar (1997). Kumar and Singh (2008) have studied the stability of superposed Maxwellian viscoelastic fluids through porous media in hydromagnetics. The problem of double-diffusive convection and cross-diffusion in a Maxwell fluid in a horizontal layer in porous media is re-examined using the modified Darcy-Brinkman model by Awad et al. (2010). In another study, Kumar (2013) has studied the thermal instability of Maxwellian heterogeneous viscoelastic fluid layer through porous medium. Kumar (2013) has also studied the slow, immiscible, Maxwellian viscoelastic liquid-liquid displacement in a permeable medium. Malashetty and Bharati (2016) have studied the onset of double diffusive convection in a binary Maxwell fluid saturated porous layer with cross diffusion effects using linear and weakly non-linear stability analyses. Centrifugal instability of a pulsed flow in a viscoelastic fluid confined in a Taylor-Couette system has been investigated by Riahi et al. (2016).

Motivation for studying the problem is likely to have some industrial and modern technology application on the problems of transport of vorticity in magnetic Maxwellian viscoelastic fluid-particle mixtures.

2. Basic Assumptions and Magnetic Body Force

The particles of magnetic material are much larger than the molecules of the carrier liquid. Accordingly, we consider the limit of a microscopic volume element in which the fluid can be assumed to be a continuous medium and the magnetic particles must be treated as discrete entities. If we consider a cell of magnetic fluid containing a larger number of magnetic particles, then we must consider the microrotation of the cell in addition to its translations as a point mass. We must therefore assign the average velocity $\vec{q}_d$ and the average angular velocity $\vec{\omega}$ to the cell. But we here neglect the effect of micro rotation as an approximation. We also assume the following:

(i) Most ferro fluids are relatively poor conductors. Hence, the free current density $\vec{J}$ is negligible and $\vec{J}\times\vec{B}$ is insignificant.

(ii) The magnetic field is curl free i.e. $\nabla\times\vec{H}=0$.

(iii) The liquid compressibility is unimportant in many situations. Hence, the contribution due to magnetic friction can be neglected. The remaining force of the magnetic field is called the magnetization force.

(iv) All time-dependent magnetization effects in the fluid (such as hysteresis) are negligible, and the magnetization $\vec{M}$ is collinear with $\vec{H}$.

From electromagnetic theory, the force per unit volume (in MKS units) on a piece of magnetized material of magnetization $\vec{M}$ (i.e. dipole moment per unit volume) in the field of magnetic intensity $\vec{H}$ is $\mu_0\left(\vec{M}.\nabla\right)\vec{H}$, where μ_0 is the free space permeability. Using assumption (iv), we get

$$\mu_0\left(\vec{M}.\nabla\right)\vec{H}=\frac{\mu_0 M}{H}\left(\vec{H}.\nabla\right)\vec{H},\ \text{ where } M=\left|\vec{M}\right| \text{ and } H=\left|\vec{H}\right| \quad (1)$$

$$\text{But } \left(\vec{H}.\nabla\right)\vec{H}=\frac{1}{2}\nabla\left(\vec{H}.\vec{H}\right)-\vec{H}\times\left(\nabla\times\vec{H}\right)=\frac{1}{2}\nabla\left(\vec{H}.\vec{H}\right)$$

$$\text{[by assumption (ii)]} \quad (2)$$

$$\text{Hence} \quad \mu_0\left(\vec{M}.\nabla\right)\vec{H}=\left(\frac{\mu_0 M}{H}\right)\frac{1}{2}\nabla\left(\vec{H}.\vec{H}\right)=\mu_0 M\nabla H.$$

Thus the magnetic body force assumes the form [Rosensweig (1997)].

$$f_m=\mu_0 M\nabla H. \quad (3)$$

3. Derivation of Equations Governing Transport of Vorticity in Magnetic Maxwellian Viscoelastic Fluid

Let $\Gamma_{ij}, \tau_{ij}, e_{ij}, \mu, \lambda, p, \delta_{ij}, q_i, x_i$ *and* d/dt denote respectively the total stress tensor, the shear stress tensor, the rate-of-strain tensor, the viscosity, the stress relaxation time, the isotropic pressure, the Kroneckor delta, the velocity vector, the position vector and the convective derivative. Then the Maxwellian viscoelastic fluid is described by the constitutive relations

$$\left.\begin{aligned} T_{ij}&=-p\delta_{ij}+\tau_{ij} \quad ,\\ \left(1+\lambda\frac{d}{dt}\right)\tau_{ij}&=2\mu e_{ij} \quad ,\\ e_{ij}&=\frac{1}{2}\left(\frac{\partial q_i}{\partial x_j}+\frac{\partial q_j}{\partial x_i}\right) \quad ,\end{aligned}\right\} \quad (4)$$

whereas $\lambda=0$ gives the Newtonian viscous fluid.

To describe the flow of a magnetic fluid with the body force $\mu_0 M\nabla H$ acting on the suspended magnetic particles taken into account, the Saffman equations for the flow of suspensions were modified by Wagh (1991). Now the equations expressing the flow of suspended magnetic particles and the flow of Maxwellian viscoelastic fluid in which magnetic particles are suspended are therefore written as

$$\begin{aligned} mN\left[\frac{\partial\vec{q}_d}{\partial t}+\left(\vec{q}_d.\nabla\right)\vec{q}_d\right]&=mN\vec{g}+\mu_0 M\nabla H\\ &\quad +KN\left(\vec{q}-\vec{q}_d\right),\end{aligned} \quad (5)$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\rho\left[\frac{\partial\vec{q}}{\partial t}+(\vec{q}.\nabla)\vec{q}\right]=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\nabla P+\rho\vec{g}+KN(\vec{q}_d-\vec{q})\right]+\mu\nabla^2\vec{q}, \tag{6}$$

where $P, \rho, \mu, \vec{q}(u, v, w), \vec{g}(0, 0, -g); \vec{q}_d(l, r, s), m, N(\bar{x}, t)$ denote, respectively, the pressure less the hydrostatic pressure, density, viscosity, velocity of the pure fluid, gravity force; velocity, mass and number density of the particles; $\bar{x}=(x, y, z), K=6\pi\mu\eta, \eta$ being the particle radius, is the Stokes' drag coefficient and mN is the mass of particles per unit volume.

If we assume that the particle has a uniform spherical shape and that the particle velocity relative to the fluid is small, then in the equations of motion for the fluid, because of the presence of suspended particles, an additional force term appears proportional to the velocity difference between the suspended particles and the fluid. Since the force exerted by the fluid on the suspended particles is equal and opposite to that exerted by the particles on the fluid, there must be an extra force term, equal in magnitude but opposite in sign appears in the equations of motion of the suspended particles. The buoyancy force on the particles is also neglected. This force is proportional to the quotient of ρ and the particle density, and an analysis for the case of free-free boundary conditions shows that its small stabilizing effect is negligible. We assume that the distances between particles are quite large compared with their diameter, so interparticle reactions are also ignored.

If we use the Lagrange's vector identities

$$(\vec{q}_d.\nabla)\vec{q}_d=\frac{1}{2}\nabla q_d^2-\vec{q}_d\times\vec{\Omega},\ (\vec{q}.\nabla)\vec{q}=\frac{1}{2}\nabla q^2-\vec{q}\times\vec{\Omega}_1, \tag{7}$$

equations (5) and (6) become

$$mN\left[\frac{\partial\vec{q}_d}{\partial t}-\left(\vec{q}_d\times\vec{\Omega}\right)\right]=-mNgz-mN\nabla q_d^2+\mu_0M\nabla H+KN(\vec{q}-\vec{q}_d), \tag{8}$$

$$\rho\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{\partial\vec{q}}{\partial t}-\left(\vec{q}\times\vec{\Omega}_1\right)\right]=\left(1+\lambda\frac{\partial}{\partial t}\right)\left[-\nabla P-\nabla\rho gz-\frac{1}{2}\rho\nabla q^2+KN(\vec{q}_d-\vec{q})\right]+\mu\nabla^2\vec{q}, \tag{9}$$

where $\vec{\Omega}=\nabla\times\vec{q}_d$ and $\vec{\Omega}_1=\nabla\times\vec{q}$ are respectively the solid vorticity and fluid vorticity.

Taking the curl of these equations and keeping that the curl of a gradient is identically equal to zero, we get

$$mN\left[\frac{\partial\vec{\Omega}}{\partial t}-\left(\nabla\times\vec{q}_d\times\vec{\Omega}\right)\right]=\mu_0\nabla\times M\nabla H+KN\left(\vec{\Omega}_1-\vec{\Omega}\right), \tag{10}$$

$$\rho\left(1+\lambda\frac{\partial}{\partial t}\right)\left[\frac{\partial\vec{\Omega}_1}{\partial t}-\left(\nabla\times\vec{q}\times\vec{\Omega}_1\right)\right]=\mu\nabla^2\vec{\Omega}_1+\left(1+\lambda\frac{\partial}{\partial t}\right)KN\left(\vec{\Omega}-\vec{\Omega}_1\right). \tag{11}$$

By making use of the vector identities

$$\nabla\times\left(\vec{q}_d\times\vec{\Omega}\right)=\left(\vec{\Omega}.\nabla\right)\vec{q}_d-\left(\vec{q}_d.\nabla\right)\vec{\Omega}+\vec{q}_d\nabla.\vec{\Omega}-\vec{\Omega}\nabla.\vec{q}_d=\left(\vec{\Omega}.\nabla\right)\vec{q}_d-\left(\vec{q}_d.\nabla\right)\vec{\Omega}, \tag{12}$$

$$\nabla\times\left(\vec{q}\times\vec{\Omega}_1\right)=\left(\vec{\Omega}_1.\nabla\right)\vec{q}-\left(\vec{q}.\nabla\right)\vec{\Omega}_1+\vec{q}\nabla.\vec{\Omega}_1-\vec{\Omega}_1\nabla.\vec{q}=\left(\vec{\Omega}_1.\nabla\right)\vec{q}-\left(\vec{q}.\nabla\right)\vec{\Omega}_1, \tag{13}$$

equations (10) and (11) become

$$mN\frac{D\vec{\Omega}}{Dt}=\mu_0\nabla\times M\nabla H+mN\left(\vec{\Omega}.\nabla\right)\vec{q}_d+KN\left(\vec{\Omega}_1-\vec{\Omega}\right), \tag{14}$$

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{D\vec{\Omega}_1}{Dt}=\nu\nabla^2\vec{\Omega}_1+\left(1+\lambda\frac{\partial}{\partial t}\right)\left(\vec{\Omega}_1.\nabla\right)\vec{q}+\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{KN}{\rho}\left(\vec{\Omega}-\vec{\Omega}_1\right), \tag{15}$$

where ν is the kinematic viscosity and $\frac{D}{Dt}=\frac{\partial}{\partial t}+(\vec{q}_d.\nabla)$ is the convective derivative.

In equation (14),

$$\nabla\times(M\nabla H)=(\nabla M\times\nabla H)+(M\nabla\times\nabla H). \tag{16}$$

Since the curl of the gradient is zero, the last term in equation (16) is zero. Also since $M=M(H,T)$. We have

$$\nabla M=\left(\frac{\partial M}{\partial H}\right)\nabla H+\left(\frac{\partial M}{\partial T}\right)\nabla T. \tag{17}$$

By making use of (17), equation (16) becomes

$$\nabla\times(M\nabla H)=\left(\frac{\partial M}{\partial H}\right)\nabla H\times\nabla H+\left(\frac{\partial M}{\partial T}\right)\nabla T\times\nabla H \tag{18}$$

The first term on the right hand side of this equation is zero, hence

$$\nabla\times\left(M\nabla H\right)=\left(\frac{\partial M}{\partial T}\right)\nabla T\times\nabla H\,. \tag{19}$$

Putting this expression in equation (14), we obtain

$$mN\frac{D\vec{\Omega}}{Dt}=\mu_0\left(\frac{\partial M}{\partial T}\right)\nabla T\times\nabla H+\ mN\left(\vec{\Omega}.\nabla\right)\vec{q}_d+KN\left(\vec{\Omega}_1-\vec{\Omega}\right). \tag{20}$$

Here (15) and (20) are the equations governing the transport of vorticity in magnetic Maxwellian viscoelastic fluid-particle mixtures.

In equation (20), the first term in the right hand side $\mu_0\left(\frac{\partial M}{\partial T}\right)\nabla T\times\nabla H$ describes the production of vorticity due to thermo-kinetic processes. The last term $KN\left(\vec{\Omega}_1-\vec{\Omega}\right)$. gives the change in solid vorticity on account of exchange of vorticity between the liquid and solid.

From equations (15) and (20), it follows that the transport of solid vorticity $\vec{\Omega}$ is coupled with the transport of fluid vorticity $\vec{\Omega}_1$.

From equation (20), we see that if solid vorticity $\vec{\Omega}$ is zero, then the fluid vorticity $\vec{\Omega}_1$ is not zero, but it is given by

$$\vec{\Omega}_1=-\frac{\mu_0}{KN}\left(\frac{\partial M}{\partial T}\right)\nabla T\times\nabla H. \tag{21}$$

This implies that due to thermo-kinetic processes, fluid vorticity can exist in the absence of solid vorticity.

From equation (14), we find that if $\vec{\Omega}_1$ is zero, then $\vec{\Omega}$ is also zero. This implies that when fluid vorticity is zero, then solid vorticity is necessarily zero.

In the absence of suspended magnetic particles, N is zero, and the magnetization M is also zero. Then equation (20) is identically satisfied, and equation (15) reduces to

$$\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{D\vec{\Omega}_1}{Dt}=\nu\nabla^2\vec{\Omega}_1+\left(1+\lambda\frac{\partial}{\partial t}\right)\left(\vec{\Omega}_1.\nabla\right)\vec{q}. \tag{22}$$

This equation is vorticity transport equation. The last term on the right hand side of equation (22) represents the rate at which $\vec{\Omega}_1$ varies for a given particle, when the vortex lines move with the fluid (the strengths of the vortices remaining constant) and the rate of change of vorticity, which varies for a given particle due to stress relaxation time. The first term represents the rate of dissipation of vorticity through friction (resistance).

4. Two-Dimensional Case

Here we consider the two-dimensional case:

$$\text{Let}\quad \begin{aligned}\vec{q}_d&=q_{d_x}\left(x,y\right)\hat{i}+q_{d_y}\left(x,y\right)\hat{j}\ ,\\ \vec{q}&=q_x\left(x,y\right)\hat{i}+q_y\left(x,y\right)\hat{j}\end{aligned} \tag{23}$$

where components q_{d_x},q_{d_y} *and* q_x,q_y are functions of x,y *and* t, then

$$\vec{\Omega}=\Omega_z\hat{k}\ ,\ \vec{\Omega}_1=\Omega_{1z}\hat{k} \tag{24}$$

In two-dimensional case, equation (21) becomes

$$\begin{aligned}\frac{D\Omega_z}{Dt}=\frac{\mu_0\varepsilon}{mN}\left(\frac{\partial M}{\partial T}\right)\left(\frac{\partial T}{\partial x}\frac{\partial H}{\partial y}-\frac{\partial H}{\partial x}\frac{\partial T}{\partial y}\right)\\ +\frac{K}{m}\left(\Omega_{1z}-\Omega_z\right).\end{aligned} \tag{25}$$

Similarly, equation (16) becomes

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{D\Omega_{1z}}{Dt}=\nu\nabla^2(\Omega_{1z})+\frac{KN}{\rho}(\Omega_z-\Omega_{1z})\\ +\frac{KN\lambda}{\rho}\frac{\partial}{\partial t}(\Omega_z-\Omega_{1z}),\end{aligned} \tag{26}$$

since it can be easily verified that

$$\left(\vec{\Omega}.\nabla\right)\vec{q}_d=0\ and\ \left(\vec{\Omega}_1.\nabla\right)\vec{q}=0. \tag{27}$$

The first term on the right hand side of equation (26) is the change of fluid vorticity due to internal friction (resistance), the second term is change in fluid vorticity on account of exchange of vorticity between solid and liquid and the third term rate of change in fluid vorticity on account of exchange of vorticity between solid and liquid due to stress relaxation time. Equation (26) does not involve explicitly the term representing change of vorticity due to magnetic field gradient and/or temperature gradient. But equation (25) shows that solid vorticity Ω_z depends on these factors. Hence, it follows that fluid vorticity is indirectly influenced by the temperature and the magnetic field gradient.

In the absence of magnetic particles, N is zero and magnetization M is also zero, so equation (25) is identically satisfied and equation (26) reduces to classical equation for the transport of fluid vorticity. If we consider a suspension of non-magnetic particles instead of magnetic fluid, then the corresponding equation for the transport of vorticity may be obtained by setting M equal to zero in the equations governing the transport of vorticity in magnetic fluids. If the magnetization M of the magnetic particles is independent of temperature, then the first term in equations (20) and (25) vanishes and so the equations governing the transport of vorticity in magnetic fluid become the same as those governing the transport of vorticity in non-magnetic suspensions.

If the temperature gradient ∇T vanishes or if the magnetic field gradient ∇H vanishes or if ∇T is parallel to ∇H, then also the first term in equations (20) and (25) vanishes. Thus, we see that in this case also the transport of vorticity in magnetic fluid is same as transport of vorticity in non-magnetic suspension.

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