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Sum Types of Uncertainty Relations for Generalized Quasi-metric Adjusted Skew Informations

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Abstract: It is well known that almost all uncertainty relations including Heisenberg uncertainty relation and Schrödinger uncertainty relation were given by product types of trace inequalities. This is why these results were proved by Schwarz's inequality. These product types of uncertainty relations were extended to the case of not necessarily hermitian quantum mechanical observables and positive operators representing quantum states. On the other hand sum types of uncertainty relations were given for arbitrary finite N not necessarily hermitian quantum mechanical observables. Some uncertainty relations are presented by generalized quasi-metric adjusted skew informations for two different generalized states. These uncertainty relations are nontrivial as long as the observables are mutually noncommutative. The relations among these new and existing uncertainty inequalities have been investigated. Finally, the reverse inequalities of the sum types of uncertainty relations are obtained.

Keywords: Trace Inequality, Metric Adjusted Skew Information, Generalized Quasi-metric Adjusted Skew Information

1. Introduction

The famous uncertainty relations in quantum mechanics were founded by Heisenberg and Schrödinger independently. These results were proved by Schwarz's inequality. Then these are product type relations. On the other hand, sum types of relations were given as inequalities related to entropy. Recently several sum types of uncertainty relations were obtained by [1, 2, 3, 4, 5, 6, 7]. Now we state two examples of typical sum types of uncertainty relations.

Proposition 1.1 ([8]) Assume that observables A, B have the following spectral decompositions.

$$A = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle\langle\phi_i|, \quad B = \sum_{j=1}^{n} \mu_j |\psi_j\rangle\langle\psi_j|.$$

For any state $|\varphi\rangle$, the probability distributions are defined by

$$P = (p_1, p_2, \dots, p_n), \quad Q = (q_1, q_2, \dots, q_n),$$

where

$$p_i = |\langle\phi_i|\varphi\rangle|^2, \quad q_j = |\langle\psi_j|\varphi\rangle|^2.$$

Let $H(P), H(Q)$ be Shannon entropies of P and Q, respectively.

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i, \quad H(Q) = -\sum_{i=1}^{n} q_i \log q_i.$$

Then the following sum type of uncertainty relation is obtained.

$$H(P) + H(Q) \geq -2\log c,$$

where $c = \max_{i,j} |\langle\phi_i|\psi_j\rangle|$.

Definition 1.1 The Fourie transformation of $\psi \in L^2(\mathbb{R})$ is defined by

$$\hat{\psi}(\omega) = \int_{-\infty}^{\infty} \psi(t) e^{-2\pi i\omega t} dt.$$

And let

$$Q(\mathbb{R}) = \left\{ f \in L^2(\mathbb{R}); \int_{-\infty}^{\infty} t^2 |f(t)|^2 dt < \infty \right\}.$$

Then it is well known that the following result holds.

Proposition 1.2 ([9]) If $\psi \in L^2(\mathbb{R})$, $\|\psi\|^2 = 1$ satisfies $\psi, \hat{\psi} \in Q(\mathbb{R})$, then

$$S(\psi) + S(\hat{\psi}) \geq \log \frac{e}{2},$$

where

$$S(\psi) = -\int_{-\infty}^{\infty} |\psi(t)|^2 \log |\psi(t)|^2 dt,$$
$$S(\hat{\psi}) = -\int_{-\infty}^{\infty} |\hat{\psi}(t)|^2 \log |\hat{\psi}(t)|^2 dt.$$

Now the Heisenberg uncertainty relation [10] is stated as follows;

$$V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2$$

for a quantum state (density operator) ρ and two observables (self-adjoint operators) A and B, where $[A,B] = AB - BA$. The further stronger result was given by Schrödinger in [11, 12]:

$$V_\rho(A)V_\rho(B) - |Re\{Cov_\rho(A,B)\}|^2 \geq \frac{1}{4}|Tr[\rho[A,B]]|^2,$$

where the covariance is defined by

$$Cov_\rho(A,B) \equiv Tr[\rho\,(A - Tr[\rho A]I)\,(B - Tr[\rho B]I)].$$

The Wigner-Yanase skew information represents a measure for non-commutativity between a quantum state ρ and an observable A. Luo introduced the quantity $U_\rho(A)$ representing a quantum uncertainty excluding the classical mixture [13]:

$$U_\rho(A) \equiv \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_\rho(A))^2},$$

with the Wigner-Yanase skew information [14]:

$$I_\rho(A) \equiv \frac{1}{2}Tr\left[(i[\rho^{1/2}, A_0])^2\right]$$
$$= Tr[\rho A_0^2] - Tr[\rho^{1/2}A_0\rho^{1/2}A_0],$$
$$A_0 \equiv A - Tr[\rho A]I$$

and then he successfully showed a new Heisenberg-type uncertainty relation on $U_\rho(A)$ in [13]:

$$U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A,B]]|^2. \qquad (1)$$

As stated in [13], the physical meaning of the quantity $U_\rho(A)$ can be interpreted as follows. For a mixed state ρ, the variance $V_\rho(A)$ has both classical mixture and quantum uncertainty. Also, the Wigner-Yanase skew information $I_\rho(A)$ represents a kind of quantum uncertainty [15, 16]. Thus, the difference $V_\rho(A) - I_\rho(A)$ has a classical mixture so that we can regard that the quantity $U_\rho(A)$ has a quantum uncertainty excluding a classical mixture. Therefore it is meaningful and suitable to study an uncertainty relation for a mixed state by the use of the quantity $U_\rho(A)$. After then a one-parameter extension of the inequality (1) was given in [17]:

$$U_{\rho,\alpha}(A)U_{\rho,\alpha}(B) \geq \alpha(1-\alpha)|Tr[\rho[A,B]]|^2,$$

where

$$U_{\rho,\alpha}(A) \equiv \sqrt{V_\rho(A)^2 - (V_\rho(A) - I_{\rho,\alpha}(A))^2},$$

with the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(A)$ is defined by

$$I_{\rho,\alpha}(A) \equiv \frac{1}{2}Tr\left[(i[\rho^\alpha, A_0])(i[\rho^{1-\alpha}, A_0])\right]$$
$$= Tr[\rho A_0^2] - Tr[\rho^\alpha A_0\rho^{1-\alpha}A_0].$$

It is notable that the convexity of $I_{\rho,\alpha}(A)$ with respect to ρ was successfully proved by Lieb in [18]. The further generalization of the Heisenberg-type uncertainty relation on $U_\rho(A)$ has been given in [19] using the generalized Wigner-Yanase-Dyson skew information introduced in [20]. Then it is shown that these skew informations are connected to special choices of quantum Fisher information in [21]. The family of all quantum Fisher informations is parametrized by a certain class of operator monotone functions $\mathcal{F}_{op}$ which were justified in [22]. The Wigner-Yanase skew information and Wigner-Yanase-Dyson skew information are given by the following operator monotone functions

$$f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2,$$

$$f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)},\ \alpha \in (0,1),$$

respectively. In particular the operator monotonicity of the function f_{WYD} was proved in [23]. See also [24].

Definition 1.2 Let $M_n(\mathbb{C})$ be a set of all $n \times n$ complex matrices, $M_{n,sa}(\mathbb{C})$ be a set of all $n \times n$ hermitian matrices, $M_{n,+}(\mathbb{C})$ be a set of all $n \times n$ positive definite complex matrices and $M_{n,+,1}(\mathbb{C})$ be a set of all $n \times n$ density matrices. The inner product on $M_n(\mathbb{C})$ is defined by

$$(A,B)_{HS} = Tr(A^*B) = \sum_{i=1}^n \sum_{j=1}^n \overline{a_{ij}}b_{ij},$$

where $A = (a_{ij}),\ B = (b_{ij})$. For $A \in M_n(\mathbb{C})$, left multiplicative operator and right multiplicative operator are defined as follows. respectively.

$$L_A(X) = AX,\ R_A(X) = XA,\ (X \in M_n(\mathbb{C})).$$

Definition 1.3 When $f : (0, +\infty) \to \mathbb{R}$ satisfies the condition

$$A, B \in M_{n,+}(\mathbb{C}),\ 0 \leq A \leq B \Longrightarrow 0 \leq f(A) \leq f(B),$$

$f(x)$ is said to be operator monotone function. When operator monotone function $f(x)$ satisfies $f(x) = xf(x^{-1})$, it is said to be symmetric. If $f(1) = 1$, then it is said to be normarized. Let $\mathcal{F}_{op}$ be a set of all symmetric normarized operator monotone functions.

Example 1.1

$$f_{RLD}(x) = \frac{2x}{x+1},\quad f_{SLD}(x) = \frac{x+1}{2},$$

$$f_{BKM}(x) = \frac{x-1}{\log x},\quad f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2,$$

$$f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)},\ \alpha \in (0,1).$$

For $f \in \mathcal{F}_{op}$, let $f(0) = \lim_{x\to 0} f(x)$. The regular function and non-regular function are defined by

$$\mathcal{F}_{op}^r = \{f \in \mathcal{F}_{op} | f(0) \neq 0\}, \quad \mathcal{F}_{op}^n = \{f \in \mathcal{F}_{op} | f(0) = 0\},$$

respectively.

Definition 1.4 ([25, 26, 27]) For $f \in \mathcal{F}_{op}^r$, $\tilde{f}(x)$ is defined by

$$\tilde{f}(x) = \frac{1}{2}\left\{(x+1) - (x-1)^2\frac{f(0)}{f(x)}\right\}, \quad x > 0.$$

Example 1.2

$$\tilde{f}_{WY}(x) = \sqrt{x}, \quad \tilde{f}_{WYD}(x) = \frac{x^\alpha + x^{1-\alpha}}{2},$$
$$\tilde{f}_{SLD}(x) = \frac{2x}{x+1}.$$

Proposition 1.3 ([21, 25]) $f \to \tilde{f}$ is an one-to-one correspondence between $\mathcal{F}_{op}^r$ and $\mathcal{F}_{op}^n$.

According to Kubo-Ando theory([28]), matrix mean m_f is combined with operator monotone functions in the following way.

For $f \in \mathcal{F}$,

$$m_f(A,B) = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}.$$

Then the monotone metric is defined as follows. For $\rho = \sum_{i=1}^{n} \lambda_i |\phi_i\rangle\langle\phi_i| \in M_{n,+,1}(\mathbb{C})$,

$$\langle X, Y\rangle_f = Tr[X^* m_f(L_\rho, R_\rho)^{-1} Y], \quad X, Y \in M_n(\mathbb{C}),$$

where

$$m_f(L_\rho, R_\rho)^{-1} = \sum_{i,j} m_f(\lambda_i, \lambda_j)^{-1} L_{|\phi_i\rangle\langle\phi_i|} R_{|\phi_j\rangle\langle\phi_j|}.$$

2. Generalized Quasi-metric Adjusted Skew Information

For $g, f \in \mathcal{F}_{op}^r$, the condition (A) is defined by .

$$g(x) \geq k\frac{(x-1)^2}{f(x)}, \text{ for some } k > 0.$$

Then let

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} \in \mathcal{F}_{op}.$$

Furthermore the condition (B) is defined by

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \text{ for some } \ell > 0.$$

Definition 2.1 For $X, Y \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$,

$$\begin{aligned}\Gamma_{A,B}^{(g,f)}(X,Y) &= k\langle(L_A - R_B)X, (L_A - R_B)Y\rangle_f \\ &= kTr[X^*(L_A - R_B)m_f(L_A, R_B)^{-1}(L_A - R_B)Y] \\ &= Tr[X^* m_g(L_A, R_B)Y] - Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y],\end{aligned}$$

$$I_{A,B}^{(g,f)}(X) = \Gamma_{A,B}^{(g,f)}(X,X),$$

$$\begin{aligned}\Psi_{A,B}^{(g,f)}(X,Y) &= Tr[X^* m_g(L_A, R_B)Y] \\ &\quad + Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y],\end{aligned}$$

$$J_{A,B}^{(g,f)}(X) = \Psi_{A,B}^{(g,f)}(X,X),$$

$$U_{A,B}^{(g,f)}(X) = \sqrt{I_{A,B}^{(g,f)}(X) \cdot J_{A,B}^{(g,f)}(X)}.$$

$\Gamma_{A,B}^{(g,f)}(X,Y)$ is called a generalized quasi-metric adjusted correlation measure and $I_{A,B}^{(g,f)}(X)$ is called a generalized quasi-metric adjusted skew information, respectively.

The folllowing results were given.

Theorem 2.1 ([29]) Under condition (A), we have (1), (2).

(1) For $X, Y \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$, the following product type uncertainty relations hold.

$$\begin{aligned}&I_{A,B}^{(g,f)}(X) \cdot I_{A,B}^{(g,f)}(Y) \geq |\Gamma_{A,B}^{(g,f)}(X,Y)|^2 \\ &\geq \frac{1}{16}(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y))^2.\end{aligned}$$

(2) For $X, Y \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$, if condition (B) is satisfied, then the following uncertainty relations hold.

(a) $U_{A,B}^{(g,f)}(X) \cdot U_{A,B}^{(g,f)}(Y) \geq k\ell |Tr[X^*|L_A - R_B|Y]|^2$.

(b) $U_{A,B}^{(g,f)}(X) \cdot U_{A,B}^{(g,f)}(Y) \geq \dfrac{f(0)^2\ell}{k} |\Gamma_{A,B}^{(g,f)}(X,Y)|^2$.

Proof (1) Since the first inequality is proved in [19], the second inequality has to be proved. Since

$$\begin{aligned}I_{A,B}^{(g,f)}(X \pm Y) &= Tr[(X^* \pm Y^*)m_g(L_A, R_B)(X \pm Y)] \\ &\quad - Tr[(X^* \pm Y^*)m_{\Delta_g^f}(L_A, R_B)(X \pm Y)],\end{aligned}$$

$$\begin{aligned}&I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y) \\ = \ &2Tr[X^* m_g(L_A, R_B)Y] + 2TrY^* m_g(L_A, R_B)X] \\ - \ &2Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y] - 2Tr[Y^* m_{\Delta_g^f}(L_A, R_B)X] \\ = \ &2\Gamma_{A,B}^{(g,f)}(X,Y) + 2\Gamma_{A,B}^{(g,f)}(Y,X) = 4Re\{\Gamma_{A,B}^{(g,f)}(X,Y)\}.\end{aligned}$$

Then

$$\begin{aligned}\Gamma_{A,B}^{(g,f)}(X,Y) &= Re\{\Gamma_{A,B}^{(g,f)}(X,Y)\} + iIm\{\Gamma_{A,B}^{(g,f)}(X,Y)\} \\ &= \frac{1}{4}(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y)) \\ &\quad + iIm\{\Gamma_{A,B}^{(g,f)}(X,Y)\}.\end{aligned}$$

Therefore

$$\begin{aligned}|\Gamma_{A,B}^{(g,f)}(X,Y)|^2 &= \frac{1}{16}(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y))^2 \\ &\quad + (Im\{\Gamma_{A,B}^{(g,f)}(X,Y)\})^2 \\ &\geq \frac{1}{16}(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y))^2,\end{aligned}$$

(2) Since (a) is proved in [19], (b) has to be proved. By Lemma 3.3 and Lemma 3.4 in [3],

$$m_g(x,y)^2 - m_{\Delta_g^f}(x,y)^2 \geq k\ell(x-y)^2 \geq k\ell\frac{f(0)^2}{k^2}(m_g(x,y) - m_{\Delta_g^f}(x,y))^2.$$

Then

$$m_g(x,y) + m_{\Delta_g^f}(x,y) \geq \frac{f(0)^2\ell}{k}(m_g(x,y) - m_{\Delta_g^f}(x,y)).$$

Hence

$$J_{A,B}^{(g,f)}(Y) = \sum_{i,j}\{m_g(\lambda_i,\mu_j) + m_{\Delta_g^f}(\lambda_i,\mu_j)\}|\langle\phi_i|Y|\psi_j\rangle|^2 \geq \frac{f(0)^2\ell}{k}\sum_{i,j}\{m_g(\lambda_i,\mu_j) - m_{\Delta_g^f}(\lambda_i,\mu_j)\}|\langle\phi_i|Y|\psi_j\rangle|^2 = \frac{f(0)^2\ell}{k}I_{A,B}^{(g,f)}(Y).$$

By the first inequality in (1),

$$|\Gamma_{A,B}^{(g,f)}(X,Y)|^2 \leq I_{A,B}^{(g,f)}(X)\cdot I_{A,B}^{(g,f)}(Y) \leq I_{A,B}^{(g,f)}(X)\cdot\frac{k}{f(0)^2\ell}J_{A,B}^{(g,f)}(Y).$$

Then

$$I_{A,B}^{(g,f)}(X)\cdot J_{A,B}^{(g,f)}(Y) \geq \frac{f(0)^2\ell}{k}|\Gamma_{A,B}^{(g,f)}(X.Y)|^2.$$

Similarly,

$$J_{A,B}^{(g,f)}(X)\cdot I_{A,B}^{(g,f)}(Y) \geq \frac{f(0)^2\ell}{k}|\Gamma_{A,B}^{(g,f)}(X.Y)|^2.$$

So the proof is completed. □

Remark 2.1 If $A = B = \rho \in M_{n,+,1}(\mathbb{C}), g = f_{SLD}, f = f_{WY}, k = \frac{1}{8}, \ell = 2$, then the result of Luo [13] are given.

3. Sum Types of Uncertainty Relations

Theorem 3.1 For $X, Y \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$, the following hold.

(1) $I_{A,B}^{(g,f)}(X) + I_{A,B}^{(g,f)}(Y) \geq \frac{1}{2}\max\{I_{A,B}^{(g,f)}(X+Y), I_{A,B}^{(g,f)}(X-Y)\}$.

(2) $\sqrt{I_{A,B}^{(g,f)}(X)} + \sqrt{I_{A,B}^{(g,f)}(Y)} \geq \max\{\sqrt{I_{A,B}^{(g,f)}(X+Y)}, \sqrt{I_{A.B}^{(g,f)}(X-Y)}\}$.

(3) $\sqrt{I_{A,B}^{(g,f)}(X)} + \sqrt{I_{A,B}^{(g,f)}(Y)} \leq 2\max\{\sqrt{I_{A,B}^{(g,f)}(X+Y)}, \sqrt{I_{A.B}^{(g,f)}(X-Y)}\}$.

Proof (1) Hilbert-Schmidt norm $\|\cdot\|$ satisfies

$$\begin{aligned}\|X\|^2 + \|Y\|^2 &= \frac{1}{2}(\|X+Y\|^2 + \|X-Y\|^2)\\ &\geq \frac{1}{2}\max\{\|X+Y\|^2, \|X-Y\|^2\}.\end{aligned}$$

Since $I_{A,B}^{(g,f)}(X,X)$ is second power of Hilbert-Schmidt norm, $\|X\| = \sqrt{I_{A,B}^{(g,f)}(X)}$.

Then the result is obtained by substituting the above inequality,

(2) The triangle inequality of general norm is applied for $\|X\| = \sqrt{I_{A,B}^{(g,f)}(X)}$.

(3) The following norm inequality is proved:

$$\|X\| + \|Y\| \leq \|X+Y\| + \|X-Y\|. \qquad (2)$$

Since

$$\|X\| = \|\frac{1}{2}(X+Y) + \frac{1}{2}(X-Y)\| \leq \frac{1}{2}\|X+Y\| + \frac{1}{2}\|X-Y\|$$

and

$$\|Y\| = \|\frac{1}{2}(Y+X) + \frac{1}{2}(Y-X)\| \leq \frac{1}{2}\|Y+X\| + \frac{1}{2}\|Y-X\|,$$

the aimed result is given by adding two inequalities. □

Theorem 3.2 *The following hold.*

(1) $I_{A,B}^{g,f)}(X)$ is convex with respect to X, that is, for $\alpha, \beta \geq 0, \alpha+\beta = 1$ and $X, Y \in M_n(\mathbb{C})$,

$$I_{A,B}^{(g,f)}(\alpha X + \beta Y) \leq \alpha I_{A,B}^{(g,f)}(X) + \beta I_{A,B}^{(g,f)}(Y).$$

(2) $\sqrt{I_{A,B}^{(g,f)}(X)}$ is convex with respect to X, that is, for $\alpha, \beta \geq 0, \alpha+\beta = 1$ and $X, Y \in M_n(\mathbb{C})$,

$$\sqrt{I_{A,B}^{(g,f)}(\alpha X + \beta Y)} \leq \alpha\sqrt{I_{A,B}^{(g,f)}(X)} + \beta\sqrt{I_{A,B}^{(g,f)}(Y)}.$$

Proof (1) Since

$$\begin{aligned}\|\alpha X + \beta Y\|^2 &\leq (\alpha\|X\| + \beta\|Y\|)^2\\ &= \alpha^2\|X\|^2 + 2\alpha\beta\|X\|\|Y\| + \beta^2\|Y\|^2,\end{aligned}$$

$$\begin{aligned}&\alpha\|X\|^2 + \beta\|Y\|^2 - \|\alpha X + \beta Y\|^2\\ \geq\ &\alpha\|X\|^2 + \beta\|Y\|^2 - \alpha^2\|X\|^2 - 2\alpha\beta\|X\|\|Y\|\\ -\ &\beta^2\|Y\|^2 = \alpha\beta(\|X\| - \|Y\|)^2 \geq 0.\end{aligned}$$

Then $\|\alpha X + \beta Y\|^2 \leq \alpha\|X\|^2 + \beta\|Y\|^2$.

(2) It is clear. □

Theorem 3.3 For $\{X_i\}_{i=1}^N, \{Y_j\}_{j=1}^N \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$, if the condition $X_i^*|L_A - R_B|Y_j = \delta_{ij}C$ and the condition (B) are satisfied, then the following hold.

(1) $$\left(\sum_{i=1}^N U_{A,B}^{(g,f)}(X_i)\right)\left(\sum_{j=1}^N U_{A,B}^{(g,f)}(Y_j)\right) \geq Nk\ell|Tr[C]|^2.$$

(2) $$\left(\sum_{i=1}^N \sqrt{U_{A,B}^{(g,f)}(X_i)}\right)\left(\sum_{j=1}^N \sqrt{U_{A,B}^{(g,f)}(Y_j)}\right) \geq N\sqrt{k\ell}|Tr[C]|.$$

Proof (1) By Theorem **??**(2)(a),

$$U_{A,B}^{(g,f)}(X_i)\cdot U_{A,B}^{(g,f)}(Y_j) \geq k\ell|Tr[X_i^*|L_A - R_B|Y_j]|^2. \quad (3)$$

Then

$$\left(\sum_{i=1}^{N} U_{A,B}^{(g,f)}(X_i)\right)\cdot\left(\sum_{j=1}^{N} U_{A,B}^{(g,f)}(Y_j)\right) \geq \sum_{i,j} k\ell |Tr[X_i^*|L_A - R_B|Y_j]|^2$$

$$= \sum_{i,j} k\ell|Tr[\delta_{ij}C]|^2 = \sum_{i=1}^{N} k\ell|Tr[C]|^2 = Nk\ell|Tr[C]|^2.$$

(2) By (3)

$$\sqrt{U_{A,B}^{(g,f)}(X_i)}\cdot\sqrt{U_{A,B}^{(g,f)}(Y_j)} \geq N\sqrt{k\ell}|Tr[X_i^*|L_A - R_B|Y_j]|.$$

Then the result is given by the same method as (1). □

Theorem 3.4 For $\{X_i\}_{i=1}^N \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$ the followings hold.

(1) $\displaystyle\sum_{i=1}^{N} I_{A,B}^{(g,f)}(X_i) \geq \frac{1}{N-2}\sum_{1\leq i<j\leq N} I_{A,B}^{(g,f)}(X_i+X_j) - \frac{1}{(N-1)^2(N-2)}\left(\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)}\right)^2.$

(2) $\displaystyle\sum_{i=1}^{N}\sqrt{I_{A,B}^{(g,f)}(X_i)} \geq \frac{1}{N-2}\left(\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)} - \sqrt{I_{A,B}^{(g,f)}\left(\sum_{i=1}^{N} X_i\right)}\right) \geq \frac{1}{N-1}\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)}$

$\displaystyle\geq \max\left\{\frac{1}{N-2}\left(\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)} - \sum_{i=1}^{N}\sqrt{I_{A,B}^{(g,f)}(X_i)}\right), \sqrt{I_{A,B}^{(g,f)}(\sum_{i=1}^{N} X_i)}\right\}.$

(3) $\displaystyle\frac{1}{N-1}\left(\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)} + \sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i-X_j)}\right) \geq \sum_{i=1}^{N}\sqrt{I_{A,B}^{(g,f)}(X_i)}$

$\displaystyle\geq \frac{1}{2(N-1)}\left(\sum_{i<j}\sqrt{I_{A.B}^{(g,f)}(X_i+X_j)} + \sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i-X_j)}\right).$

(4) $\displaystyle\frac{1}{N(N-1)^2}\left\{\left(\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)}\right)^2 + \left(\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i-X_j)}\right)^2\right\} \leq \sum_{i=1}^{N} I_{A,B}^{(g,f)}(X_i)$

$\displaystyle\leq \frac{1}{N}\sum_{i<j} I_{A,B}^{(g,f)}(X_i-X_j) + \frac{1}{N(N-1)^2}\left(\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)}\right)^2.$

The following lemma is used in order to prove these inequalities. Lemma 3.1 is proved in Appendix.

Lemma 3.1 Let $\|\cdot\|$ be the Hilbert Schmidt norm in $M_n(\mathbb{C})$, For $\{A_i\}_{i=1}^N \subset M_n(\mathbb{C})$, the followings hold.

(1) $\displaystyle\|\sum_{i=1}^{N} A_i\| \leq \frac{1}{N-1}\sum_{i<j}\|A_i+A_j\| \leq \sum_{i=1}^{N}\|A_i\|.$

(2) $\displaystyle\|\sum_{i=1}^{N} A_i\| + (N-2)\sum_{i=1}^{N}\|A_i\| \geq \sum_{i<j}\|A_i+A_j\|.$

(3) $\displaystyle\frac{1}{N-2}\sum_{i<j}\|A_i+A_j\| - \frac{1}{N-2}\|\sum_{i=1}^{N} A_i\| \geq \frac{1}{N-1}\sum_{i<j}\|A_i+A_j\|$

$$\geq \max\left\{\frac{1}{N-2}\sum_{i<j}\|A_i+A_j\| - \frac{1}{N-2}\sum_{i=1}^{N}\|A_i\|, \|\sum_{i=1}^{N}A_i\|\right\}.$$

(4) $\frac{1}{2(N-1)}\left(\sum_{i<j}\|A_i+A_j\|+\sum_{i<j}\|A_i-A_j\|\right) \leq \sum_{i=1}^{N}\|A_i\| \leq \frac{1}{N-1}\left(\sum_{i<j}\|A_i+A_j\|+\sum_{i<j}\|A_i-A_j\|\right).$

(5) $\|\sum_{i=1}^{N}A_i\|^2 + (N-2)\sum_{i=1}^{N}\|A_i\|^2 = \sum_{i<j}\|A_i+A_j\|^2.$

(6) $\sum_{i=1}^{N}\|A_i\|^2 \leq \frac{1}{N}\left(\sum_{i<j}\|A_i-A_j\|^2 + \left(\frac{1}{N-1}\sum_{i<j}\|A_i+A_j\|\right)^2\right).$

(7) $\sum_{i=1}^{N}\|A_i\|^2 \geq \frac{1}{N}\left\{\left(\frac{1}{N-1}\sum_{i<j}\|A_i+A_j\|\right)^2 + \left(\frac{1}{N-1}\sum_{i<j}\|A_i-A_j\|\right)^2\right\}.$

Lemma 3.1 *Lemma 3.1 (4) is refined by the following.*

$$\begin{aligned}\frac{1}{N-1}\sum_{i<j}\max\{\|A_i+A_j\|, \|A_i-A_j\|\} &\leq \sum_{i=1}^{N}\|A_i\| \\ \leq \quad \frac{1}{2(N-1)}\sum_{i<j}(\|A_i+A_j\|+\|A_i-A_j\|) &+ \frac{1}{N-1}\sum_{i<j}\min\{\|A_i\|, \|A_j\|\}.\end{aligned}$$

Then Theorem 3.4 (3) is given by the stronger inequality

$$\begin{aligned}\frac{1}{N-1}\sum_{i<j}\max\{\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)}, \sqrt{I_{A,B}^{(g,f)}(X_i-X_j)}\} &\leq \sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i)} \\ &\leq \frac{1}{2(N-1)}\sum_{i<j}(\sqrt{I_{A,B}^{(g,f)}(X_i+X_j)}+\sqrt{I_{A,B}^{(g,f)}(X_i-X_j)}) \\ &\quad + \frac{1}{N-1}\sum_{i<j}\min\{\sqrt{I_{A,B}^{(g,f)}(X_i)}, \sqrt{I_{A,B}^{(g,f)}(X_j)}\}.\end{aligned}$$

Theorem 3.5 *For* $\{X_i\}_{i=1}^{N} \in M_n(\mathbb{C}), A, B \in M_{n,+}(\mathbb{C})$*, the followings hold.*

(1) $\sum_{i=1}^{N}\sqrt{I_{A,B}^{(g,f)}(X_i)} \leq \frac{\sqrt{2}}{N-1}\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i \pm X_j)}\left\{\frac{\sqrt{I_{A,B}^{(g,f)}(X_i)I_{A,B}^{(g,f)}(X_j)}}{\sqrt{I_{A,B}^{(g,f)}(X_i)I_{A,B}^{(g,f)}(X_j)} \pm \mathrm{Re}\left\{\Gamma_{A,B}^{(g,f)}(X_i, X_j)\right\}}\right\}^{1/2}.$

(2) $\sum_{i=1}^{N}I_{A,B}^{(g,f)}(X_i) \leq \frac{2}{N-1}\sum_{i<j}\sqrt{I_{A,B}^{(g,f)}(X_i)I_{A,B}^{(g,f)}(X_j)}\left\{\frac{I_{A,B}^{(g,f)}(X_i \pm X_j)}{\sqrt{I_{A,B}^{(g,f)}(X_i)I_{A,B}^{(g,f)}(X_j)} \pm \mathrm{Re}\left\{\Gamma_{A,B}^{(g,f)}(X_i, X_j)\right\}} - 1\right\}.$

Proof (1) For $X, Y \in M_n(\mathbb{C})$,

$$\Gamma_{A,B}^{(g,f)}(X,Y) = \langle X, Y\rangle, \quad \sqrt{I_{A,B}^{(g,f)}(X)} = \|X\|.$$

Then by the equality

$$\|\frac{X_i}{\|X_i\|} \pm \frac{X_j}{\|X_j\|}\| = \sqrt{2}\sqrt{1 \pm \frac{\mathrm{Re}\langle X_i, X_j\rangle}{\|X_i\|\|X_j\|}}$$

and Dunkl-Williams inequality

$$\|X_i\| + \|X_j\| \leq \frac{2\|X_i \pm X_j\|}{\|\frac{X_i}{\|X_i\|} \pm \frac{X_j}{\|X_j\|}\|},$$

$$\begin{aligned}\sum_{i=1}^{N} \|X_i\| &= \frac{1}{N-1}\sum_{i<j}(\|X_i\| + \|X_j\|) \leq \frac{2}{N-1}\sum_{i<j}\frac{\|X_i \pm X_j\|}{\|\frac{X_i}{\|X_i\|} \pm \frac{X_j}{\|X_j\|}\|} \\ &= \frac{\sqrt{2}}{N-1}\sum_{i<j}\frac{\|X_i \pm X_j\|}{\sqrt{1 \pm \frac{\mathrm{Re}\langle X_i, X_j\rangle}{\|X_i\|\|X_j\|}}} = \frac{\sqrt{2}}{N-1}\sum_{i<j}\frac{\|X_i \pm X_j\|\sqrt{\|X_i\|\|X_j\|}}{\sqrt{\|X_i\|\|X_j\| \pm \mathrm{Re}\langle X_i, X_j\rangle}}.\end{aligned}$$

(2) By the equality

$$\|\frac{X_i}{\|X_i\|} \pm \frac{X_j}{\|X_j\|}\|^2 = 2\left\{1 \pm \frac{\mathrm{Re}\langle X_i, X_j\rangle}{\|X_i\|\|X_j\|}\right\}$$

and the second power of Dunkl-Williams inequality,

$$\|X_i\|^2 + \|X_j\|^2 \leq \frac{4\|X_i \pm X_j\|^2}{\|\frac{X_i}{\|X_i\|} \pm \frac{X_j}{\|X_j\|}\|^2} - 2\|X_i\|\|X_j\|,$$

$$\begin{aligned}\sum_{i=1}^{N} \|X_i\|^2 &= \frac{1}{N-1}\sum_{i<j}(\|X_i\|^2 + \|X_j\|^2) \leq \frac{1}{N-1}\sum_{i<j}\left\{\frac{4\|X_i \pm X_j\|^2}{\|\frac{X_i}{\|X_i\|} \pm \frac{X_j}{\|X_j\|}\|^2} - 2\|X_i\|\|X_j\|\right\} \\ &= \frac{1}{N-1}\sum_{i<j}\left\{\frac{2\|X_i \pm X_j\|^2}{1 \pm \frac{\mathrm{Re}\langle X_i, X_j\rangle}{\|X_i\|\|X_j\|}} - 2\|X_i\|\|X_j\|\right\} = \frac{2}{N-1}\sum_{i<j}\|X_i\|\|X_j\|\left\{\frac{\|X_i \pm X_j\|^2}{\|X_i\|\|X_j\| \pm \mathrm{Re}\langle X_i, X_j\rangle} - 1\right\}.\end{aligned}$$

□

Remark 3.2 *Theorem 3.1 (3), Theorem 3.4 (3), (4) and Theorem 3.5 (1), (2) are considered as the reverse inequalities of sum types of uncertainty relations.*

4. Result

The product types of uncertainty relations for generalized quasi-metric adjusted skew informations are obtained in Theorem 2.1. The sum types of uncertainty relations for generalized quasi-metric adjusted skew informations are obtained in Theorem 3.1, Theorem 3.2, Theorem 3.3 and Theorem 3.4. The reverse inequalities of the sum types of uncertainty relations are obtained in Theorem 3.5.

5. Discussion

Let A, B, g, f, k, ℓ be taken by the following examples;

$$A = B = \rho \in M_{n,+,1}(\mathbb{C}), g = f_{SLD},$$
$$f = f_{WYD}, k = \frac{f(0)}{2}, \ell = 2.$$

Then product types of uncertainty relations for Wigner-Yanase-Dyson skew informations are obtained. So the results are most general in all of previous results. For sum types of uncertainty relations for generalized quasi-metric addjusted skew informations it is important to obtain different types of uncertanty relations which modify the results obtained in this paper. Though it is easy to calculate some fomulas in the case of Hilbertian norms, it is difficult to do in the case of general norms. Then the refinement of triangle inequality is needed to give the new inequality. Also the reverse type of triangle inequality is important to get the different type of Dunkl-William inequality.

6. Conclution

All of results obtained in this paper are most general inequalities representing product or sum types uncertainty relations. Almost all of uncertainty relations obtained ago are in the case of pure states and the results can be given by simple version of our results. They are obtained as corollaries of the main theorems in this paper by taking the values of A, B, f, g, k, ℓ concletely. The product types of uncertainty relations can be obtained by using Schwarz's inequality and the sum types of uncertainty relations can be obtained by using refined properties of norms or square norms.

Appendix

Proof of Lemma 3.1 (1) In general, for $\{x_i\}_{i=1,\cdots,N} \subset \mathbb{R}$

$$\begin{aligned}\sum_{i=1}^{N} x_i &= \frac{1}{2}(\sum_{i=1}^{N} x_i + \sum_{j=1}^{N} x_j) = \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i + x_j) = \frac{1}{2N}\{2\sum_{i=1}^{N} x_i + \sum_{i\neq j}(x_i + x_j)\} \\ &= \frac{1}{N}\sum_{i=1}^{N} x_i + \frac{1}{2N}\{\sum_{i<j}(x_i + x_j) + \sum_{i>j}(x_i + x_j)\} = \frac{1}{N}\sum_{i=1}^{N} x_i + \frac{1}{N}\sum_{i<j}(x_i + x_j).\end{aligned}$$

Then

$$\left(1 - \frac{1}{N}\right)\sum_{i=1}^{N} x_i = \frac{1}{N}\sum_{i<j}(x_i + x_j).$$

Therefore

$$\sum_{i=1}^{N} x_i = \frac{1}{N-1}\sum_{i<j}(x_i + x_j).$$

Let $x_i = \|A_i\|,\ i = 1, 2, \cdots, N$.

$$\sum_{i=1}^{N} \|A_i\| = \frac{1}{N-1}\sum_{i<j}(\|A_i\| + \|A_j\|) \geq \frac{1}{N-1}\sum_{i<j}\|A_i + A_j\|.$$

By summarized in both sides,

$$\|\sum_{i=1}^{N} A_i\| = \|\frac{1}{N-1}\sum_{i<j}(A_i + A_j)\| \leq \frac{1}{N-1}\sum_{i<j}\|A_i + A_j\|.$$

By combing the above two equalities, (1) is given.

(2) It is clear by Hlawka's inequality. ([30, 31])

(3) It is clear by $\|\sum_{i=1}^{N} A_i\| \leq \frac{1}{N-1}\sum_{i<j}\|A_i + A_j\| \leq \sum_{i=1}^{N}\|A_i\|$ in (1).

(4) By (2),

$$\|A_i\| + \|A_j\| \leq \|A_i + A_j\| + \|A_i - A_j\| \leq 2(\|A_i\| + \|A_j\|).$$

Then

$$\sum_{i<j}(\|A_i\| + \|A_j\|) \leq \sum_{i<j}\|A_i + A_j\| + \sum_{i<j}\|A_i - A_j\| \leq 2\sum_{i<j}(\|A_i\| + \|A_j\|).$$

Since

$$\sum_{i=1}^{N}\|A_i\| = \frac{1}{N-1}\sum_{i<j}(\|A_i\| + \|A_j\|),$$

$$\frac{1}{2(N-1)}(\sum_{i<j}\|A_i + A_j\| + \sum_{i<j}\|A_i - A_j\|) \leq \sum_{i=1}^{N}\|A_i\| \leq \frac{1}{N-1}(\sum_{i<j}\|A_i + A_j\| + \sum_{i<j}\|A_i - A_j\|).$$

(5) By $\|A_i + A_j\|^2 = \langle A_i + A_j | A_i + A_j\rangle = \|A_i\|^2 + \langle A_i|A_j\rangle + \langle A_j|A_i\rangle + \|A_i\|^2$,

$$\sum_{i,j}\|A_i + A_j\|^2 = N\sum_{i=1}^{N}\|A_i\|^2 + 2\sum_{i,j}\langle A_i|A_j\rangle + N\sum_{i=1}^{N}\|A_i\|^2.$$

And

$$\sum_{i,j} \|A_i + A_j\|^2 = 4\sum_{i=1}^{N} \|A_i\|^2 + 2\sum_{i<j} \|A_i + A_j\|^2.$$

By combing the above two equalities,

$$\sum_{i<j} \|A_i + A_j\|^2 = (N-2)\sum_{i=1}^{N} \|A_i\|^2 + \|\sum_{i=1}^{N} A_i\|^2.$$

(6) By the same equalities as (5),

$$\sum_{i<j} \|A_i - A_j\|^2 = N\sum_{i=1}^{N} \|A_i\|^2 - \|\sum_{i=1}^{N} A_i\|^2.$$

Then

$$\sum_{i=1}^{N} \|A_i\|^2 = \frac{1}{N}\left(\sum_{i<j} \|A_i - A_j\|^2 + \|\sum_{i=1}^{N} A_i\|^2\right) \leq \frac{1}{N}\left(\sum_{i<j} \|A_i - A_j\|^2 + \left(\frac{1}{N-1}\sum_{i<j} \|A_i + A_j\|\right)^2\right).$$

(7) $\frac{2}{N(N-1)}\sum_{i<j} \|A_i \pm A_j\|^2 \geq \left(\frac{2}{N(N-1)}\sum_{i<j} \|A_i \pm A_j\|\right)^2.$

That is,

$$\sum_{i<j} \|A_i \pm A_j\|^2 \geq \frac{2}{N(N-1)}\left(\sum_{i<j} \|A_i \pm A_j\|\right)^2.$$

Hence

$$\begin{aligned}\sum_{i=1}^{N} \|A_i\|^2 &= \frac{1}{2(N-1)}\left\{\sum_{i<j} \|A_i + A_j\|^2 + \sum_{i<j} \|A_i - A_j\|^2\right\} \\ &\geq \frac{1}{N}\left\{\left(\frac{1}{N-1}\sum_{i<j} \|A_i + A_j\|\right)^2 + \left(\frac{1}{N-1}\sum_{i<j} \|A_i - A_j\|\right)^2\right\}.\end{aligned}$$

□

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