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Continuous Mean - Variance Portfolio Problem Is Studied with Time Delay Using Stochastic LQ Control Theory

Yuquan Cui*, Linlin Li, Cong Liu

School of Mathematics, Shandong University, Jinan, P.R. of China

Email address

cuiyq@sdu.edu.cn (Yuquan Cui), lilinlinwfzc@163.com (Linlin Li),
1378962120@qq.com (Cong Liu)

*Corresponding author

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Abstract

The mean - variance portfolio selection model based on the expectation and variance of return on assets to measure the expected return and risk of investment. Due to the financial sector complicated variety of events, each financial problems from changes to know its essence, the change rule, from the change of strategy to formulate relevant policy and policy into effect, etc, the process inevitably has a certain lag. Therefore, in order to better reflect the actual situation, we study the portfolio model with delays in this paper. By joining our delay control item, the optimization model was established, the goal is to maximize earnings expectations. In this paper, it studies the continuous time without delay the mean - variance portfolio problems on the basis of existing research. It established auxiliary problem using the stochastic linear quadratic optimal control theory. Using the maximum principle, the solution of the optimal investment strategy are given and it analysis the case, the conclusion is in conformity with the actual. It studies the existing time delay portfolio strategy problem in discrete time case. Based on the stochastic LQ optimal control theory, it established the discrete mean - variance time model with time delay. The paper has carried on the solution and example analysis. And according to the maximum principle, the optimal control model of the general form of the input delay stochastic LQ problem are obtained. The final result shows that when the delay is zero, the results is the same as the model without time delay

1. Introduction

Since China's reform and opening up, China's capital market unceasing development and perfection, has been basically mature. China's capital market play a role of the financing to raise asset allocation for many of China's listed companies, and also make the public investment fans involved in the economic development, share the achievements of economic development; Especially with the popularity of Internet plus, more and more online financial products such as Yu Ebao And Shou Yibao, earnings treasure emerge in endlessly. It makes more and more investors financial management idea change, investment diversification. Due to the investment for risky assets, it needs to solve two major problems: expected return and risk, therefore, investors need to consider how to measure the risks and benefits of investment portfolio and how to balance these two indicators in the capital markets, and then invest in asset allocation. According to the two main problem, the portfolio investment theory was put forward by the American scholar Markowitz in the 1950 s and 1960 s. The theoretical basis for the following assumptions: 1) the investors consider every investment options according to probability distribution of stock returns to on a certain position time; 2) investors estimate the risk of portfolio according to the expected return of securities; 3) investors make a decision only on the basis of risks and benefits of securities; 4) the investor's purpose is to choose the portfolio on a certain level of risk, expected revenue is the largest, or is in a certain income level, the expected risk is minimal. Based on the above assumptions, Markowitz established the calculation method of Portfolio expected return and risk and efficient frontier theory. He published paper in the Journal of Finance of Portfolio selection in 1952, it put forward the expectation and variance of return on assets to measure the expected return and risk of investment, the mean-variance Portfolio selection model is established and opened the prelude to the development of modern financial theory.

Markowitz gives the numerical solution of mean-variance model in the market of short sales restrictions. Then many scholars has carried on the promotion and development on this basis. When markets allow short-selling, Merton [2] and Szego [3] are given the model of single phase that does not contain the risk-free asset and contain the risk-free asset, in that situation it analysis the solution of the optimal portfolios and efficient frontiers. Among them, it mainly use Lagrange multiplier method to do that in the situation of does not contain risk-free assets. In the case of risk-free assets, it mainly use iterative method and genetic algorithm to do that, etc. See [4]. In the case of not allow short-selling, it can be solved by numerical method. Such as, Wen-jing Guo [5] gives a method of neural network algorithm, De-quan Yang [6] gives a range search method to solve the efficient portfolio frontier, etc. Zhu-wu wu [7] analysis the effective frontier drift in the situation of joining effective securities or eliminating invalid securities cases. Markowitz established the mean - variance model, using variance to measure the investment risk. The advantage is that the model is easy to understand, the relationship between the returns and risk can be directly said, but the variance deal with excess returns that higher than the average as risk, obviously it is not reasonable. Therefore, many scholars consider the second half of the variance is used to measure risk, Markowitz [8] study the average - the second half of the variance model. On average - the second half of the variance combination optimization problem, O. L. V. Costa presents a linear matrix Inequality method, Enrique Ballestero [10] study the effective frontier of the average - the second half of the variance problem.

As capital markets increasingly complex, single stage portfolio model cannot satisfy the actual needs, in order to obtain high yield, investors tend to adjust investment strategy according to the change of market environment, the multi-stage investment, the resulting the multi-stage mean variance portfolio model. F. C. Jen, S. Zionts [11] study found that compared with the single phase model, objective function in the multi-phase planning model contains Integral item $(EX_T)^2$ which has dynamic planning sense, make model is unable to get analytic solution. The following nearly 30 years, dynamic variance model studies were not made greater progress. Until 1998, D. L I, l. Wan [12] used maximize the expected utility function method to solve a mean - variance model, That is the biggest $E[U(X(T))]$, U is the utility function, can be a form of quadratic form, the logarithmic form or index, etc., thus embedded in the question to another problem that can be solved by dynamic programming method, this is multi-phase mean - variance problem study of major breakthrough. In addition, Yao Haixiang [13] study the of multi-stage mean variance portfolio selection problem in the risky asset return on related cases. M. V. A raujo [14] study the parameter multi-phase mean - variance model in the market in line with the random market state transition process cases, get effective strategy is a closed set of conclusions, and the optimal strategy by solving a set of coupled Riccati difference equation.

If market transactions is continuous, investors can adjust investment strategy at any point in time, it is called a continuous time mean - variance model, this is a multi-stage model for further extension. G. Yin, X. Zhou [15] reveals the discrete time mean - variance model and the nature of continuous time variance model, makes the study of multiphase portfolio naturally transition to the continuous time model. Capital market is full of randomness and uncertainty, how to seek the optimal strategy in the random environment become a research focus in the academic circles. X. Zhou [16] in a random linear quadratic optimal control theory for the tool, through the establishment of auxiliary problem, studies the continuous variance optimization problem under the complete market. This is the first time to embed the variance problem solving stochastic LQ optimal control problem, for subsequent use of the optimal control problem to solve the problem of portfolio laid a foundation. X. Zhou [17] gives the exact form of continuous time mean - variance model efficient frontiers. A. E. B. Lim, X. Zhou [18] and W. Guo [19] studies the investment decision making problems in the random market parameters, respectively, considering the stock prices have two kinds of situations of jumped and continuous, also using the stochastic LQ optimal control and backward stochastic differential equation method, getting the analytical form of effective investment strategies and efficient frontiers. X. Li, X. Zhou, A. E. B. Lim [20] studies the mean - variance optimal combination problem In case of not allow short-selling. For the continuous-time portfolio selection optimization problem in the case of liabilities, S. Xie [21] gives the corresponding research results. L. Liu [22] considering the optimal portfolio selection problem in case of liabilities and not allow short-selling restrictions. In addition, R. Bielecki [23] and X. Zhou [24] respectively studied continuous time mean - variance optimal combination problem in case of not allowed to fail on the complete market and the market under the state transition. Yunhui Xu, zhongfei Li [25] takes a different approach to study the conclusions that got by solving the model in case of yield sequence related. Since the mean-variance model only considering the investors' investment behavior, without considering the consumption behavior of investors and the influence of consumption to investment, it's inspired researchers through the utility function combined investors' investment behavior and consumer behavior. The purpose of the investors is pursuit of consumer utility and eventually the expected utility of wealth is the largest. Merton [28] [29] studied the optimal consumption investment strategy in the case of continuous time, investment spending under stochastic interest rate environment, investment spending under stochastic volatility model systematically, such as specific see S. H. Wang [26], etc.

At present, the research of time-delay stochastic LQ optimal control problem has made great progress and gets some effective method. For example, L. Chen, Z. Wu [27] gave maximum principle and its application of time-delay stochastic optimal control problem. X. Song, H. Zhang, L. Xie [28] studied for a class of discrete-time stochastic systems with input delay and satisfy the conditions of the optimal controller is given, and got the result of the linear quadratic control, Juanjuan Xu [29] studied continuous-time linear systems with input time-delay optimal control problem systematically. She get the necessary and sufficient condition for existence and uniqueness of the optimal solution and the display solution of the optimal controller based on the stochastic maximum principle. H. Zhang, L. Li [30] studied LQ optimal control problem with multiplicative noise and Input delay discrete time stochastic systems and get the display solution of the optimal controller based on the discrete stochastic maximum principle. More research about stochastic control are S. Chen, X. Li, X. Zhou [31], C. Li [32], etc.

Due to the Complexity variability of financial sector and financial event, the releasing and spread of policies and message need a certain amount of time, which makes a time lag when investors adjust the investment strategy. In order to better reflect the actual situation, in this paper, we study portfolio model with time delay in order to realize the expected utility maximization goals. In theory, research mean - variance portfolio problem with time-delay with the help of LQ optimal control method with input delay in control theory. On the one hand, adding time delay widen the depth of the mean - variance portfolio problem, which makes the model more close to the actual situation and improves the theoretical model applied in the real problems. On the other hand, using the control theory research results to solve the problem of financial theory, which makes control theory having a new place and also inspires further research. On the basis of existing research, this paper presents continuous time mean-variance model without time delay, through establishing of auxiliary problem, the original problem is transformed into LQ control problem, and then use stochastic LQ optimal control method to get the analytic solution of the optimal portfolio strategy. According to multi-phase mean-variance portfolio model, established continuous time variance model with time-delay, gave the solution of general form continuous LQ optimal control problem with time delay, obtained the optimal portfolio strategy applying the maximum principle.

2. Continuous Time Variance Model with Time Delay

Front part of the study is discrete systems mean-variance portfolio model with time-delay, this part considers that the market is continuous changes over time, investors can adjust the investment strategy at any point in time. The next we study continuous systems mean-variance optimal portfolio problem time-delay by the continuous linear quadratic optimal control with time delay.

2.1. Description of Model

Then consider continuous systems mean-variance portfolio model with time-delay, the wealth $x(t)$ of investors at t time:

$$\begin{cases} dx(t) = \left\{ r(t)x(t) + \sum_{i=1}^{m} [b_i(t) - r(t)]u_i(t-h) \right\} dt \\ \quad + \sum_{j=1}^{m} \sum_{i=1}^{m} \sigma_{ij}(t)u_i(t-h)dw^j(t) \\ x(0) = x_0 \succ 0 \end{cases} \tag{1}$$

Modeled on the front part of the structure of auxiliary problem method, make that

$$\gamma = \frac{\lambda}{2\mu}, y(t) = x(t) - \gamma \tag{2}$$

The continuous mean-variance model with input time delay can be represented as:

$$\begin{cases} \min E[\frac{1}{2}\mu y(T)^2] \\ dy(t) = [r(t)y(t) + (b(t) - r(t))u(t-h) + \gamma r(t)]dt + \sum_{j=1}^{m} \sum_{i=1}^{m} \sigma_{ij}(t)u_i(t-h)dw^j(t) \\ y(0) = x_0 - \gamma \end{cases} \tag{3}$$

2.2. Continuous System Stochastic Maximum Principle with Time-Delay

Consider stochastic systems with time-delay:

$$dx(t) = [A(t)x(t) + B(t)u(t-h) + f)dt + [\overline{A}(t)x(t) + \overline{B}u(t-h) + \overline{f}]dw(t) \tag{4}$$

$$x(0) = x_0, u(\tau) = \mu(\tau), \tau \in [-h, 0)$$

$x(t) \in R^n$ is the state, $u(t) \in R^m$ is control, $w(t)$ is a one-dimensional standard Brownian motion, $h \succ 0$ represents the input delay, T is the constant matrix of appropriate dimension. Remember $F_t = \{w_s, 0 \le s \le t\}$ is no less σ - algebra generated by Brownian motion, then $F_{t1} \subseteq F_{t2}, t1 \le t2$.

The corresponding quadratic value function is:

$$J_T(u(.)) = \frac{1}{2} E[\int_0^T [x(t)'Q(t)x(t) + \int_h^T u(t-h)'R(t)u(t-h)]dt + x(T)'P(T)x(T) \tag{5}$$

This problem can be described as: the controller $u(t-h)$ of F_{t-h} measurable, make the minimum of J_T.

Lemma 2.1 the problem of the optimal controller satisfy that:

$$0 = E[B'p(t+h) + \overline{B}'q(t+h)\big|F_t] + Ru(t) \tag{6}$$

The $E[.\big|F_t]$ said about conditional expectation of F_t, $(p(t), q(t))$ is the solution of following backward stochastic differential equations:

$$\begin{cases} dp(t) = -[A'p(t) + \overline{A}'q(t) + Qx(t)]dt + q(t)dw(t) \\ p(T) = P(T)x(T) \end{cases} \tag{7}$$

Proof: Assume that feasible control field is convex, make that $u(t), \delta u(t) \in U$, then take $\varepsilon \in [0,1]$, we can get $u^{\varepsilon} = u(t) + \varepsilon\delta u(t) \in U$, $x^{\varepsilon}, J^{\varepsilon}$ are corresponding state and performance index; x, J are the corresponding status indicators of $u(t)$. Y = 1, then δx satisfy stochastic differential equations as follows:

$$\begin{cases} d\delta x(t) = (A\delta x(t) + B\varepsilon\delta u(t))ds + (C\delta x(t) + D\varepsilon\delta u(t))dw_t \\ \delta x(0) = 0 \end{cases} \tag{8}$$

The solution of (8) can be expressed as

$$\delta x(t) = \Phi(t)\int_0^t \Phi(\tau)^{-1}(B - CD)\varepsilon\delta u(\tau)d\tau + \Phi(t)\int_0^t \Phi(\tau)^{-1} D\varepsilon\delta u(\tau)dw(\tau) \tag{9}$$

The following equation for $\Phi(t)$:

$$\begin{cases} d\Phi(t) = A\Phi(t)dt + C\Phi(t)dw(t) \\ \Phi(0) = \mathrm{I} \end{cases} \tag{10}$$

The following equation for $\Phi(t)^{-1} = \Psi(t)$:

$$\begin{cases} d\Psi(t) = \Psi(t)(-A + C^2)dt - \Psi(t)Cdw(t) \\ \Psi(0) = \mathrm{I} \end{cases} \tag{11}$$

Using the Taylor expansion, we can get that

$$2J_T^{\varepsilon} - 2J_T$$

$$= E[\int_0^T x(t)'Q\delta x(t)dt + \int_0^T u(t)'R\varepsilon\delta u(t)dt + x(T)'P(T)\delta x(T)] + O(\varepsilon^2)$$

$$= E\left\{\int_0^T x(t)'Q[\Phi(t)\int_0^t \Phi^{-1}(\tau)(B-CD)\varepsilon\delta u(\tau)d\tau + \Phi(t)\int_0^t \Phi(\tau)^{-1}D\varepsilon\delta u(\tau)dw(\tau)]dt + \int_0^T u(t)'R\varepsilon\delta u(t)dt\right.$$

$$\left. + x(T)'P(T)[\Phi(T)\int_0^T \Phi^{-1}(\tau)(B-CD)\varepsilon\delta u(\tau)d\tau + \Phi(T)\int_0^t \Phi(\tau)^{-1}D\varepsilon\delta u(\tau)dw(\tau)]\right\} + O(\varepsilon^2)$$

$$= E\left\{\int_0^T [\int_\tau^T x(t)'Q\Phi(t)dt\Phi^{-1}(\tau)(B-CD) + u(\tau)'R + x(T)'P(T)[\Phi(T)\Phi^{-1}(\tau)(B-CD)\varepsilon\delta u(\tau)d\tau\right.$$

$$\left. + \int_0^T x(t)'Q\Phi(t)dt\int_0^T \Phi(\tau)^{-1}D\varepsilon\delta u(\tau)dw(\tau) + x(T)'P(T)\Phi(T)\int_0^T \Phi^{-1}(\tau)D\varepsilon\delta u(\tau)dw(\tau)\right\} + O(\varepsilon^2)$$

$$= E\left\{\int_0^T (u(\tau)'R + [\int_\tau^T x(t)'Q\Phi(t)dt + x(T)'P(T)\Phi(T)] + \Phi^{-1}(\tau)(B-CD)\varepsilon\delta u(\tau)d\tau + [\int_0^T x(t)'Q\Phi(t)dt\right.$$

$$\left. + x(T)'P(T)\Phi(T)]\int_0^T \Phi^{-1}(\tau)D\varepsilon\delta u(\tau)dw(\tau)\right\} + O(\varepsilon^2) \tag{12}$$

Make that $\zeta = \int_0^T x(t)'Q\Phi(t)dt + x(T)'P(T)\Phi(T)$, $E[\zeta|F_\tau]$ is a martingale about the time variable τ apparently, therefore, by the expression of martingale theorem, existing only F_t measurable process η_t satisfy that:

$$E[\zeta|F_\tau] = E[\zeta] + \int_0^\tau \eta(t)'dw(t)$$

In particular, when $\tau = T$, there are

$$\zeta = E[\zeta|F_T] = E[\zeta] + \int_0^T \eta(t)'dw(t) \tag{13}$$

Put (13) into (12), we can get that:

$$2J_T^\varepsilon - 2J_T = E\left\{\int_0^T \{u(\tau)'R + \int_\tau^T [x(t)'Q\Phi(t)dt + x(T)'P(T)\Phi(T)] \times \Phi^{-1}(\tau)(B-CD)\}\varepsilon\delta u(\tau)d\tau\right.$$

$$\left. + E[\zeta]\int_0^T \Phi^{-1}(\tau)D\varepsilon\delta u(\tau)dw(\tau) + \int_0^T \eta(t)'dw(t)\int_0^T \Phi(\tau)^{-1}D\varepsilon\delta u(\tau)dw(\tau)\right\} + O(\varepsilon^2)$$

$$= E\left\{\int_0^T \{u(\tau)'R + [\int_\tau^T x(t)'Q\Phi(t)dt + x(T)'P(T)\Phi(T)] \times \Phi^{-1}(\tau)(B-CD)\}\varepsilon\delta u(\tau)d\tau + \int_0^T \eta(\tau)'\Phi(\tau)^{-1}D\varepsilon\delta u(\tau)d\tau\right\} + O(\varepsilon^2)$$

$$= E\left\{\int_0^T \{u(\tau)'R + [\int_\tau^T x(t)'Q\Phi(t)dt + x(T)'P(T)\Phi(T)]\Phi(\tau)^{-1}(B-CD) + \eta(\tau)'\Phi(\tau)^{-1}D)\varepsilon\delta u(\tau)\}d\tau\right\} + O(\varepsilon^2) \tag{14}$$

Denote

$$G(\tau)' = u(\tau)'R + [\int_\tau^T x(t)'Q\Phi(t)dt + x(T)'P(T)\Phi(T)]\Phi(\tau)^{-1}(B-CD) + \eta(\tau)'\Phi(\tau)^{-1}D$$

(14) can be represented as:

$$2J_T^\varepsilon - 2J_T = E\int_0^T G(\tau)'\varepsilon\delta u(\tau)d\tau + O(\varepsilon^2) = E\int_0^T E[G(\tau)'|F_\tau]\varepsilon\delta u(\tau)d\tau + O(\varepsilon^2) \tag{15}$$

Due to $\delta u(\tau)$ is an arbitrary, the necessary conditions of value function J_T is:

$$E[G(\tau)'|F_\tau] = 0$$

That is

$$0 = Ru(\tau) + E[(B-CD)'\Phi(\tau)'^{-1}[\int_\tau^T \Phi(t)'Qx(t)dt + \Phi(\tau)'^{-1}\Phi(T)'P(T)x(T)] + D'\Phi(\tau)'^{-1}\eta(\tau)|F_\tau] \tag{16}$$

Denote

$$p(\tau)=E[\Phi(\tau)^{'-1}[\int_{\tau}^{T}\Phi(t)^{'}Qx(t)dt+\Phi(\tau)^{'-1}\Phi(T)^{'}P(T)x(T)]\big|F_{\tau}]$$

$$q(\tau)=-Cp(\tau)+\Phi(\tau)^{'-1}\eta(\tau)$$

By the formula $It\hat{o}$, $(p(\tau),q(\tau))$ satisfy backward differential equation, so the (16) can be written as:

$$0=Ru(\tau)+E[B^{'}p(\tau)+D^{'}q(\tau)\big|F_{\tau}] \tag{17}$$

That is lemma 2.1 is tenable. □

2.3. Solution of the General Continuous LQ Problem with Time Delay

Review the situation without delay, that is $h=0$, assume that $p(t)=P(t)x(t)+n(t)$, $P(t)$ satisfies a stochastic Riccati equation, $n(t)$ satisfies the equation. Based on the $P(t)$, we can get the analytical solution of the optimal control. Because of time lag, this relationship was not established, we need to establish a new relationship in this section.

Define the coupled Riccati equations as follows:

$$-\dot{P}(t)=P(t)+A^{'}P(t)+Q+\overline{A}^{'}P(t)\overline{A}-\Pi(t,t+h) \tag{18}$$

$$-\dot{n}(t)=P(t)f+A^{'}n(t)+\overline{A}^{'}P(t)\overline{f}-[P(t)B(t)+\overline{A}^{'}P(t)\overline{B}]\Omega^{-1}(t)H(t) \tag{19}$$

Have the following relations

$$\Pi(t,s)=e^{A^{'}(s-t)}\Pi(s,s)e^{A(s-t)},s\in[t,t+h] \tag{20}$$

$$\Omega(t)=R+\overline{B}^{'}P(t)\overline{B} \tag{21}$$

$$K(t)=-\Omega^{-1}(t)[B^{'}P(t)+\overline{B}^{'}P(t)\overline{A}-B^{'}\int_{t}^{t+h}\Pi(t,s)ds] \tag{22}$$

$$\Pi(t,t)=K^{'}(t)\Omega(t)K(t) \tag{23}$$

$$H(t)=B^{'}n(t)+\overline{B}^{'}P(t)\overline{f} \tag{24}$$

Terminal value $\Pi(T,T+\theta)=0$, $P(T)$ was given earlier and $n(T)=0$.

Based on the above equation, assuming $p(t)$ and $x(t)$ satisfy the following relations:

$$p(t)=P(t)x(t)-\int_{0}^{h}\Pi(t,t+h)\hat{x}(t\big|t+\theta-h)d\theta+n(t),t\geq h \tag{25}$$

Among them $\hat{x}(t|s)$ satisfies the following relations

$$\hat{x}(t\big|s)=E(x(t)\big|F_{s})=e^{A(t-s)}x(s)+\int_{s}^{t}e^{A(t-\theta)}Bu(\theta-h)d\theta,t-h\leq s\leq t$$

Hypothesis 2. Any $h\leq t\leq T,\Omega_{t}\succ 0$

Theorem 2.1 in view of the above problem, that is researching the F_{t-h} measurable controller $u(t-h)$, makes the value function J_{T} is minimum, under the premise of satisfies the assumption 2, the optimal controller is

$$u(t-h)=K(t)\hat{x}(t\big|t-h)-\Omega^{-1}(t)H(t) \tag{26}$$

Among them

$$\hat{x}(t\big|t-h) = e^{Ah}x(t-h) + \int_{t-h}^{t} e^{A(t-\theta)}Bu(\theta-h)d\theta \tag{27}$$

And $\Omega(t), K(t), H(t)$ are given by (21), (22) and (24) respectively.

Proof: make that $\Theta_t = -\int_0^h \Pi(t,t+h)\hat{x}(t\big|t+\theta-h)d\theta$ then

$$d\Theta_t = -\Big\{A'\Theta_t + \Pi(t,t+h)x(t) - [P(t)B + \overline{A}'P(t)\overline{B}]\Omega^{-1}(t)[B'P(t)$$

$$+\overline{B}'P(t)\overline{A}]\hat{x}(t\big|t-h) - [P(t)B + \overline{A}'P(t)\overline{B}]\Omega^{-1}(t)E[B'\Theta_t\big|F_{t-h}]\Big\}dt \triangleq \Lambda dt \tag{28}$$

To (25), using the formula $It\hat{o}$, we can get that:

$$dp(t) = d(P(t)x(t)) + d(-\int_0^h \Pi(t,t+h)\hat{x}(t\big|t+\theta-h)d\theta) + \dot{n}(t)dt$$

$$= \dot{P}(t)x(t)dt + P(t)dx(t) + \Lambda dt + \dot{n}(t)dt$$

$$= \Big\{\dot{P}(t)x(t) + P(t)[Ax(t) + Bu(t-h) + f(t)] + \Lambda + \dot{n}(t)\Big\}dt + P(t)[\overline{A}x(t) + \overline{B}u(t-h) + \overline{f}]dw(t) \tag{29}$$

By (7) and (16), comparing the corresponding coefficient is equal on both sides, we can get that:

$$q(t) = P(t)[\overline{A}x(t) + \overline{B}u(t-h) + \overline{f}] \tag{30}$$

Notice (7) and using the Riccati equation (18), also can get the following relations:

$$-[A'p(t) + \overline{A}'q(t) + Qx(t) = -\Big\{A'[P(t)x(t) + \Theta_t + n(t)] + \overline{A}'P(t)[\overline{A}x(t) + \overline{B}u(t-h) + \overline{f}] + Qx(t)\Big\}$$

$$= \dot{P}(t)x(t) + P(t)[Ax(t) + Bu(t-h) + f] + \Lambda + \dot{n}(t) \tag{31}$$

By the equilibrium condition (6) of stochastic maximum principle, we can get that:

$$0 = Ru(t-h) + E[B'p(t) + \overline{B}'q(t)\big|F_{t-h}]$$

$$= Ru(t-h) + E\Big\{B'[P(t)x(t) + \Theta_t + n(t)] + \overline{B}'p(t)[\overline{A}x(t) + \overline{B}u(t-h) + \overline{f}]\big|F_{t-h}\Big\}$$

$$= Ru(t-h) + B'P(t)\hat{x}(t\big|t-h) + B'E[\Theta_t\big|_{t-h}] + B'n(t) + \overline{B}'p(t)\overline{A}\hat{x}(t\big|t-h) + \overline{B}'P(t)\overline{B}u(t-h) + \overline{B}'P(t)\overline{f}$$

$$= [R + \overline{B}'P(t)\overline{B}]u(t-h) + [B'P(t) + \overline{B}'P(t)\overline{A}]\hat{x}(t\big|t-h) + B'E[\Theta_t\big|_{t-h}] + B'n(t) + \overline{B}'P(t)\overline{f}$$

$$= [R + \overline{B}'P(t)\overline{B}]u(t-h) + [B'P(t) + \overline{B}'P(t)\overline{A}]\hat{x}(t\big|t-h) - B'\int_t^{t+h}\Pi(t,s)\hat{x}(t\big|t-h)ds + B'n(t) + \overline{B}'P(t)\overline{f}$$

$$= [R + \overline{B}'P(t)\overline{B}]u(t-h) + [B'P(t) + \overline{B}'P(t)\overline{A} - B'\int_t^{t+h}\Pi(t,s)ds]\hat{x}(t\big|t-h) + B'n(t) + \overline{B}'P(t)\overline{f} \tag{32}$$

Let $\Omega(t) = R + \overline{B}'P(t)\overline{B}$, $H(t) = B'n(t) + \overline{B}'P(t)\overline{f}$,

$$K(t) = -\Omega^{-1}(t)[B'P(t) + \overline{B}'P(t)\overline{A} - B'\int_t^{t+h}\Pi(t,s)ds]$$

Easy to get $u(t-h) = K(t)\hat{x}(t\big|t-h) - \Omega^{-1}(t)h(t)$ (33)

The following still need to verify the $P(t)$, $n(t)$ coupled Riccati equation (18), (19).

In fact, we put $u(t-h)$ into (11) and reduction, we can get that:

$$[-A'P(t)-\overline{A}'P(t)\overline{A}-Q]x(t)-A'P(t)\overline{B}K(t)\hat{x}(t|t-h)+\overline{A}'P(t)\overline{B}\Omega^{-1}(t)H(t)-A'\Theta_t-A'n(t)-\overline{A}'P(t)\overline{f}$$

$$=(\dot{P}(t)+P(t)A)x(t)+P(t)BK(t)\hat{x}(t|t-h)-P(t)B\Omega^{-1}(t)H(t)+P(t)f+\dot{n}(t) \tag{34}$$

Due to the corresponding coefficient is equal, we can get that:

$$0=\dot{P}(t)+P(t)A+A'P(t)+\overline{A}'P(t)\overline{A}+Q-\Pi(t,t+h) \tag{35}$$

$$0=\dot{n}(t)+P(t)f+A'n(t)+\overline{A}'P(t)\overline{f}-[P(t)B(t)+\overline{A}'P(t)\overline{B}]\Omega^{-1}(t)H(t) \tag{36}$$

That is (18), (19)

End of theorem proving □

2.4. Model Solution

We deal with the mean-variance model (1) - (3) by using theorem 2.1 in the following.

It is easy to know that the model is a special case of the problem "research the controller $u(t-h)$ of F_{t-h} measurable to make the minimum J_T ". For the system (4), (5), make that

$$A(t)=r(t), B(t)=(b(t)-r(t))=(b_1(t)-r(t),...,b_m(t)-r(t)),$$

$$f(t)=\gamma r(t), \overline{A}(t)=0, \overline{B}(t)=\sigma(t)=(\sigma_{1j}(t),...,\sigma_{mj}(t)),$$

$$\overline{f}=0, (Q(t),R(t))=(0,0), P(T)=\mu.$$

Using theorem 2.1 we can get that the optimal portfolio strategy of problem is:

$$u^*(t-h)=K(t)\hat{x}(t|t-h)-\Omega^{-1}(t)H(t) \tag{37}$$

Among them

$$\hat{x}(t|t-h)=e^{r(t)h}x(t-h)+\int_{t-h}^{t}e^{r(t)(t-\theta)}(b(t)-r(t))u(\theta-h)d\theta$$

$$H(t)=(b(t)-r(t))'n(t),$$

$$\Omega(t)=\sigma(t)'u\sigma(t),$$

$$K(t)=-\Omega^{-1}(t)[(b(t)-r(t))'u-(b(t)-r(t))'\int_t^{t+h}\Pi(t,s)ds]$$

$$-\dot{P}(t)=P(t)r(t)+r(t)'P(t)-\Pi(t,t+h), \quad P(T)=\mu,$$

$$-\dot{n}(t)=P(t)\gamma r(t)+r(t)'n(t)-[P(t)(b(t)-r(t))]\Omega^{-1}(t)H(t),$$
$$n(T)=0.$$

Obviously, from theorem 2.1, when $h=0$, that is degrade to the front part three without delay; When $h\succ 0$, by theorem 2.1, in order to solve the optimal controller, we need to solve the $P(t)$ and $n(t)$ firstly, then solving Riccati equation. The mean - variance model analytical solution of the corresponding Riccati equation generally difficult to solve, often we can use numerical algorithm to get $P(t)$ and $n(t)$ by programming, then calculate the $K(t)$, $H(t)$ and optimal $u^*(t), t\in[h,T]$.

3. Conclusion

With the rapid development of economy in our country, the capital market gradually perfect, the financial product is increasing, especially some risks such as balance of treasure of wealth management products, make the broad masses of investors will not put money only one basket, will be carried out in a certain portfolio investment, in order to obtain greater benefits. The mean - variance model is one of the basic calculation model of portfolio strategy, previous researchers did not consider delay problems, on the basis of existing research. In this paper it established the delay discrete mean - variance model, stochastic optimal control with the aid of LQ problem, to get the analytical solution of the optimal portfolio. Although the significance of this research is more important, the result is satisfactory, but its research conclusion and the actual still has a gap, there are still many aspects need further research, such as:

1) on the basis of solving the optimal strategy, to further study the effective frontier of portfolio, analysis the relationship between the expected return and risk; 2) Considering it contains transaction fees and consumption in the model; 3) considering investing object contains options, etc.

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