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Exponential Attractor For a class of Higher-Order Coupled Kirchhoff-type Equations

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Abstract: This paper mainly studies a class of higher-order coupled Kirchhoff-type equations with strongly damping and nonlinear source terms. In the process of research, first of all, Lipschitz property of the nonlinear semi-group related to the initial boundary value problem is proved by applying the Young inequality, Poincare inequality, mean value theorem, Gronwall's inequality and so on. Then, the Discrete Squeezing property of the problem is proved by applying the boundedness of this problem in infinite dimensional space and making some changes on the left hand side of the inequality to be proved. Finally, the existence of exponential attractor is proved by using the Lipschitz property, Discrete Squeezing property and other relevant proofs of the problem.

Keywords: Higher-order Coupled Kirchhoff-type Equations, Exponential Attractor, Lipschitz Property, Discrete Squeezing Property

1. Introduction

In this paper, the existence of exponential attractor is considered for a class of higher-order coupled Kirchhoff-type equations:

$$u_{tt}+M(\|\nabla u\|^2+\|\nabla v\|^2)(-\Delta)^m u+\beta(-\Delta)^m u_t+g_1(u,v)=f_1(x), \quad (1)$$

$$v_{tt}+M(\|\nabla u\|^2+\|\nabla v\|^2)(-\Delta)^m v+\beta(-\Delta)^m v_t+g_2(u,v)=f_2(x), \quad (2)$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega, \quad (3)$$

$$v(x,0)=v_0(x), v_t(x,0)=v_1(x), x\in\Omega, \quad (4)$$

$$u\big|_{\partial\Omega}=0, \frac{\partial^i u}{\partial\mu^i}\big|_{\partial\Omega}=0, i=1,2,3\cdots m-1, \quad (5)$$

$$v\big|_{\partial\Omega}=0, \frac{\partial^i v}{\partial v^i}\big|_{\partial\Omega}=0, i=1,2,3\cdots m-1, \quad (6)$$

where $m>1$ is an integer constant, Ω is a bounded domain of R^n with a smooth Dirichlet boundary $\partial\Omega$, $u_0(x),u_1(x),v_0(x),v_1(x)$ are the initial value, μ_i and v_i are the unit outward normal on $\partial\Omega$, $M(s)$ is a nonnegative C^1 function, $(-\Delta)^m u_t$ and $(-\Delta)^m v_t$ are strongly damping, $g_1(u,v)$ and $g_2(u,v)$ are nonlinear source terms, $f_1(x)$ and $f_2(x)$ are given forcing functions.

Shangya Dong and Bolin Guo [1] considered the asymptotic behavior of solutions for a class of nonclassical diffusion equation:

$$u_t-v\Delta u_t-\lambda\Delta u+g(u)=f(x),(x,t)\in\Omega\times R^+, \quad (7)$$

$$u(x,0)=u_0(x), x\in\Omega, \quad (8)$$

$$u(x,t)=0,(x,t)\in\partial\Omega\times R^+. \quad (9)$$

Some assumptions are made for the nonlinearity term $g(s)$ to satisfy the following:

(G1) $g(s)\in C^1(R), \exists\mu\in R$, such that $\lim\limits_{|s|\to\infty}\frac{g(s)}{s}\geq\mu.$

(G2) $\exists c,r>0$ such that

$$|g'(s)|<c(1+|s|^r), \quad (10)$$

where $0 < r < \infty, (n = 1, 2.), r \le \frac{2n}{n-2}, (n \ge 3)$.

According to above assumptions, the squeezing property and existence of the exponential attractor are proved for this equation and estimation on its fractal dimension and exponential attraction are also made.

Zhijian Yang, Zhiming Liu and Panpan Niu [2] studied the existence of an exponential attractor for the wave equation with structural damping and supercritical nonlinearity:

$$u_{tt} - \Delta u + r(-\Delta)^{\alpha} u_t + f(u) = g(x), \tag{11}$$

$$u\big|_{\partial\Omega} = 0, u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \tag{12}$$

The existence of an exponential attractor is obtained in the natural energy space by constructing a bounded absorbing set with higher global regularity (rather than the long-standing partial regularity) and by using the weak quasi-stability estimates (rather than the strong ones as usual).

Guoguang Lin, Penhui Lv and Ruijin Lou [3] studied the global dynamics for a class of nonlinear generalized Kirchhoff-Boussinesq equations with damping term:

$$u_{tt} + \alpha u_t - \beta \Delta u_t + \Delta^2 u = div(g(|\nabla u|^2)\nabla u) + \Delta h(u) + f(x), \tag{13}$$

$$u(x,0) = u_0(x); u_t(x,0) = u_1(x), \tag{14}$$

$$u(x,t)\big|_{\partial\Omega} = 0; \Delta u(x,t)\big|_{\partial\Omega} = 0. \tag{15}$$

Under some reasonable assumptions, the squeezing property of the nonlinear semi-group associated with this equation and the existence of exponential attractors and inertial manifolds are proved.

Lin Chen, Wei Wang and Guoguang Lin [4] studied the higher-order Kirchhoff-type equation with nonlinear strong dissipation in n dimensional space:

$$u_{tt} + (-\Delta)^m u_t + \phi(\|\nabla u\|^2)(-\Delta)^m u + g(u) = f(x), x \in \Omega, t > 0, m > 1, \tag{16}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1, 2, \cdots m-1, x \in \partial\Omega, t > 0, \tag{17}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x). \tag{18}$$

For the above equation, some suitable assumptions about $\phi(s)$ and $g(u)$ are made to get the existence of exponential attractors and inertial manifolds.

More researches on the existence of exponential attractors, these can be seen in the literature [9-14].

In this paper, first, Lipschitz property is proved. Then, Discrete Squeezing property of the nonlinear semi-group related to the initial boundary value problem is obtained. At last, based on the above properties, the existence of the exponential attractor is obtained.

This article is arranged as follows. The second part is that mainly some notations and basic concepts are established. In the third part of the article, the existence of exponential attractor is proved by proving several lemmas.

2. Preliminaries

$\|\cdot\|$ and (,) stand for norm and inner product of H . some symbols are introduced as: $H^m = H^m(\Omega)$, $H_0^m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega)$, $H_0^{2m}(\Omega) = H^{2m}(\Omega) \cap H_0^1(\Omega)$, $H = L^2(\Omega), \upsilon = \|\nabla u\|^2 + \|\nabla v\|^2$, $\tilde{\upsilon} = \|\nabla \tilde{u}\|^2 + \|\nabla \tilde{v}\|^2$, $\|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $V_1 = H_0^m(\Omega) \times H_0^m(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, $V_2 = H_0^{2m}(\Omega) \times H_0^{2m}(\Omega) \times H_0^m(\Omega) \times H_0^m(\Omega)$. And $C_i (i = 1, 2, 3, \cdots)$ are various positive constants.

The inner product and the norm are defined in V_1 space as follows:

$\forall U_i \in (u_i, v_i, p_i, q_i) \in V_1, i = 1, 2,$ it follows that

$$(U_1, U_2) = (\nabla^m u_1, \nabla^m u_2) + (\nabla^m v_1, \nabla^m v_2) + (p_1, p_2) + (q_1, q_2). \tag{19}$$

$$\|U\|_{V_1}^2 = (U, U)_{V_1} = \|\nabla^m u\|^2 + \|\nabla^m v\|^2 + \|p\|^2 + \|q\|^2 . \tag{20}$$

Let $U = (u, v, p, q) \in V_1, p = u_t + \varepsilon u, q = v_t + \varepsilon v,$ equations (1)-(2) are equivalent to the following the evolution equation:

$$U_t + H(U) = F(U) , \tag{21}$$

where

$$H(U) = \begin{pmatrix} \varepsilon u - p \\ \varepsilon v - q \\ \varepsilon^2 u - \varepsilon p + \beta(-\Delta)^m p + (1 - \varepsilon\beta)(-\Delta)^m u \\ \varepsilon^2 v - \varepsilon q + \beta(-\Delta)^m q + (1 - \varepsilon\beta)(-\Delta)^m v \end{pmatrix}, \tag{22}$$

$$F(U) = \begin{pmatrix} 0 \\ 0 \\ f_1(x) - g_1(u,v) + (1 - M(\upsilon))(-\Delta)^m u \\ f_2(x) - g_2(u,v) + (1 - M(\upsilon))(-\Delta)^m v \end{pmatrix}. \tag{23}$$

The following notations are going to be used. Let V_1, V_2 be two Hilbert space. $V_1 \to V_2$ be dense and continuous injection. $V_1 \to V_2$ is compact. $S(t)$ is a map from V_i into $V_i, i = 1, 2.$

Next, some assumptions needed for problem (1)-(6) are gave.

(H1) $M(s) \in C^2(R), 0 < m_0 \le M(s) \le m_1, M'(s) \ge 0;$

(H2) $g_i(u,v) \in C^1(R)$;

(H3). $\beta \ge (2 + 2m_1)\lambda_{n_0+1}^{\frac{m}{2}} + 2\frac{\varepsilon^2}{\lambda_1^m} + C\lambda_1^{-\frac{m}{2}}$

Then, the basic concepts are gave below.

Definition 2.1. [5] The semi-group $S(t)$ possesses a (V_2,V_1) compact attractor A, if it exists a compact set $A\subset V_1, A$ attracts all bounded subsets of V_2, and $S(t)A=A,\ \forall t\ge 0$.

Definition 2.2. [6] A compact set M is called a (V_2,V_1) exponential attractor for the system $(S(t),B)$, if $A\subseteq \mathrm{M}\subseteq B$ and

(1) $S(t)\mathrm{M}\subseteq \mathrm{M}, \forall t\ge 0$,

(2) M has finite fractal dimension, $d_f(\mathrm{M})<+\infty$;

(3) There exist positive constants C_2,C_3 such that $dist(S(t)B,\mathrm{M})\le C_2e^{-C_3t}, t>0$,

where $dist_{V_1}(A,B)=\sup_{x\in A}\inf_{y\in B}|x-y|_{V_1}, B\subset V_1$ is a positive invariant set of $S(t)$.

Definition 2.3. [7] $S(t)$ is said to satisfy the discrete squeezing property on B if there exists an orthogonal projection P_N of rank N such that for every u and v in B, either

$$|S(t_*)u-S(t_*)v|_{V_1}\le \delta|u-v|_{V_1}, \delta\in(0,\frac{1}{8}), \tag{24}$$

or

$$|Q_N(S(t_*)u-S(t_*)v)|_{V_1}\le |P_N(S(t_*)u-S(t_*)v)|_{V_1}, \tag{25}$$

where $Q_N=I-P_N$.

Theorem 2.1. [6] Assuming that

(1) $S(t)$ possesses a (V_2,V_1) compact attractor A;

(2) It exists a positive invariant compact set $B\subset V_1$ of $S(t)$;

(3) $S(t)$ is a Lipschitz continuous map with Lipschitz constant l on B, and satisfies the discrete squeezing property on B.

Then $S(t)$ has a (V_2,V_1) exponential attractor $\mathrm{M}\supseteq A$ on B, and $\mathrm{M}=\bigcup_{0\le t\le t^*}S(t)\mathrm{M}_*$,

$$\mathrm{M}_*=A\cup(\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty}S(t_*)^j(E^{(k)})). \tag{26}$$

Moreover, the fractal dimension of M satisfies $d_f(\mathrm{M})\le C_4N_0+1$, where N_0, $E^{(k)}$ are defined as in [6].

Theorem 2.2.[15] Assuming the nonlinear function $g(u,v), M(s)$ satisfy the condition (H1)-(H3), $(u_0,v_0,p_0,q_0)\in V_k, k=1,2$, then the problem (1)-(6) admits a unique solution $(u,v,p,q)\in L^\infty(R^+,V_k)$. This solution possesses the following properties:

$$\|(u,v,p,q)\|_{V_1}^2=\|\nabla^m u\|^2+\|\nabla^m v\|^2+\|p\|^2+\|q\|^2\le C_5(R_0), \tag{27}$$

$$\|(u,v,p,q)\|_{V_2}^2=\|\Delta^m u\|^2+\|\Delta^m v\|^2+\|\nabla^m p\|^2+\|\nabla^m q\|^2\le C_6(R_1). \tag{28}$$

The solution is denoted in Theorem 2.1, by $S(t)(u_0,v_0,p_0,q_0)=(u(t),v(t),p(t),q(t))$, then $S(t)$ is continuous semigroup in V_1, balls are shown:

$$B_1=\{(u,v,p,q)\in V_1:\|(u,v,p,q)\|_{V_1}^2\le C_5(R_0)\}, \tag{29}$$

$$B_2=\{(u,v,p,q)\in V_2:\|(u,v,p,q)\|_{V_2}^2\le C_6(R_1)\}. \tag{30}$$

Respectively is a absorbing set of $S(t)$ in V_1 and V_2.

There exists $t_0(B_2)$ such that $B=\overline{\bigcup_{0\le t\le t_0}S(t)B_2}$ is the positive in variant set of $S(t)$ in V_1, and B attracts all bounded subset of V_2, where B_2 is a closed bounded absorbing set for $S(t)$ in V_2. According to [8] and Theorem 2.1, it holds that the semigroup $\{S(t)\}_{t\ge 0}$ possesses (V_2,V_1) compact attractor $A=\bigcap_{s\ge 0}\overline{\bigcup_{t\ge s}S(t)B_2}$, where the bar means the closure in V_1 and A is bounded in V_2.

3. Exponential Attractors

Lemma 3.1. For any $U=(u,v,p,q)\in V_1$, there holds

$$(H(U),U)\ge k_1\|U\|_{V_1}^2+k_2\|\nabla^m p\|^2+k_2\|\nabla^m q\|^2. \tag{31}$$

Proof. According to (19) and (22), it follows that

$$(H(U),U)=(\nabla^m(\varepsilon u-p),\nabla^m u)+(\nabla^m(\varepsilon v-q),\nabla^m v)+(A_1,p)+(A_2,q), \tag{32}$$

where $A_1=-\varepsilon p+\varepsilon^2u+(1-\varepsilon\beta)(-\Delta)^m u+\beta(-\Delta)^m p,\ A_2=-\varepsilon q+\varepsilon^2 v+(1-\varepsilon\beta)(-\Delta)^m v+\beta(-\Delta)^m q.$

$$\begin{aligned}(A_1,p)&=(-\varepsilon p+\varepsilon^2u+(1-\varepsilon\beta)(-\Delta)^m u+\beta(-\Delta)^m p,p)\\&=-\varepsilon\|p\|^2+\varepsilon^2(u,p)+(1-\varepsilon\beta)(\nabla^m u,\nabla^m p)+\beta\|\nabla^m p\|^2.\end{aligned} \tag{33}$$

$$\begin{aligned}(A_2,q)&=(-\varepsilon q+\varepsilon^2 v+(1-\varepsilon\beta)(-\Delta)^m v+\beta(-\Delta)^m q,q)\\&=-\varepsilon\|q\|^2+\varepsilon^2(v,q)+(1-\varepsilon\beta)(\nabla^m v,\nabla^m q)+\beta\|\nabla^m q\|^2.\end{aligned} \tag{34}$$

According to using Holder inequality, Young inequality and Poincare inequality, for the following items, there holds

$$-\varepsilon\beta(\nabla^m u,\nabla^m p)\ge-\frac{\beta}{2}\|\nabla^m p\|^2-\frac{\varepsilon^2\beta}{2}\|\nabla^m u\|^2. \tag{35}$$

$$\begin{aligned}\varepsilon^2(u,p)&\ge-\varepsilon^2\lambda_1^{-m}\|\nabla^m u\|\|\nabla^m p\|\\&\ge-\frac{\varepsilon^2\lambda_1^{-m}}{2}\|\nabla^m u\|^2-\frac{\varepsilon^2\lambda_1^{-m}}{2}\|\nabla^m p\|^2.\end{aligned} \tag{36}$$

Similarly, it follows that

$$-\varepsilon\beta(\nabla^m v,\nabla^m q)\ge-\frac{\beta}{2}\|\nabla^m q\|^2-\frac{\varepsilon^2\beta}{2}\|\nabla^m v\|^2. \tag{37}$$

$$\begin{aligned}\varepsilon^2(v,q)&\ge-\varepsilon^2\lambda_1^{-m}\|\nabla^m v\|\|\nabla^m q\|\\&\ge-\frac{\varepsilon^2\lambda_1^{-m}}{2}\|\nabla^m v\|^2-\frac{\varepsilon^2\lambda_1^{-m}}{2}\|\nabla^m q\|^2,\end{aligned} \tag{38}$$

where $\lambda_1(>0)$ is the first eigenvalue of the operator $(-\Delta)$.

$$\begin{aligned}(H(U),U)\ge&(\varepsilon-\frac{\varepsilon^2\beta}{2}-\frac{\varepsilon^2\lambda_1^{-m}}{2})\|\nabla^m u\|^2+(\frac{\beta}{4}-\frac{\varepsilon^2\lambda_1^{-m}}{2})\|\nabla^m p\|^2+(\frac{\beta}{4}\lambda_1^m-\varepsilon)\|p\|^2\\&+(\varepsilon-\frac{\varepsilon^2\beta}{2}-\frac{\varepsilon^2\lambda_1^{-m}}{2})\|\nabla^m v\|^2+(\frac{\beta}{4}-\frac{\varepsilon^2\lambda_1^{-m}}{2})\|\nabla^m q\|^2+(\frac{\beta}{4}\lambda_1^m-\varepsilon)\|q\|^2.\end{aligned}$$

Therefore, there holds

$$(H(U),U)\ge k_1\|U\|_{V_1}^2+k_2\|\nabla^m p\|^2+k_2\|\nabla^m q\|^2, \tag{39}$$

where

$$a_1=\varepsilon-\frac{\varepsilon^2\beta}{2}-\frac{\varepsilon^2\lambda_1^{-m}}{2}>0,a_2=\frac{\beta}{4}-\varepsilon,k_1=\min\{a_1,a_2\},k_2=\frac{\beta}{4}-\frac{\varepsilon^2\lambda_1^{-m}}{2}>0.\cdot$$

The proof is completed.

Let

$$S(t)U_0=U(t)=(u(t),v(t),p(t),q(t))^T, \tag{40}$$

where $p(t)=u_t+\varepsilon u,q(t)=v_t+\varepsilon v.$

$$S(t)V_0=V(t)=(\tilde u(t),\tilde v(t),\tilde p(t),\tilde q(t))^T, \tag{41}$$

where $\tilde p(t)=\tilde u_t+\varepsilon\tilde u,\tilde q(t)=\tilde v_t+\varepsilon\tilde v.$

Set

$$W(t)=S(t)U_0-S(t)V_0=U(t)-V(t)=(w_1(t),w_2(t),z_1(t),z_2(t)), \tag{42}$$

where $z_1(t)=w_{1t}+\varepsilon w_1,z_2(t)=w_{2t}+\varepsilon w_2$, then $W(t)$ satisfies

$$W_t(t)+H(U)-H(V)=F(U)-F(V). \tag{43}$$

$$W(0)=U_0-V_0, \tag{44}$$

In order to show that (1)-(6) has an exponential attractor, it is proved that the dynamical system $S(t)$ of (1)-(6) is Lipschitz continuous on B.

Lemma 3.2. (Lipschitz property) For any $U_0, V_0 \in B, T \geq 0,$ there holds

$$\|S(t)U_0-S(t)V_0\|_{V_1}^2 \leq e^{kt}\|U_0-V_0\|_{V_1}^2. \tag{45}$$

Proof. Taking the inner product of the equation (43) with $W(t)$ in V_1, it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|W(t)\|_{V_1}^2+(H(U)-H(V),W(t))\\&+(z_1(t),g_1(u,v)-g_1(\tilde{u},\tilde{v})+(M(\upsilon)-1)(-\Delta)^m u-(M(\tilde{\upsilon})-1)(-\Delta)^m\tilde{u})\\&+(z_2(t),g_2(u,v)-g_2(\tilde{u},\tilde{v})+(M(\upsilon)-1)(-\Delta)^m v-(M(\tilde{\upsilon})-1)(-\Delta)^m\tilde{v})=0.\end{aligned} \tag{46}$$

According to the Lemma 3.1, it follows that

$$(H(U)-H(V),W(t))_{V_1}=(H(W(t)),W(t))_{V_1}\geq k_1\|W(t)\|_{V_1}^2+k_2\left\|\nabla^m z_1(t)\right\|^2+k_2\left\|\nabla^m z_2(t)\right\|^2. \tag{47}$$

According to theorem 2.2, using Young inequality, Poincare inequality and Lagrange's mean value Theorem, it follows that

$$\begin{aligned}\left|(g_1(u,v)-g_1(\tilde{u},\tilde{v}),z_1(t))\right|&\leq\|g_{1u}(\theta u+(1-\theta)\tilde{u},v)\|_\infty\|w_1(t)\|\|z_1(t)\|+\|g_{1v}(\tilde{u},\theta\tilde{v}+(1-\theta)v)\|_\infty\|w_2(t)\|\|z_1(t)\|\\&\leq C_7\lambda_1^{-\frac{m}{2}}(\left\|\nabla^m w_1(t)\right\|\|z_1(t)\|+\left\|\nabla^m w_2(t)\right\|\|z_1(t)\|)\\&\leq\frac{C_7\lambda_1^{-\frac{m}{2}}}{2}(\left\|\nabla^m w_1(t)\right\|^2+\left\|\nabla^m w_2(t)\right\|^2+2\|z_1(t)\|^2),\end{aligned} \tag{48}$$

$$\begin{aligned}&\left|(1-M(\upsilon))(-\Delta)^m u-(1-M(\tilde{\upsilon}))(-\Delta)^m\tilde{u},p-\tilde{p})\right|\\&=\left|((-\Delta)^m w_1+M(\tilde{\upsilon})(-\Delta)^m\tilde{u}-M(\upsilon))(-\Delta)^m u,p-\tilde{p})\right|\\&=\left|((-\Delta)^m w_1-M(\tilde{\upsilon})(-\Delta)^m w_1+M'(\xi_1)(\|\nabla u\|^2+\|\nabla v\|^2-\|\nabla\tilde{u}\|^2-\|\nabla\tilde{v}\|^2)(-\Delta)^m u,z_1)\right|\\&\leq\left\|\nabla^m w_1\right\|\left\|\nabla^m z_1\right\|+M(\tilde{\upsilon})\left\|\nabla^m w_1\right\|\left\|\nabla^m z_1\right\|+C_8\lambda_1^{-\frac{m}{2}}(\left\|\nabla^m w_1\right\|+\left\|\nabla^m w_2\right\|)\left\|\nabla^m z_1\right\|\\&\leq(\frac{1}{2}\lambda_1^{-\frac{m}{2}}+\frac{m_1}{2}\lambda_1^{-\frac{m}{2}}+\frac{C_8}{2}\lambda_1^{-\frac{m}{2}})\left\|\nabla^m w_1\right\|^2+\frac{C_8}{2}\lambda_1^{-\frac{m}{2}}\left\|\nabla^m w_2\right\|^2+(\frac{1}{2}\lambda_1^{\frac{m}{2}}+\frac{m_1}{2}\lambda_1^{\frac{m}{2}}+C_8\lambda_1^{-\frac{m}{2}})\left\|\nabla^m z_1\right\|^2,\end{aligned} \tag{49}$$

where $0<\theta<1, \xi_1=\theta(\|\nabla u\|^2+\|\nabla v\|^2)+(1-\theta)(\|\nabla\tilde{u}\|^2+\|\nabla\tilde{v}\|^2)$.

Similarly, it follows that

$$\left|(g_2(u,v)-g_2(\tilde{u},\tilde{v}),z_2(t))\right|\leq\frac{C_9\lambda_1^{-\frac{m}{2}}}{2}(\left\|\nabla^m w_1(t)\right\|^2+\left\|\nabla^m w_2(t)\right\|^2+2\|z_2(t)\|^2). \tag{50}$$

$$\begin{aligned}&\left|(1-M(\upsilon))(-\Delta)^m v-(1-M(\tilde{\upsilon}))(-\Delta)^m\tilde{v},q-\tilde{q})\right|\\&\leq(\frac{1}{2}\lambda_1^{-\frac{m}{2}}+\frac{m_1}{2}\lambda_1^{-\frac{m}{2}}+\frac{C_{10}}{2}\lambda_1^{-\frac{m}{2}})\left\|\nabla^m w_2\right\|^2+\frac{C_{10}}{2}\lambda_1^{-\frac{m}{2}}\left\|\nabla^m w_1\right\|^2+(\frac{1}{2}\lambda_1^{\frac{m}{2}}+\frac{m_1}{2}\lambda_1^{\frac{m}{2}}+C_{10}\lambda_1^{-\frac{m}{2}})\left\|\nabla^m z_2\right\|^2,\end{aligned} \tag{51}$$

Therefore, there holds

$$\frac{d}{dt}\|W(t)\|_{V_1}^2+2k_1\|W(t)\|_{V_1}^2+2(k_2-k_3)\left\|\nabla^m z_1(t)\right\|^2+2(k_2-k_3)\left\|\nabla^m z_2(t)\right\|^2\le C_{11}\lambda_1^{-\frac{m}{2}}\|W(t)\|_{V_1}^2, \tag{52}$$

where $k_3=\max\{\frac{1}{2}\lambda_1^{\frac{m}{2}}+\frac{m_1}{2}\lambda_1^{\frac{m}{2}}+C_8\lambda_1^{\frac{m}{2}},\frac{1}{2}\lambda_1^{\frac{m}{2}}+\frac{m_1}{2}\lambda_1^{\frac{m}{2}}+C_{10}\lambda_1^{\frac{m}{2}}\}$.

According to Gronwall's inequality, it follows that

$$\|W(t)\|_{V_1}^2\le e^{C_{11}\lambda_1^{-\frac{m}{2}}t}\|W(0)\|_{V_1}^2=e^{kt}\|W(0)\|_{V_1}^2, \tag{53}$$

where $k=C_{11}\lambda_1^{-\frac{m}{2}}$. Therefore, there holds

$$\|S(t)U_0-S(t)V_0\|_{V_1}^2\le e^{kt}\|U_0-V_0\|_{V_1}^2. \tag{54}$$

The proof is completed.

Now, the operator $\mathrm{A}=-\Delta:D(\mathrm{A}^m)\to H^{2m}$ is denoted as

$$D(\mathrm{A}^m)=\{u\in H\left|\mathrm{A}^m u,\in H\right\}=\{u\in H^{2m}\left|u\right|_{\partial\Omega}=\nabla^{2m-1}u\big|_{\partial\Omega}=0\}. \tag{55}$$

A is an unbounded self-adjoint positive operator and A^{-1} is compact, it can be found by elementary spectral theory the existence of an orthogonal basis of H consisting of eigenvectors ω_j of A such that

$$\mathrm{A}\omega_j=\lambda_j\omega_j\quad 0<\lambda_1\le\lambda_2\le\cdots\le\lambda_j\to+\infty.$$

$\forall N$ denote by $P=P_n:H^{2m}\to span\{\omega_1,\cdots\omega_N\}$ the projector,

$$Q=Q_N=I-P_N. \tag{56}$$

Then, it follows that

$$\left\|\mathrm{A}^m u\right\|=\left\|(-\Delta)^m u\right\|\ge\lambda_{n+1}^m\|u\|,\forall u\in Q_n(H_0^{2m}(\Omega)),\ \|Q_n u\|\le\|u\|,u\in H.$$

Lemma 3.3. For any $U_0,V_0\in B$,

Let $W_{n_0}(t)=Q_{n_0}(U(t)-V(t))=Q_{n_0}W(t)=(w_{1n_0},w_{2n_0},z_{1n_0},z_{2n_0})^T$, then

$$\left\|W_{n_0}(t)\right\|_{V_1}^2\le(e^{-2k_1t}+\frac{C_{18}}{2k_1+k}\lambda_{n_0+1}^{-\frac{m}{2}}e^{kt})\|W(0)\|_{V_1}^2. \tag{57}$$

Proof. Taking Q_{n_0} in equation (43), it follows that

$$W_{n_0t}(t)+Q_{n_0}(H(U)-H(V))=Q_{n_0}(F(U)-F(V)). \tag{58}$$

Taking the inner product of the equation (58) with W_{n_0} in V_1, there holds

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|W_{n_0}\right\|_{V_1}^2+k_1\left\|W_{n_0}\right\|_{V_1}^2+k_2\left\|\nabla^m z_{1n_0}\right\|^2+k_2\left\|\nabla^m z_{2n_0}\right\|^2\\&+(Q_{n_0}(g_1(u,v)-g_1(\tilde{u},\tilde{v})+(M(\upsilon)-1)(-\Delta)^m u-(M(\tilde{\upsilon})-1)(-\Delta)^m\tilde{u}),z_{1n_0})\\&+(Q_{n_0}(g_2(u,v)-g_2(\tilde{u},\tilde{v})+(M(\upsilon)-1)(-\Delta)^m v-(M(\tilde{\upsilon})-1)(-\Delta)^m\tilde{v}),z_{2n_0})=0.\end{aligned} \tag{59}$$

where $a_1' = \varepsilon - \frac{\varepsilon^2\beta}{2} - \frac{\varepsilon^2 \lambda_{n_0+1}^{-m}}{2} > 0, a_2' = \frac{\beta\lambda_{n_0+1}^{m}}{4} - \varepsilon > 0, k_1' = \min\{a_1', a_2'\}, k_2' = \frac{\beta}{4} - \frac{\varepsilon^2\lambda_{n_0+1}^{-m}}{2} > 0.$

By Theorem 2.2, the mean value theorem and Holder's inequality, it follows that

$$\begin{aligned}\left|(Q_{n_0}(g_1(u,v)-g_1(\tilde{u},\tilde{v})),z_{1n_0})\right| &\le \|g_{1u}(\xi_2,v_{n_0})\|_\infty \|w_{1n_0}\|\|z_{1n_0}\| + \|g_{1v}(\tilde{u}_{n_0},\eta)\|_\infty \|w_{2n_0}\|\|z_{1n_0}\| \\ &\le C_{12}\lambda_{n_0+1}^{-\frac{m}{2}}\|\nabla^m w_{1n_0}\|\|z_{1n_0}\| + C_{13}\lambda_{n_0+1}^{-\frac{m}{2}}\|\nabla^m w_{2n_0}\|\|z_{1n_0}\|.\end{aligned} \tag{60}$$

$$\begin{aligned}&\left|(Q_{n_0}((1-M(\upsilon))(-\Delta)^m u-(1-M(\tilde{\upsilon}))(-\Delta)^m\tilde{u}),z_{1n_0})\right| \\ &= \left|(Q_{n_0}((-\Delta)^m w_1 - M(\tilde{\upsilon})(-\Delta)^m w_1 + M'(\xi_1)((\nabla\tilde{u}+\nabla u,\nabla u-\nabla\tilde{u})+(\nabla\tilde{v}+\nabla v,\nabla v-\nabla\tilde{v}))(-\Delta)^m u),z_{1n_0})\right| \\ &\le (\frac{1}{2}\lambda_{n_0+1}^{-\frac{m}{2}}+\frac{m_1}{2}\lambda_{n_0+1}^{-\frac{m}{2}}+\frac{C_{14}}{2}\lambda_{n_0+1}^{-\frac{m}{2}})\|\nabla^m w_{1n_0}\|^2 + \frac{C_{14}}{2}\lambda_{n_0+1}^{-\frac{m}{2}}\|\nabla^m w_{2n_0}\|^2 + (\frac{1}{2}\lambda_{n_0+1}^{\frac{m}{2}}+\frac{m_1}{2}\lambda_{n_0+1}^{\frac{m}{2}}+C_{14}\lambda_{n_0+1}^{-\frac{m}{2}})\|\nabla^m z_{1n_0}\|^2,\end{aligned} \tag{61}$$

Where $\xi_2 = \theta u_{n_0} + (1-\theta)\tilde{u}_{n_0}$, $\eta = \theta v_{n_0} + (1-\theta)\tilde{v}_{n_0}$, $0 < \theta < 1$.

Similarly, it follows that

$$\left|(Q_{n_0}(g_2(u,v)-g_2(\tilde{u},\tilde{v})),z_{2n_0})\right| \le C_{15}\lambda_{n_0+1}^{-\frac{m}{2}}\|\nabla^m w_{1n_0}\|\|z_{2n_0}\| + C_{16}\lambda_{n_0+1}^{-\frac{m}{2}}\|\nabla^m w_{2n_0}\|\|z_{2n_0}\|. \tag{62}$$

$$\begin{aligned}&\left|(Q_{n_0}((1-M(\upsilon))(-\Delta)^m v-(1-M(\tilde{\upsilon}))(-\Delta)^m\tilde{v}),z_{2n_0})\right| \\ &\le (\frac{1}{2}\lambda_{n_0+1}^{-\frac{m}{2}}+\frac{m_1}{2}\lambda_{n_0+1}^{-\frac{m}{2}}+\frac{C_{17}}{2}\lambda_{n_0+1}^{-\frac{m}{2}})\|\nabla^m w_{2n_0}\|^2 + \frac{C_{17}}{2}\lambda_{n_0+1}^{-\frac{m}{2}}\|\nabla^m w_{1n_0}\|^2 + (\frac{1}{2}\lambda_{n_0+1}^{\frac{m}{2}}+\frac{m_1}{2}\lambda_{n_0+1}^{\frac{m}{2}}+C_{17}\lambda_{n_0+1}^{-\frac{m}{2}})\|\nabla^m z_{2n_0}\|^2\end{aligned} \tag{63}$$

According to the above, there holds

$$\begin{aligned}\frac{d}{dt}\|W_{n_0}\|_{V_1}^2 + 2k_1\|W_{n_0}\|_{V_1}^2 &\le C_{18}\lambda_{n_0+1}^{-\frac{m}{2}}\|W_{n_0}\|_{V_1}^2 \\ &\le C_{18}\lambda_{n_0+1}^{-\frac{m}{2}}\|S(t)U_0 - S(t)V_0\|_{V_1}^2 \le C_{18}\lambda_{n_0+1}^{-\frac{m}{2}}e^{kt}\|W(0)\|_{V_1}^2.\end{aligned} \tag{64}$$

By Gronwall's inequality, it follows that

$$\begin{aligned}\|W_{n_0}(t)\|_{V_1}^2 &\le \|W_{n_0}(0)\|_{V_1}^2 e^{-2k_1 t} + \frac{C_{18}}{2k_1+k}\lambda_{n_0+1}^{-\frac{m}{2}}e^{kt}\|W(0)\|_{V_1}^2 \\ &\le (e^{-2k_1 t} + \frac{C_{18}}{2k_1+k}\lambda_{n_0+1}^{-\frac{m}{2}}e^{kt})\|W(0)\|_{V_1}^2.\end{aligned} \tag{65}$$

The proof is completed.

Lemma 3.4. (Discrete squeezing property) For any $U_0, V_0 \in B$, if

$$\|P_{n_0}(S(T^*)U_0 - S(T^*)V_0)\|_{V_1} \le \|(I-P_{n_0})(S(T^*)U_0 - S(T^*)V_0)\|_{V_1}. \tag{66}$$

then

$$\|S(T^*)U_0 - S(T^*)V_0\|_{V_1} \le \frac{1}{8}\|U_0 - V_0\|_{V_1}. \tag{67}$$

Proof. If

$$\|P_{n_0}(S(T^*)U_0 - S(T^*)V_0)\|_{V_1} \le \|(I-P_{n_0})(S(T^*)U_0 - S(T^*)V_0)\|_{V_1}. \tag{68}$$

then

$$\begin{aligned}\left\|S(T^*)U_0 - S(T^*)V_0\right\|_{V_1}^2 &\le 2\left\|(I-P_{n_0})(S(T^*)U_0 - S(T^*)V_0)\right\|_{V_1}^2 + 2\left\|P_{n_0}(S(T^*)U_0 - S(T^*)V_0)\right\|_{V_1}^2 \\ &\le 4\left\|(I-P_{n_0})(S(T^*)U_0 - S(T^*)V_0)\right\|_{V_1}^2 \\ &\le 4(e^{-2k_1T^*} + \frac{C_{18}}{2k_1+k}\lambda_{n_0+1}^{-\frac{m}{2}}e^{kT^*})\left\|U_0 - V_0\right\|_{V_1}^2 .\end{aligned} \tag{69}$$

Let T^* be large enough

$$e^{-2k_1T^*} \le \frac{1}{512}. \tag{70}$$

Similarly, let n_0 be large enough, then

$$\frac{C_{18}}{2k_1+k}\lambda_{n_0+1}^{-\frac{m}{2}}e^{kT^*} \le \frac{1}{512}. \tag{71}$$

Substituting (70), (71) into (69), it follows that

$$\left\|S(T^*)U_0 - S(T^*)V_0\right\|_{V_1} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{V_1}. \tag{72}$$

The proof is completed.

Theorem 3.2. Under of the above assume, $(u_0, v_0, p_0, q_0) \in V_k, (k = 1,2.), p = u_t + \varepsilon u,$ $q = v_t + \varepsilon v, f_i \in H, (i = 1,2), \exists \varepsilon > 0,$ then, the initial boundary value problem (1)-(6) the solution semi-group has a (V_2, V_1) -exponential attractor on B ,

$$M = \bigcup_{0\le t\le T^*} S(t)(A \cup (\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} S(T^*)^j(E^{(k)}))). \tag{73}$$

And the fractal dimension is satisfied $d_f(M) \le 1 + C_4N_0$.

Proof. By Theorem 2.1, Lemma 3.2 and Lemma 3.4, Theorem 3.2 is proved.

4. Conclusion

In this paper, the existence of the exponential attractor is studied for a class of higher-order coupled Kirchhoff-type equations. In the process of research, first, some reasonable assumptions of $M(s)$ and $g_i(u,v),(i=1,2.)$ are made to prove the Lipschitz property and Discrete Squeezing property of the nonlinear semi-group related to the initial boundary value problem. Secondly, based on the above two properties and other relevant proofs of problem (1)-(6), the existence of the exponential attractor is obtained for a class of higher-order coupled Kirchhoff-type equations. In addition, for such problems, our research direction is relatively monotonous. In order to make the research of the problem more comprehensive and thorough, the research of pull back attractor, random attractor, blow-up and so on will be the main directions of our researches of next.

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