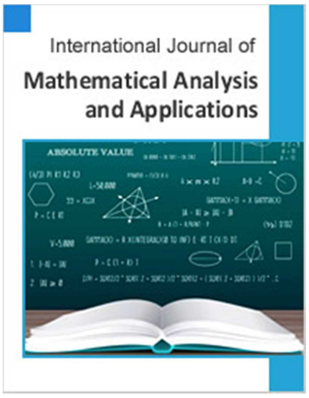
International Journal of Mathematical Analysis and Applications
2018; 5(1): 24-34
http://www.aascit.org/journal/ijmaa
ISSN: 2375-3927

American Association for Science and Technology



Keywords
Gamma Function,
Beta Function,
Hypergeometric Function,
Confluent Hypergeometric Function,
Pfaff-Saalschütz Theorem,
Generalized Hypergeometric Functions

Received: November 17, 2017
Accepted: December 15, 2017
Published: January 18, 2018

Some Applications on Generalized Hypergeometric and Confluent Hypergeometric Functions

Salma Ibrahim El-Soubhy*, Mareiah Mansoor Al-Khalaf, Aziza Salamah Al-Rasheedi, Ghofran Abdul-Rahman Al-Hendi, Sayfiah Karazim Al-Juhani, Sumayyah Ahmed Al-Ahmadi

Department of Mathematics, Faculty of Science, Taibah University, Al-Medinah Al-munawwarh, Saudi Arabia

Email address

sisubhi@taibahu.edu.sa (S. I. El-Soubhy)
*Corresponding author

Citation

Salma Ibrahim El-Soubhy, Mareiah Mansoor Al-Khalaf, Aziza Salamah Al-Rasheedi, Ghofran Abdul-Rahman Al-Hendi, Sayfiah Karazim Al-Juhani, Sumayyah Ahmed Al-Ahmadi. Some Applications on Generalized Hypergeometric and Confluent Hypergeometric Functions. *International Journal of Mathematical Analysis and Applications.* Vol. 5, No. 1, 2018, pp. 24-34.

Abstract

Recently, some generalizations of the generalized famous special functions (e.g. Gamma function, Beta function, Gauss hypergeometric function,…etc) have been studied in recent literature. The main object of this paper is to express explicitly the generalization of the classical generalized hypergeometric function pFq in terms of the classical generalized hypergeometric function itself; moreover, the Pfaff-Saalschütz theorem is given as special case from it, and some new integrals using the generalized Gauss hypergeometric functions are obtained and many important results are noted.

1. Introduction

Special functions have extensive applications in pure mathematics, as well as in applied areas such as acoustics, electrical current, fluid dynamics, heat conduction, solutions of wave equations, moments of inertia and quantum mechanics [1]. Hypergeometric functions have explicit series and integral representations, and thus provide ideal tools for establishing useful summation and transformation formulae. In addition, applied problems frequently require solutions of a function in terms of parameters, rather than merely in terms of a variable. As a result, the hypergeometric function can be used to solve physical problems in diverse areas of applied mathematics [1, 2]. Hypergeometric functions have also been shown to have applications in group theory, algebraic geometry, algebraic K-theory, and conformal field theory. The extended q-hypergeometric series are related to elliptic and theta functions, and are thus useful in partition theory, difference equations and Lie algebras [1].

In the eighteenth century, the problem of interpolating between the numbers

$$n! = \int_0^\infty e^{-t}\ t^n dt, n = 0,1,2, \dots$$

with nonintegral values of n, led Euler in (1729) to the now Gamma function, a generalization of the factorial function that gives meaning to $x!$ when x is any positive number [2]. The integral representation of now widely accepted Gamma and Beta functions are

$$\Gamma(x)=\int_{0}^{\infty} t^{x-1}e^{-t}dt;\ Re(x)>0.$$

$$\beta(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}dt;\ Re(x)>0, Re(y)>0. \tag{1}$$

In (1994), by inserting a regularization factor $e^{-\frac{p}{t}}$, Chaudhry and Zubair [3] have introduced the following extension of Gamma function,

$$\Gamma_p(x)=\int_0^\infty t^{x-1}e^{-t-\frac{p}{t}}dt;\ Re(p)>0. \tag{2}$$

The extension of Euler's Beta function is considered by Chaudhry et al. in (1997) [4], in the following form:

$$\beta_p(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}e^{\frac{-p}{t(1-t)}}dt;\ Re(p)>0, Re(x)>0, Re(y)>0. \tag{3}$$

It is clearly seen that

$$\Gamma_0(x)=\Gamma(x) \text{ and } \beta_0(x,y)=\beta(x,y).$$

Following this, Chaudhry et al. (1994) [3] used $\beta_p(x,y)$ to extend the hypergeometric function, known as the Extended Gauss hypergeometric functions (EGHF), as follows:

$$F_p(a,b;c;z)=\sum_{n=0}^{\infty}(a)_n\frac{\beta_p(b+n,c-b)}{\beta(b,c-b)}\frac{z^n}{n!};\ p\geq 0, Re(c)>Re(b)>0, |z|<1. \tag{4}$$

where $(a)_n$ denotes the Pochhammer symbol defind by:

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1, n=0; a\in\mathbb{C}\setminus\{0\}\\ a(a+1)(a+2)\dots(a+n-1), n\in\mathbb{N}, a\in\mathbb{C}.\end{cases}$$

This series is known to converge where $|z|<1$, provided that c is not negative integer or zero. For the (EGHF), we have the following integral representation:

$$F_p(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}e^{\frac{-p}{t(1-t)}}dt;$$

$$p\geq 0, Re(c)>Re(b)>0 \text{ and } |arg(1-z)|<\pi<p. \tag{5}$$

Also for $p=0$ in (EGHF), it reduces to the usual Gauss hypergeometric function (GHF).

The Extended Confluent Hypergeomtric function (ECHF) [3] is defined as

$$\varphi_p(b;c;z)=\sum_{n=0}^{\infty}\frac{\beta_p(b+n,c-b)}{\beta(b,c-b)}\frac{z^n}{n!};\ p\geq 0, Re(c)>Re(b)>0. \tag{6}$$

In addition, the integral representation of (ECHF) is

$$\varphi_p(b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}e^{\left(zt-\frac{p}{t(1-t)}\right)}dt;$$

$$p>0; p=0 \text{ and } Re(c)>Re(b)>0. \tag{7}$$

The following generalized Euler's Gamma function (GEGF) is defined in [5] as

$$\Gamma_p^{(\alpha,\beta)}(x)=\int_0^\infty t^{x-1}\,{}_1F_1\left(\alpha;\beta;-t-\frac{p}{t}\right)dt;$$

$$Re(\alpha)>0, Re(\beta)>0, Re(p)>0, Re(x)>0, \tag{8}$$

while, the generalized Euler's Beta function (GEBF) is given by

$$\beta_p^{(\alpha,\beta)}(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt;$$

$$Re(\alpha) > 0, Re(\beta) > 0, Re(p) > 0, Re(x) > 0, Re(y) > 0, \tag{9}$$

respectively,

It is obvious from (2) and (8), (3) and (9) that,

$$\Gamma_p^{(\alpha,\alpha)}(x) = \Gamma_p(x) \text{ and } \Gamma_0^{(\alpha,\alpha)}(x) = \Gamma(x),$$

$$\beta_p^{(\alpha,\alpha)}(x,y) = \beta_p(x,y) \text{ and } \beta_0^{(\alpha,\beta)}(x,y) = \beta(x,y).$$

Now, the new generalization of Beta function (9) can be used to generalize the Hypergeometric and Confluent Hypergeometric functions as defind by [5]:

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \sum_{n=0}^{\infty}(a)_n \frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{z^n}{n!}, \tag{10}$$

and

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \sum_{n=0}^{\infty} \frac{\beta_p^{(\alpha,\beta)}(b+n,c-b)}{\beta(b,c-b)}\frac{z^n}{n!}, \tag{11}$$

respectively.

The integral representations for the generalized Gauss hypergeometric function (GGHF) and the generalized Confluent Hypergeometric function (GCHF), are defined as [5]:

$$F_p^{(\alpha,\beta)}(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt;$$

$$Re(p) \geq 0, Re(c) > Re(b) > 0 \text{ and } |arg(1-z)| < \pi < p. \tag{12}$$

and

$${}_1F_1^{(\alpha,\beta;p)}(b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\, e^{zt}\, {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt;$$

$$p \geq 0, Re(c) > Re(b) > 0. \tag{13}$$

It is to be noted here that

$$F_p^{(\alpha,\alpha)}(a,b;c;z) = F_p(a,b;c;z), F_0^{(\alpha,\beta)}(a,b;c;z) = {}_2F_1(a,b;c;z).$$

and

$${}_1F_1^{(\alpha,\alpha;p)}(b;c;z) = {}_1F_1^{(p)}(b;c;z),\ {}_1F_1^{(\alpha,\beta;0)}(b;c;z) = {}_1F_1(b;c;z).$$

The generalized hypergeometric function with p numerator and q denominator parameters is defined by [1, 6]:

$${}_pF_q\left(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots,b_q;z\right) = \sum_{r=0}^{\infty}\frac{(a_1)_r(a_2)_r\ldots(a_p)_r}{(b_1)_r(b_2)_r\ldots(b_q)_r}\frac{z^r}{r!},$$

$$= \frac{\Gamma(b_1)\ldots\Gamma(b_q)}{\Gamma(a_1)\ldots\Gamma(a_p)}\sum_{r=0}^{\infty}\frac{\Gamma(a_1+r)\Gamma(a_2+r)\ldots\Gamma(a_p+r)}{\Gamma(b_1+r)\Gamma(b_2+r)\ldots\Gamma(b_q+r)}\frac{z^r}{r!}. \tag{14}$$

Where $|z| < 1, a_i, b_j \in \mathbb{C}, b_j \neq 0,-1,-2,\ldots, i = 1,2,\ldots p, j = 1,2,\ldots,q$.

In the recent years [7-11], Various extensions of some special functions were studied for introducing some new weighted hypergeometric functions and fractional derivative [9] and more applications on the generalization of hypergeometric functions and orthogonal polynomials, such as Jacobi polynomials, these polynomials can be expressed explicitly in terms of Gauss hypergeometric function and confluent hypergeomeric function, and express explicitly the derivatives of generalized Jacobi polynomials in terms of Jacobi polynomials themselves, by using generalized hypergeometic functions of any degree that have been differentiated an arbitrary numbers of times [10], and all orthogonal polynomials such as Laguerre, Bessel, Hermit [11] are expected to be useful in studying the differentiation (or integration) of these familiar in the future [10].

Up to now, and to the best of our knowledge, many formulae corresponding to those mentioned previously are not known and are traceless in the literature, in particular, for the generalization of the generalized classical hypergeometric function and the generalization of the famous theorem of Pfaff-Saalschütz.

The structure of this article is as follows. In Section 2, we give the generalization of classical generalized hypergeometric function. We (re-derive) the Pfaff-Saalschütz theorem as special case and give the generalization of it. More special cases of the generalization of classical generalized hypergeometric function are given as results. New applications and recurrence relations for the generalized Gauss hypergeometric function (GGHF) are given; furthermore, many important special cases are given in corollaries in Section 3. Finally, Conclusion is noted in Section 4.

2. The Generalization of Classical Generalized Hypergeometric Function

In this section, the generalization of Beta function (9) is used to find the Generalization of classical generalized hypergeometric function. Furthermore; we express explicitly the generalization of classical generalized hypergeometric function in terms of classical generalized hypergeometric function itself. Many useful results are considered.

Theorem 2.1. For the generalization of classical generalized hypergeometric functions, we have

$$
\begin{aligned}
&{}_{r+1}F_{s+1}^{(p,\alpha,\beta)}(a_1,a_2,\ldots,a_r,a_{r+1};b_1,b_2,\ldots,b_s,b_{s+1};x)\\
&=\frac{1}{\beta(a_{r+1},b_{s+1}-a_{r+1})}\int_0^1 t^{a_{r+1}-1}(1-t)^{b_{s+1}-a_{r+1}-1}\\
&\times\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\,{}_rF_s(a_1,a_2,\ldots,a_r;b_1,b_2,\ldots,b_s;xt)dt;\\
&r,s\in\mathbb{N}, Re(b_{s+1})>Re(a_{r+1})>0. \qquad (15)
\end{aligned}
$$

Proof: Direct use of (14), with the aid of (10), enables one to write the left hand side of (15) in the form

$$
\begin{aligned}
&{}_{r+1}F_{s+1}^{(p,\alpha,\beta)}(a_1,a_2,\ldots,a_r,a_{r+1};b_1,b_2,\ldots,b_s,b_{s+1};x)\\
&=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n}\frac{\beta_p^{(\alpha,\beta)}(a_{r+1}+n,b_{s+1}-a_{r+1})}{\beta(a_{r+1},b_{s+1}-a_{r+1})}\frac{x^n}{n!},\\
&=\frac{1}{\beta(a_{r+1},b_{s+1}-a_{r+1})}\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n}\frac{x^n}{n!}\int_0^1 t^{a_{r+1}+n-1}(1-t)^{b_{s+1}-a_{r+1}-1}\times\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt,\\
&=\frac{1}{\beta(a_{r+1},b_{s+1}-a_{r+1})}\int_0^1 t^{a_{r+1}-1}(1-t)^{b_{s+1}-a_{r+1}-1}\times\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\ldots(a_r)_n}{(b_1)_n(b_2)_n\ldots(b_s)_n}\frac{(xt)^n}{n!}dt,\\
&=\frac{1}{\beta(a_{r+1},b_{s+1}-a_{r+1})}\int_0^1 t^{a_{r+1}-1}(1-t)^{b_{s+1}-a_{r+1}-1}\times\ {}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\,{}_rF_s(a_1,a_2,\ldots,a_r;b_1,b_2,\ldots,b_s;xt)dt,
\end{aligned}
$$

and this completes the proof of Theorem 2.1.

The particular expressions for the generalization of classical generalized hypergeometric function may be derived as special cases. These special cases are given in the following corollaries:

Corollary 1.

$$
\begin{aligned}
{}_3F_2^{(p,\alpha,\beta)}(-n,a,b;c,d;1)&=\frac{(c-a)_n}{(c)_n\beta(d-b,b)}\int_0^1 t^{d-b-1}(1-t)^{b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)\\
&\times\ {}_2F_1(-n,a;-n+a+1-c;t)dt;\ n\text{ is an even positive integer.} \qquad (16)
\end{aligned}
$$

Proof: Making use of the generalization of classical generalized hypergeometric function (15), yields

$$
\begin{aligned}
{}_3F_2^{(p,\alpha,\beta)}(-n,a,b;c,d;1)&=\frac{1}{\beta(b,d-b)}\int_0^1 x^{b-1}(1-x)^{d-b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{x(1-x)}\right)\\
&\times\ {}_2F_1(-n,a;c;x)dx,
\end{aligned}
$$

Let $t=1-x$, then

$$ {}_3F_2^{(p,\alpha,\beta)}(-n,a,b;c,d;1)=\frac{1}{\beta(d-b,b)}\int_0^1 t^{d-b-1}(1-t)^{b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) \times\ {}_2F_1(-n,a;c;1-t)dt, $$

and after recalling the fact,

$$ {}_2F_1(-n,a;c;1-t)=\frac{(c-a)_n}{(c)_n}\,{}_2F_1(-n,a;-n+a+1-c;t), $$

we have,

$$ {}_3F_2^{(p,\alpha,\beta)}(-n,a,b;c,d;1)=\frac{(c-a)_n}{(c)_n\beta(d-b,b)}\int_0^1 t^{d-b-1}(1-t)^{b-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right) \times\ {}_2F_1(-n,a;-n+a+1-c;t)dt, $$

and this completes the proof of the corollary.

Corollary 2. Putting $p=0$ and $d=1-c+a+b-n$ into (16), give the Pfaff-Saalschütz theorem, namely

$$ {}_3F_2(-n,a,b;c,1-c+a+b-n;1)=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}. \tag{17} $$

Proof:

$$ {}_3F_2(-n,a,b;c,1-c+a+b-n;1)=\sum_{r=0}^{\infty}\frac{(-n)_r(a)_r(b)_r(1)^r}{(c)_r(1-c+a+b-n)_r r!}, $$

$$ =\sum_{r=0}^{\infty}\frac{(-n)_r(b)_r}{(c)_r r!}\frac{\beta(a+r,1-c+b-n)}{\beta(a,1-c+b-n)}, $$

$$ =\frac{1}{\beta(a,1-c+b-n)}\sum_{r=0}^{\infty}\frac{(-n)_r(b)_r}{(c)_r r!}\beta(1-c+b-n,a+r), $$

$$ =\frac{1}{\beta(a,1-c+b-n)}\sum_{r=0}^{\infty}\frac{(-n)_r(b)_r}{(c)_r r!}\int_0^1 t^{1-c+b-n-1}(1-t)^{a+r-1}dt, $$

$$ =\frac{1}{\beta(a,1-c+b-n)}\int_0^1 t^{1-c+b-n-1}(1-t)^{a-1}\sum_{r=0}^{\infty}\frac{(-n)_r(b)_r}{(c)_r r!}(1-t)^r\,dt, $$

$$ =\frac{1}{\beta(a,1-c+b-n)}\int_0^1 t^{1-c+b-n-1}(1-t)^{a-1}\,{}_2F_1(-n,b;c;1-t)dt, $$

and knowing that

$$ {}_2F_1(-n,b;c;1-t)=\frac{(c-b)_n}{(c)_n}\,{}_2F_1(-n,b;-n+b+1-c;t), $$

then,

$$ {}_3F_2(-n,a,b;c,1-c+a+b-n;1)=\frac{1}{\beta(a,1-c+b-n)}\int_0^1 t^{1-c+b-n-1}(1-t)^{a-1} \times\frac{(c-b)_n}{(c)_n}\,{}_2F_1(-n,b;-n+b+1-c;t)dt, $$

$$ =\frac{(c-b)_n}{(c)_n}\frac{1}{\beta(a,1-c+b-n)}\int_0^1 t^{-c+b-n}(1-t)^{a-1}\sum_{r=0}^{\infty}\frac{(-n)_r(b)_r}{(-n+b+1-c)_r r!}t^r\,dt, $$

$$= \frac{(c-b)_n}{(c)_n} \frac{1}{\beta(a, 1-c+b-n)} \sum_{r=0}^{\infty} \frac{(-n)_r (b)_r}{(-n+b+1-c)_r r!} \int_0^1 t^{c+b-n+r} (1-t)^{a-1} dt,$$

$$= \frac{(c-b)_n}{(c)_n} \sum_{r=0}^{\infty} \frac{(-n)_r (b)_r}{(-n+b+1-c)_r r!} \frac{\beta(1-c+b-n+r, a)}{\beta(a, 1-c+b-n)},$$

$$= \frac{(c-b)_n}{(c)_n} \sum_{r=0}^{\infty} \frac{(-n)_r (b)_r}{(-n+b+1-c)_r r!} \frac{\Gamma(1-c+b-n+r)\Gamma(1-c+a+b-n)}{\Gamma(1-c+b-n+r+a)\Gamma(1-c+b-n)},$$

$$= \frac{(c-b)_n}{(c)_n} \sum_{r=0}^{\infty} \frac{(-n)_r (b)_r}{(-n+b+1-c)_r r!} \frac{(-n+b+1-c)_r}{(-n+a+b+1-c)_r},$$

$$= \frac{(c-b)_n}{(c)_n} \sum_{r=0}^{\infty} \frac{(-n)_r (b)_r}{(-n+a+b+1-c)_r r!},$$

$$= \frac{(c-b)_n}{(c)_n} \, {}_2F_1(-n, b; 1+a+b-n-c; 1),$$

and recalling the Chu-Vandermonde formula

$${}_2F_1(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n},$$

enables one to write

$${}_3F_2(-n, a, b; c, 1-c+a+b-n; 1) = \frac{(c-b)_n}{(c)_n} \frac{(1+a+b-n-c-b)_n}{(1+a+b-n-c)_n},$$

$$= \frac{(c-b)_n}{(c)_n} \frac{(1+a-n-c)_n}{(1+a+b-n-c)_n},$$

$$= \frac{(c-b)_n}{(c)_n} \frac{\Gamma(1+a-n-c+n)}{\Gamma(1+a-n-c)} \frac{\Gamma(1+a+b-n-c)}{\Gamma(1+a+b-n-c+n)},$$

$$= \frac{(c-b)_n}{(c)_n} \frac{\Gamma(1+a-c)}{\Gamma(1+a-n-c)} \frac{\Gamma(1+a+b-n-c)}{\Gamma(1+a+b-c)},$$

$$= \frac{(c-b)_n}{(c)_n} \frac{(1+a+b-c)_{-n}}{(1+a-c)_{-n}},$$

and in view of,

$$(a)_{-k} = \frac{(-1)^k}{(1-a)_k},$$

we get,

$${}_3F_2(-n, a, b; c, 1-c+a+b-n; 1) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}.$$

and this completes the proof of corollary 2.

Corollary 3. Putting $\alpha = \beta$ into (15), gives the extension of generalized hypergeometric function in the integral form:

$${}_{r+1}F_{s+1}^{(p)}(a_1, a_2, \dots, a_r, a_{r+1}; b_1, b_2, \dots, b_s, b_{s+1}; x)$$

$$= \frac{1}{\beta(a_{r+1}, b_{s+1} - a_{r+1})} \int_0^1 t^{a_{r+1}-1} (1-t)^{b_{s+1}-a_{r+1}-1} e^{\frac{-p}{t(1-t)}}$$

$$\times \, {}_rF_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; xt) dt;$$

$$r, s \in \mathbb{N}, R(b_{s+1}) > R(a_{r+1}) > 0. \tag{18}$$

The proof of this corollary is not difficult, but what is worthy noting here is that (18) is in complete agreement with (4.6) given in Luo Minjie and Raina (2013) [2].

Corollary 4. Putting $p = 0$ into (15), gives the classical generalized hypergeometric function in the integral form:

$$_{r+1}F_{s+1}(a_1, a_2, \dots, a_r, a_{r+1}; b_1, b_2, \dots, b_s, b_{s+1}; x) = \frac{1}{\beta(a_{r+1}, b_{s+1} - a_{r+1})}$$

$$\times \int_0^1 t^{a_{r+1}-1} (1-t)^{b_{s+1}-a_{r+1}-1} {}_rF_s(a_1, a_2, \dots, a_r; b_1, b_2, \dots, b_s; xt)dt;$$

$$r, s \in \mathbb{N}, Re(b_{s+1}) > Re(a_{r+1}) > 0. \tag{19}$$

Corollary 5. Putting $r = 1$ and $s = 0$ into (15), gives generalized Gauss hypergeometric function (GGHF) in the integral form:

$$F_p^{(\alpha,\beta)}(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt;$$

$$Re(p) \geq 0, Re(c) > Re(b) > 0 \text{ and } |arg(1-z)| < \pi < p.$$

Corollary 6. Setting $p = 0, r = 1$ and $s = 0$ into (15), gives the classical Hypergeometric function in the integral form:

$$_2F_1(a_1, a_2; b; x) = \frac{1}{\beta(a_2, b - a_2)} \int_0^1 t^{a_2-1} (1-t)^{b-a_2-1} (1-xt)^{-a_1} dt;$$

$$Re(b) > Re(a_2) > 0. \tag{20}$$

Corollary 7. If $r = 0$ and $s = 0$, then (15) gives the generalized Confluent Hypergeometric functions in the integral form:

$$_1F_1^{(\alpha,\beta;p)}(\mathrm{b}; \mathrm{c}; \mathrm{z}) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} e^{zt} {}_1F_1\left(\alpha; \beta; \frac{-p}{t(1-t)}\right) dt;$$

$$p \geq 0, Re(c) > Re(b) > 0.$$

Corollary 8. Set $p = 0, r = 0$ and $s = 0$ in (15), gives classical Confluent Hypergeometric function in the integral form:

$$_1F_1(\alpha; \beta; z) = \frac{1}{\beta(\alpha, \beta-\alpha)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1} e^{zt} dt\, ; \beta > \alpha > 0. \tag{21}$$

It is worthy noting that all the special cases of Theorem 2.1 are in complete agreement with those obtained in [1-5].

3. New Applications and Recurrence Relations for Generalized Gauss Hypergeometric Function (GGHF)

In this section, new recurrence relations using the generalized Beta function (GEBF) (9) and the generalized Gauss hypergeometric function (GGHF) (12) are stated in the following theorem.

Theorem 3.1.

$$\frac{(b)_n}{(c)_n}(-x)^n F_p^{(\alpha,\beta)}\left(-n, -c-n+1; -b-n+1; \frac{1}{x}\right)$$

$$= \frac{\beta(b, 1-b)}{\beta(c, 1-c)} \sum_{w=0}^{\infty} (-n)_w \frac{(b)_w}{(c)_w} \frac{\beta_p^{(\alpha,\beta)}(-c-w+1, c-b)}{\beta(b+w, c-b)} \frac{x^w}{w!},$$

$$Re(p) \geq 0, Re(c) > Re(b) > 0 \text{ and } \left|arg(1-\tfrac{1}{x})\right| < \pi < p. \tag{22}$$

Proof: Direct substitution of (10) into the left hand side of (22), yields

$$\frac{(b)_n}{(c)_n}(-x)^n F_p^{(\alpha,\beta)}\left(-n, -c-n+1; -b-n+1; \frac{1}{x}\right)$$

$$= \frac{(b)_n}{(c)_n}(-x)^n \sum_{r=0}^{\infty} \frac{(-n)_r}{r!} \frac{\beta_p^{(\alpha,\beta)}(-c-n+1+r,-b-n+1+c+n-1)}{\beta(-c-n+1,-b-n+1+c+n-1)} (\frac{1}{x})^r,$$

$$= \frac{\Gamma(b+n)\Gamma(c)}{\Gamma(b)\Gamma(c+n)}(-1)^n \sum_{r=0}^{\infty} \frac{(-1)^r n!}{(n-r)!\,r!} \frac{\beta_p^{(\alpha,\beta)}(-c-n+1+r,c-b)}{\beta(-c-n+1,c-b)} (x)^{n-r},$$

writing $w = n - r; r \leq n$, gives

$$\frac{(b)_n}{(c)_n}(-x)^n F_p^{(\alpha,\beta)}\left(-n,-c-n+1;-b-n+1;\frac{1}{x}\right)$$

$$= \frac{\Gamma(b+n)\Gamma(c)}{\Gamma(b)\Gamma(c+n)} \sum_{w=n}^{\infty} \frac{(-1)^{2n-w} n!}{(n-w)!\,w!} \frac{\beta_p^{(\alpha,\beta)}(-c-w+1,c-b)}{\beta(-c-n+1,c-b)} (x)^w,$$

and in view of

$$(-n)_w = \frac{(-1)^w n!}{(n-w)!}, \Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}, sin\,\pi(n-k) = (-1)^k sin(n\pi)$$

and $-n\Gamma(-n) = \Gamma(1-n)$.
give,

$$\frac{(b)_n}{(c)_n}(-x)^n F_p^{(\alpha,\beta)}\left(-n,-c-n+1;-b-n+1;\frac{1}{x}\right)$$

$$= \sum_{w=n}^{\infty} \frac{\Gamma(b+n)\Gamma(c)}{\Gamma(b)\Gamma(c+n)} \frac{\Gamma(b+w)\Gamma(c+w)}{\Gamma(b+w)\Gamma(c+w)} (-n)_w \frac{\beta_p^{(\alpha,\beta)}(-c-w+1,c-b)}{\beta(-c-n+1,c-b)} \frac{x^w}{w!},$$

$$= \sum_{w=n}^{\infty} \frac{(b)_w}{(c)_w}(-n)_w \frac{\Gamma(b+n)\Gamma(c+w)}{\Gamma(b+w)\Gamma(c+n)} \frac{\Gamma(-b-n+1)}{\Gamma(-c-n+1)\Gamma(c-b)} \beta_p^{(\alpha,\beta)}(-c-w+1,c-b)\frac{x^w}{w!},$$

$$= \sum_{w=n}^{\infty} \frac{(b)_w}{(c)_w}(-n)_w \frac{\Gamma(c+w)}{\Gamma(b+w)\Gamma(c-b)} \frac{\pi}{\sin\pi(b+n)} \frac{\sin\pi(c+n)}{\pi} \beta_p^{(\alpha,\beta)}(-c-w+1,c-b)\frac{x^w}{w!},$$

$$= \sum_{w=n}^{\infty} \frac{(b)_w}{(c)_w}(-n)_w \frac{1}{\beta(b+w,c-b)} \frac{\pi}{(-1)^n \sin(\pi b)} \frac{(-1)^n \sin(\pi c)}{\pi} \beta_p^{(\alpha,\beta)}(-c-w+1,c-b)\frac{x^w}{w!},$$

$$= \sum_{w=n}^{\infty} \frac{(b)_w}{(c)_w}(-n)_w \frac{\Gamma(b)\Gamma(1-b)}{\Gamma(c)\Gamma(1-c)} \frac{\beta_p^{(\alpha,\beta)}(-c-w+1,c-b)}{\beta(b+w,c-b)} \frac{x^w}{w!},$$

$$= \sum_{w=n}^{\infty} \frac{(b)_w}{(c)_w}(-n)_w \frac{\beta(b,1-b)}{\beta(c,1-c)} \frac{\beta_p^{(\alpha,\beta)}(-c-w+1,c-b)}{\beta(b+w,c-b)} \frac{x^w}{w!},$$

$$= \frac{\beta(b,1-b)}{\beta(c,1-c)} \sum_{w=0}^{\infty} \frac{(b)_w}{(c)_w}(-n)_w \frac{\beta_p^{(\alpha,\beta)}(-c-w+1,c-b)}{\beta(b+w,c-b)} \frac{x^w}{w!},$$

and this completes the proof of Theorem 3.1.

The following special cases of formula (22) are worthy to be noted.

Corollary 9. Putting $\alpha = \beta$ into (22), yields

$$\frac{(b)_n}{(c)_n}(-x)^n F_p\left(-n,-c-n+1;-b-n+1;\frac{1}{x}\right) = \frac{\beta(b,1-b)}{\beta(c,1-c)} \sum_{w=0}^{\infty} \frac{(b)_w}{(c)_w}(-n)_w \frac{\beta_p(-c-w+1,c-b)}{\beta(b+w,c-b)} \frac{x^w}{w!}. \quad (23)$$

Corollary 10. Setting $p = 0$ in (22), gives

$$\frac{(b)_n}{(c)_n}(-x)^n\ {}_2F_1\left(-n,-c-n+1;-b-n+1;\frac{1}{x}\right) = {}_2F_1(-n,b;c;x); n = 0,1,2,\ldots \quad (24)$$

Corollary 11. Setting $p = 0$ and $b = c$ into (22), yields

$$ {}_2F_1(-n, b; b; x) = (1 - z)^n, |\arg(1 - z)| < \pi. \tag{25}$$

Corollary 12. Finally, putting $p = 0, b = c$ and $x = 1 - z$ into (22), lead to

$$ {}_2F_1(-n, b; b; 1 - z) = z^n, n = 0,1,2 \ldots \tag{26}$$

Note:

The binomial series is used to get (25) and (26).

It is to be noted here that the two formulas (25) and (26) are in complete agreement with those given in Lebedv (1965) [6] formulae (9.8.1) and (9.8.2), respectively, and the result (24) is also in complete agreement with Ismail (2005) [7] formula (0.6.6).

The next theorem gives a difference equation for $F_p^{(\alpha,\beta)}(a, b; c; x)$:

Theorem 3.2.

$$(b - a)F_p^{(\alpha,\beta)}(a, b; c; x) + aF_p^{(\alpha,\beta)}(a + 1, b; c; x)$$

$$= \sum_{n=0}^{\infty}(b + n)(a)_n \frac{\beta_p^{(\alpha,\beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{x^n}{n!}. \tag{27}$$

Proof: Making use of relation (10) with the left hand side of (27), enables one to write

$$(b - a)F_p^{(\alpha,\beta)}(a, b; c; x) + aF_p^{(\alpha,\beta)}(a + 1, b; c; x)$$

$$= (b - a)\sum_{n=0}^{\infty}(a)_n \frac{\beta_p^{(\alpha,\beta)}(b + n, c - b)}{\beta(b, c - b)} \frac{x^n}{n!} + a\sum_{n=0}^{\infty}(a + 1)_n \frac{\beta_p^{(a,b)}(b + n, c - b)}{\beta(b, c - b)} \frac{x^n}{n!},$$

$$= \sum_{n=0}^{\infty} \frac{\beta_p^{(\alpha,\beta)}(b+n, c-b)}{\beta(b, c-b)} \frac{x^n}{n!}[(b - c)(a)_n + a(a + 1)_n],$$

$$= \sum_{n=0}^{\infty} \frac{\beta_p^{(\alpha,\beta)}(b + n, c - b)}{\beta(b, c - b)} \frac{x^n}{n!}[(b - a)(a)_n + a\frac{(a + n)}{a}(a)_n],$$

$$= \sum_{n=0}^{\infty}(b + n)(a)_n \frac{\beta_p^{(\alpha,\beta)}(b + n, c - b)}{\beta(b, c - b)} \frac{x^n}{n!},$$

and this completes the proof of Theorem 3.2.

Corollary 13. Taking $\alpha = \beta$, then (27) yields the recurrence relation

$$(b - a)F_p(a, b; c; x) + aF_p(a + 1, b; c; x) = \sum_{n=0}^{\infty}(b + n)(a)_n \frac{\beta_p(b+n, c-b)}{\beta(b, c-b)} \frac{x^n}{n!}. \tag{28}$$

Corollary 14. If we put $p = 0$, then (27) gives

$$(b - a)\, {}_2F_1(a, b; c; x) + a\, {}_2F_1(a + 1, b; c; x) = b\, {}_2F_1(a, b + 1; c; x). \tag{29}$$

It is to be noted here that the result (29) is in complete agreement with that given in (9.2.10) [6].

Next, new integral formulas for the generalized Gauss Hypergeometric function (GGHF) are given in the following theorem.

Theorem 3.3. For the generalized Gauss hypergeometric function (GGHF), the following integral holds:

$$\int_0^1 x^{-n-1}(1 - x)^{n+d-1} F_p^{(\alpha,\beta)}(a, \mathrm{b}; \mathrm{c}; x)dx = \beta(-n, d + n) \sum_{r=0}^{\infty} \frac{(a)_r(-n)_r}{(d)_r r!} \frac{\beta_p{}^{(\alpha,\beta)}(b+r, c-b)}{\beta(b, c-b)}. \tag{30}$$

Proof: Following the same procedure of the previous theorem, we get

$$\int_0^1 x^{-n-1}(1 - x)^{n+d-1} F_p^{(\alpha,\beta)}(a, \mathrm{b}; \mathrm{c}; x)dx,$$

$$= \int_0^1 x^{-n-1}(1 - x)^{n+d-1} \sum_{r=0}^{\infty}(a)_r \frac{\beta_p{}^{(\alpha,\beta)}(b+r, c-b)}{\beta(b, c-b)} \frac{x^r}{r!} dx,$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{\beta_p{}^{(\alpha,\beta)}(b+r, c-b)}{\beta(b, c-b)} \frac{1}{r!} \int_0^1 x^{-n+r-1}(1-x)^{n+d-1}\, dx,$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{\beta_p{}^{(\alpha,\beta)}(b+r, c-b)}{\beta(b, c-b)} \frac{1}{r!} \beta(-n+r, n+d),$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{\beta_p{}^{(\alpha,\beta)}(b+r, c-b)}{\beta(b, c-b)} \frac{1}{r!} \frac{\Gamma(-n+r)\Gamma(n+d)}{\Gamma(r+d)} \frac{\Gamma(-n)}{\Gamma(-n)} \frac{\Gamma(d)}{\Gamma(d)},$$

$$= \beta(-n, d+n) \sum_{r=0}^{\infty} \frac{(a)_r(-n)_r}{(d)_r r!} \frac{\beta_p{}^{(\alpha,\beta)}(b+r, c-b)}{\beta(b, c-b)},$$

and this completes the proof of Theorem 3.3.

Again, one important special case of Theorem 3.3 is given in the following corollary:

Corollary 15. Putting $p = 0$ into (30), yields

$$\int_0^1 x^{-n-1}(1-x)^{n+d-1}\ {}_2F_1(a, \mathrm{b}; \mathrm{c}; x)dx = \beta(-n, d+n)\ {}_3F_2(a, b, -n; c; d; 1) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \beta(-n, d+n). \tag{31}$$

Theorem 3.4.

$$\int_0^1 x^{c-1}(1-x)^{\frac{a+b+1}{2}-c-1} F_p^{(\alpha,\beta)}(a, b; 2c; x)dx = \beta(c, \frac{a+b+1}{2} - c) \sum_{r=0}^{\infty} \frac{(a)_r(c)_r}{(\frac{a+b+1}{2})_r} \frac{\beta_p^{(\alpha,\beta)}(b+r, 2c-b)}{\beta(b, 2c-b)} \frac{(1)^r}{r!}. \tag{32}$$

Proof:

From relation (10), one can write write

$$F_p^{(\alpha,\beta)}(a, b; 2c; x) = \sum_{r=0}^{\infty} (a)_r \frac{\beta_p^{(\alpha,\beta)}(b+r, 2c-b)}{\beta(b, 2c-b)} \frac{x^r}{r!},$$

and accordingly, the left hand side of (32), yields

$$\int_0^1 x^{c-1}(1-x)^{\frac{a+b+1}{2}-c-1} F_p^{(\alpha,\beta)}(a, b; 2c; x)dx$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{\beta_p^{(\alpha,\beta)}(b+r, 2c-b)}{\beta(b, 2c-b)} \frac{1}{r!} \int_0^1 x^{c+r-1}(1-x)^{\frac{a+b+1}{2}-c-1} dx,$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{\beta_p^{(\alpha,\beta)}(b+r, 2c-b)}{\beta(b, 2c-b)} \frac{1}{r!} \beta\left(c+r, \frac{a+b+1}{2} - c\right),$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{\beta_p^{(\alpha,\beta)}(b+r, 2c-b)}{\beta(b, 2c-b)} \frac{1}{r!} \frac{\Gamma(c+r)\Gamma(\frac{a+b+1}{2} - c)}{\Gamma(\frac{a+b+1}{2} + r)},$$

$$= \sum_{r=0}^{\infty} (a)_r \frac{\beta_p^{(\alpha,\beta)}(b+r, 2c-b)}{\beta(b, 2c-b)} \frac{1}{r!} \frac{\Gamma(c+r)\Gamma(\frac{a+b+1}{2} - c)}{\Gamma(\frac{a+b+1}{2} + r)} \frac{\Gamma(c)}{\Gamma(c)} \frac{\Gamma(\frac{a+b+1}{2})}{\Gamma(\frac{a+b+1}{2})},$$

$$= \beta\left(c, \frac{a+b+1}{2} - c\right) \sum_{r=0}^{\infty} \frac{(a)_r(c)_r}{(\frac{a+b+1}{2})_r} \frac{\beta_p^{(\alpha,\beta)}(b+r, 2c-b)}{\beta(b, 2c-b)} \frac{1}{r!},$$

and this completes the proof of Theorem 3.4.

Next, the particular expression for the classical hypergeometric function is given in following corollary:

Corollary 16. If $p = 0$, then equation (32) gives the important integral result for classical (GHF), namely

$$\int_0^1 x^{c-1}\,(1-x)^{\frac{a+b+1}{2}-c-1}\ {}_2F_1(a,b;2c;x)\,dx$$

$$= \frac{2^{a+b+2c-2}\,\Gamma(\frac{a+b+c}{2})\Gamma(2c)\Gamma\left(\frac{a}{2}\right)}{\sqrt{\pi}}\,\frac{\Gamma(a+b-2c)}{\Gamma(a)\Gamma\left(\frac{a+b-2c}{2}\right)}\,\frac{\Gamma(2c-a-b)}{\Gamma\left(\frac{2c-a-b}{2}\right)}\,\frac{\Gamma\left(\frac{b}{2}\right)}{\Gamma(b)}\,\frac{\Gamma\left(\frac{a+b}{2}\right)}{\Gamma(a+b)}\,\frac{\Gamma\left(\frac{2c-a}{2}\right)}{\Gamma(2c-a)}\,\frac{\Gamma\left(\frac{2c-b}{2}\right)}{\Gamma(2c-b)}. \tag{33}$$

4. Conclusion

This article has dealt with formulae expressing explicitly the generalization of the classical generalized hypergeometric function in terms of classical generalized hypergeometric function itself. As an application and with aid of these formulae the Pfaff-Saalschütz theorem is obtained as special case from it. Moreover; important results as special cases are noted, some new applications and recurrence relations for the generalized Gauss hypergeometric functions are obtained.

Acknowledgements

The authors are grateful to the Editor and the referee for carefully reading the manuscript and for their valuable comments and suggestions which greatly improved this paper.

References

[1] W. W. Bell, Special Functions for Scientists and Engineers, Oxford University press, London (1968).

[2] L. Minjie, R. K. Raina, Extended Generalized Hypergeometric Functions and their applications, Bulletin of Mathematical Analysis and Applications, Vol. 5 No. 4, (2013) 65-77.

[3] M. A. Chaudhry, S. M. Zubair, Generalized incomplete gamma functions with applications, J. Comput. Appl. Math. 55 (1994) 99-124.

[4] M. A. Chaudhry, A. Qadir, M. Rafique, S. M. Zubair, Extension of Euler's Beta function, J. Comput. Appl. Math. 78 (1997) 19-32.

[5] E. özergin, M. A. özarslan, A. Altin, Extension of Gamma, Beta and Hypergeometric functions, J. Comput. Appl. Math. 235, 16 (2011) 4601-4610.

[6] N. N. Lebedev, Special Functions and Their Applications, Prentice-Hall, INC, printed in USA, (1965).

[7] M. E. Ismail, Classical and Quantum Orthogonal Polynomials in One Variable, Cambrige University press (2005).

[8] S. I. El-soubhy, Notes on Generalized Hypergeometric and Confluent Hypergeometric Functions, International Journal of Mathematical Analysis and Applications, Vol. 2, No. 3, (2015) 47-61.

[9] JE. Restrepo, A. Kilicman, P. Agarwal, and Omer Altun, Weighted hypergeometric functions and fractional derivative, Advances in Difference Equations, (2017): 105 Doi 10.1186/S13662-017-1165-7.

[10] S. I. EL-Soubhy, On the generalization of hypergeometric and confluent hypergeometric functions and their applications for finding the derivatives of the generalized Jacobi Polynomials, Intertional Journal of Mathematical Analysis and applications, vol. 2, No. 6, (2016) 79-90.

[11] M. E. Ismail, R. Zhang, A review of multivariate orthogonal polynomials, Journal of the Egyptian Mathematical Society, 25 (2017) 91–110.